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Which Fields Have No Maximal Subrings?

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Abstract - Fields which have no maximal subrings are completely determined. We observe that the quotient fields of non-field domains have maximal subrings. It is shown that for each non-maximal prime ideal P in a commutative ring R, the ring R_P has a maximal subring. It is also observed that if R is a commutative ring with $|Max(R)| > 2^{\aleph_0}$ or $|R/J(R)| > 2^{2^{\aleph_0}}$, then R has a maximal subring. It is proved that the well-known and interesting property of the field of the real numbers $\mathbb{R}$ (i.e., $\mathbb{R}$ has only one nonzero ring endomorphism) is preserved by its maximal subrings. Finally, we characterize submaximal ideals (an ideal I of a ring R is called submaximal if the ring R/I has a maximal subring) in the rings of polynomials in finitely many variables over any ring. Consequently, we give a slight generalization of Hilbert's Nullstellensatz.

Introduction

All rings in this article are commutative with $1 \neq 0$. If S is a subring of a ring R, then $1_R \in S$. The characteristic of a ring R is denoted by $c(R)$ and the algebraic closure of a field K is denoted by $\bar{K}$. A proper subring S of a ring R is said to be maximal if there is no subring of R properly between S and R. Unlike maximal ideals, whose existence is guaranteed by either Zorn's Lemma or Noetherianity of rings, maximal subrings need not always exist. We denote the set of maximal subrings (maximal ideals) of a ring by $RgMax(R)$ ($Max(R)$). Maximal subrings are also studied in [17], [5], [6], [14], [13], [16], [2], [3] and [4]. In [2], maximal subrings of commutative rings are investigated and some useful criterions for the existence of maximal subrings are given. Using these criterions one can easily see that some important rings have maximal

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subrings, see [2]. In [3], we have characterized exactly which Artinian rings have maximal subrings. In particular, in [3], it is shown that, if F is a field which has either zero characteristic or is not algebraic over some subfields (in particular every uncountable field) then F has a maximal subring. It is also interesting to note that if R is an integral domain with the quotient field K, then R is a valuation ring with a unique nonzero prime ideal if and only if R is a maximal subring of K, see [12, P. 43]. Consequently, if F/K is a function field of one variable (i.e., an extension field $K \subseteq F$ such that F is a finite algebraic extension of $K(x)$, where $x \in F$ is a transcendental element over K) then the set $RgMax(F) \neq \emptyset$ and it coincides with the set of discrete valuation subrings of F (note, by a subring R of F, we mean $K \subsetneq R \subsetneq F$), see [19, Theorem 1.1.13]. The latter fact is also an immediate consequence of a more general result, namely, if $K \subseteq F$ is a non-algebraic field extension, then F has a maximal subring containing K, which is not a field, see [3, Corollary 1.2]. We also recall that if V is a variety over an algebraically closed field k, then its coordinate ring $k[V] = \dfrac{k[x_1, \ldots, x_n]}{I(V)}$ is a domain, hence it has a field of fractions $K = k(V)$, which is called the function field of V. In particular, if $V = \mathbb{P}^1_k$, where $\mathbb{P}^1_k$ is the projective line over k, then one can easily show that $\mathbb{P}^1_k$ is bijective with the set of maximal subrings of $k(x)$ containing k, see [19, Theorem 1.2.2, Corollary 1.2.3] or [9, Problem 4.8]. More generally, if V is a hyperelliptic curve, see [10, P. 298], then by an observation which appeared recently in [21], see also [19, Appendix B] and [20], the set V is bijective with $RgMax(k(V))$.

Now let us sketch a brief outline of this paper. Our aim in Section 1, is to determine exactly which fields have no maximal subrings. In fact, we show that the only fields E which have no maximal subrings are those of the form $E = \bigcup_{n \in T} F_{p^n}$, where p is a prime number, F_{p^n} is the unique subfield, with p^n elements, of $\bar{F}_p$, the algebraic closure of $F_p = \dfrac{\mathbb{Z}}{(p)}$, and T is a subset of $\mathbb{N}$, the set of positive integers, such that $T = \{n \in \mathbb{N}\ :\ \text{every prime divisor of } n \text{ is in a fixed set of prime numbers, say } P\} \cup \{1\}$. Finally, in Section 1, we show that if $\mathcal{A}$ is the set of fields without maximal subrings, up to isomorphism, then $|\mathcal{A}| = 2^{\aleph_0}$. In Section 2, we find more rings R with $RgMax(R) \neq \emptyset$. Also in Section 2, we investigate some hereditary properties between a ring and its maximal subrings. Finally, in Section 3, we determine exactly which ideals of $R[x]$ are not submaximal, and as a consequence we give a generalization of Hilbert's Nullstellensatz.

1. Fields without maximal subrings

Before presenting our new results, let us cite the following facts from [3]. The next result is in [3, Proposition 1.1] and for a more general result, see [4, Proposition 2.1].

PROPOSITION 1.1. *Let $F \subseteq K$ be an algebraic field extension and S be a maximal subring of F which is not a field. Then for any non-unit element $x \in S$, there exists a maximal subring R_x of K such that x is not a unit element in R_x, $R_x \cap F = S$, $R_x[x^{-1}] = K$ and R_x is not a field, but it contains the integral closure of S in K.*

COROLLARY 1.2 [3, Corollary 1.2]. *Let F be a field and $F \subseteq K$ be a non-algebraic field extension of F. Then K has a maximal subring which contains F and is not a field.*

COROLLARY 1.3 [3, Corollary 1.3]. *Let K be a field. If K is of characteristic zero or uncountable then K has a maximal subring which is not a field.*

The following is now immediate.

COROLLARY 1.4. *Let E be a field without maximal subrings, then E is algebraic over F_p for some prime number p.*

Next, we recall that if K is a field and R is a domain containing K which is also algebraic over K, then R is a field. This immediately implies that whenever K is an algebraic field extension of a field F, then any subring between F and K is a field. In particular, any subring of $\bar{F}_p$ is a field (more generally, it is also well-known that any vector subspace E of $\bar{F}_p$ over F_p is a field, if and only if for any $\alpha \in E$ we have $\alpha^n \in E$, for all n).

In what follows, in this section we are going to present exactly the form of fields which have no maximal subrings.

We need the next definitions.

DEFINITION 1.5. Let $\mathbb{N}$ be the set of positive integers and $T \subseteq \mathbb{N}$. Then T is said to be a field generating set (briefly FG-set) if $E = \bigcup_{n \in T} F_{p^n}$ is a subfield of $\bar{F}_p$, where F_{p^n} is the unique subfield of $\bar{F}_p$ with p^n elements and p is a prime number. Moreover, T must be such that if $T \subseteq T' \subseteq \mathbb{N}$ and $E = \bigcup_{n \in T} F_{p^n} = \bigcup_{n \in T'} F_{p^n}$, then $T = T'$.

Definition 1.6. Let $T_1 \subset T_2$ be two FG-sets. Then T_1 is said to be a maximal FG-subset of T_2 if there is no FG-set properly between T_1 and T_2.

Remark 1.7. One can easily see that there is a one-one order preserving correspondence between the FG-subsets of $\mathbb{N}$ and subfields of $\bar{F}_p$, see also the Steinitz's numbers and their properties in either [7] or [18].

Using the previous remark and Corollary 1.4, we immediately have the following characterization of the fields without maximal subrings.

Theorem 1.8. *A field F has no maximal subrings if and only if there is a prime number p such that either $F = F_p$ or F is an infinite subfield of $\bar{F}_p$ such that $F = \bigcup_{n \in T} F_{p^n}$, where T is a FG-set which has no maximal FG-subsets.*

We know that the multiplicative group $F^*_{p^n}$ of nonzero elements of F_{p^n} is cyclic of order $p^n - 1$. This and the fact that whenever an abelian group G contains two elements of orders m and n, then it has an element of order $[m, n]$, the least common multiple of m, n, immediately yield the following characterization of FG-sets.

Proposition 1.9. *$T \subseteq \mathbb{N}$ is a FG-set if and only if it satisfies the following conditions.*

(1) $1 \in T$
(2) *If $n \in T$, $d|n$, then $d \in T$.*
(3) *If $m, n \in T$, then $[m, n] \in T$.*

Proof. Let T be a FG-set, then (1) and (2) are evident. Hence let $E = \bigcup_{n \in T} F_{p^n}$ be the subfield of $\bar{F}_p$ corresponding to T. If $m, n \in T$, then let $\alpha \in F^*_{p^n}, \beta \in F^*_{p^m}$ be elements in E which are of order $p^n - 1, p^m - 1$ respectively. Then E has an element γ of order $[p^n - 1, p^m - 1]$. But γ is a root of the equation $x^{p^k} - x = 0$, where k is the degree of $f(x)$, the minimal polynomial of γ over F_p (note, the roots of $f(x)$ are $\gamma, \gamma^p, \dots, \gamma^{p^{k-1}}$). Clearly, $\gamma^{p^k - 1} = 1$. Hence $[p^n - 1, p^m - 1] | p^k - 1$, consequently we have $p^n - 1 | p^k - 1$, $p^m - 1 | p^k - 1$. Thus $n|k$, $m|k$ imply that $[m, n]|k$. Since $F_p(\gamma) = F_{p^k} \subseteq E$ we infer that $k \in T$ and therefore $[m, n] \in T$. The converse is evident. $\square$

By the previous proposition we have the following trivial examples of FG-sets.

EXAMPLE 1.10. *$T \subseteq \mathbb{N}$ is a finite FG-set if and only if T consists of divisors of an integer $n \in \mathbb{N}$. Let P be a set of prime numbers, then $T = \{n \in \mathbb{N}$: prime divisors of n are in $P\} \cup \{1\}$ is a FG-set.*

Theorem 1.8 shows that in order to know exactly which fields are without maximal subrings we must know the *FG*-sets which have no maximal *FG*-subsets. The next result precisely determines these *FG*-sets.

PROPOSITION 1.11. *Let T be a FG-set. Then T has no maximal FG-subsets if and only if whenever q is a prime number in T, then $q^n \in T$ for all $n \in \mathbb{N}$. Moreover, T has no maximal FG-subsets if and only if T has the form $T = \{n \in \mathbb{N}$: every prime divisor of n is in a fixed set of prime numbers, say $P\} \cup \{1\}$.*

PROOF. Let T be a *FG*-set with the above property. We claim that T has no maximal *FG*-subsets. Let $S \subsetneq T$ be a maximal *FG*-subset of T and obtain a contradiction. Take $n \in T \setminus S$ and let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ be the factorization of n. Since $n \notin S$, we infer that there exists $1 \le r \le k$ with $p_r^{m_r} \notin S$. Now put $S_1 = S \cup \{[m, p_r^t] : m \in S, t \le m_r\}$. Clearly, S_1 is a *FG*-set and $S \subsetneq S_1 \subsetneq T$ (note, $p_r^m \in T$ for all m) which is the desired contradiction.

Conversely, let T have no maximal *FG*-subsets. Suppose that $q \in T$ is a prime number such that $q^n \notin T$ for some $n \ge 2$ and n is the least positive integer with this property and seek a contradiction. Since T is a *FG*-set, we infer that $q^m \notin T$ for all $m \ge n$. Now put $S = \{x \in T : \ x \neq q^{n-1}y, \ y \in T, \ (q, y) = 1\}$ and note that S is a maximal *FG*-subset of T. To see this, it is clear that S is a *FG*-set. Let $S \subseteq S_1 \subseteq T$, where S_1 is a *FG*-set. If $S \neq S_1$, then there exists $q^{n-1}y \in S_1 \setminus S$, where $y \in T$, $(y, q) = 1$. Hence $q^{n-1} \in S_1$ and therefore $q^{n-1}z \in S_1$ for all $z \in T$, $(z, q) = 1$ (note, $z \in S \subseteq S_1$). Thus $S_1 = T$ and we are done. The last part is now evident. $\square$

COROLLARY 1.12. *a FG-set T has no maximal FG-subsets if and only if whenever $t \in T$, then $t^n \in T$ for all $n \in \mathbb{N}$.*

REMARK 1.13. Trivially by the above corollary, $\mathbb{N}$ is a *FG*-set without maximal *FG*-subsets, hence $\bar{F}_p$ has no maximal subrings for any prime number p.

Next, we observe that the collection of all *FG*-sets which are without maximal *FG*-subsets is uncountable.

Proposition 1.14. *Let $\mathcal{A}$ be the collection of all FG-sets which are without maximal FG-subsets. Then $\mathcal{A}$ is uncountable.*

Proof. Let $p_1, p_2, \ldots, p_n, \ldots$ be the sequence of prime numbers, i.e., $p_1 = 2,\ p_2 = 3,\ p_3 = 5$, etc. We define the map $\phi : \mathcal{A} \longrightarrow \prod_{i=1}^{\infty} X_i$, where $X_i = \{0, 1\}$ for all i, as follows. For each $A \in \mathcal{A}$ we put $\phi(A) = <a_n>$, where $a_n = 0$ if $p_n \notin A$ and $a_n = 1$ if $p_n \in A$. It is manifest that ϕ is one-one and onto. Hence $|\mathcal{A}| = 2^{\aleph_0}$. $\square$

Corollary 1.15. *Let $\{F_i : i \in I\}$ be the family of fields, which are without maximal subrings and have the same characteristic (up to isomorphism). Then $|I| = 2^{\aleph_0}$. In particular, the set of all fields without maximal subrings (up to isomorphism) is of cardinality $2^{\aleph_0}$.*

2. More rings with maximal subrings and some hereditary properties

In this section, we begin with the following interesting result, which is a generalization of the fact that each Artinian ring with zero characteristic has a maximal subring, see [3].

Proposition 2.1. *Every zero dimensional ring R of characteristic zero has a maximal subring.*

Proof. If $S = \mathbb{Z} \setminus \{0\}$, then there exists a maximal ideal M such that $S \cap M = \emptyset$, hence R/M is a field with zero characteristic. Therefore R/M has a maximal subring, by Corollary 1.3. Thus R has a maximal subring too. $\square$

Corollary 2.2. *Let $\{R_i\}_{i=1}^{\infty}$ be a family of rings with $c(R_i) = n_i \neq 0$ and $(n_i, n_j) = 1$ for all $i \neq j$. Then $R = \prod_{i=1}^{\infty} R_i$ has a maximal subring.*

Proof. Let $M = \prod_{i=1}^{\infty} M_i$, where each M_i is a maximal ideal of R_i. Then R/M is a zero dimensional ring (in fact, a regular ring) whose characteristic is clearly zero. Hence, by the previous proposition, R/M, a fortiori R, have maximal subrings. $\square$

Proposition 2.3. *Let R be a domain and $F \neq R$ be its quotient field. Then F has a maximal subring.*

PROOF. If $c(R) = 0$, then $c(F) = 0$ and we are done by Corollary 1.3. Thus we may suppose that $c(R) = p \neq 0$ and R contains F_p. Since R is not field, we infer that R is not algebraic over F_p. Hence F is not algebraic over F_p and therefore by Corollary 1.2 we are done. □

The following is interesting.

COROLLARY 2.4. *Let R be a ring and P be a prime ideal of R which is not maximal. Then R_P has a maximal subring.*

PROOF. Let F be the quotient field of R/P. Clearly $F \neq R/P$. Hence by the previous proposition F has a maximal subring. But we have $\frac{R_P}{P^e} \cong F$, where P^e is the extension of P in R_P. This means that $\frac{R_P}{P^e}$ has a maximal subring and therefore R_P has a maximal subring too. □

PROPOSITION 2.5. *Let R be a ring which contains a field F satisfying at least one of the following conditions:*

(1) $c(F) = 0$.
(2) *F is uncountable.*
(3) *F is not algebraic over a finite field.*

Then R has a maximal subring.

PROOF. Let M be a maximal ideal in R. Then the field R/M contains a copy of F. Therefore if F satisfies in one of the above conditions, then R/M clearly satisfies it too. Hence by Corollaries 1.2, 1.3 and 1.4, R/M has a maximal subring which implies that R has a maximal subring too. □

PROPOSITION 2.6. *Let R be a ring with $|Max(R)| > 2^{\aleph_0}$. Then R has a maximal subring.*

PROOF. Put $T = \{R/M \ : \ M \in Max(R)\}$. If $R/M \ncong R/M'$ for any two maximal ideals M, M', then $|T| > 2^{\aleph_0}$ and we are done by Corollary 1.15. Hence we may assume that $R/M \cong R/M'$ for two maximal ideals M, M'. Now $\frac{R}{M \cap M'} \cong \frac{R}{M} \times \frac{R}{M'}$ and by [2, Theorem 2.2], we see that $\frac{R}{M \cap M'}$ has a maximal subring, a fortiori, R has a maximal subring. □

COROLLARY 2.7. *Let $\{R_i\}_{i \in I}$ be a family of rings with $|I| > 2^{\aleph_0}$. Then $R = \prod_{i \in I} R_i$ has a maximal subring.*

COROLLARY 2.8. *Let R be a Hilbert ring. Then either R has a maximal subring or $|Spec(R)| \leq 2^{2^{\aleph_0}}$.*

PROOF. If $|Max(R)| > 2^{\aleph_0}$, then we are done, by Proposition 2.6. Hence suppose that $|Max(R)| \leq 2^{\aleph_0}$. Since R is a Hilbert ring, every prime ideal of R is an intersection of maximal ideals, therefore $|Spec(R)| \leq 2^{2^{\aleph_0}}$. □

The following is also in order.

PROPOSITION 2.9. *Let R be a ring with $|J(R)| < |R|$ (or with $|U(R)| < |R|$, where $U(R)$ is the set of units of R). If $|R| > 2^{2^{\aleph_0}}$, then R has a maximal subring. In particular, if $|R/J(R)| > 2^{2^{\aleph_0}}$, then R has a maximal subring.*

PROOF. First we note that $|J(R)| \leq |U(R)|$. If either $|Max(R)| > 2^{\aleph_0}$ or R/M is uncountable for some maximal ideal M of R, then we are done, by Proposition 2.6 or Corollary 1.3. Hence we may assume that $|Max(R)| \leq 2^{\aleph_0}$, R/M is countable for all $M \in Max(R)$ and seek a contradiction. Clearly, $R/J(R)$ embeds in $\prod_{M \in Max(R)} \frac{R}{M}$ and therefore $|R/J(R)| \leq 2^{2^{\aleph_0}}$ which is absurd. The final part is now evident. □

The next interesting proposition shows that maximal subrings of $\mathbb{R}$ are similar to $\mathbb{R}$, regarding the existence of the unique nonzero ring endomorphism.

PROPOSITION 2.10. *Let S be a maximal subring of $\mathbb{R}$. Then S is dense in $\mathbb{R}$. Moreover, the only nonzero ring homomorphism from S into S is the identity.*

PROOF. Clearly, S is uncountable. It is well-known that any additive subgroup of $\mathbb{R}$ is either cyclic or dense in $\mathbb{R}$. But clearly, S cannot be cyclic, hence it is dense. Now for the final part, let $\phi : S \longrightarrow S$ be a nonzero homomorphism and consider the following two cases.

Case(I): $\mathbb{Q} \not\subseteq S$. In this case, we first note that S is not a field and $S \cap \mathbb{Q} = \mathbb{Z}_{(p)}$, where p is a prime number. To see this, since $\mathbb{Q} \not\subseteq S$, we infer that there exists a prime number p such that $\frac{1}{p} \notin S$. But in view of [2, Theorem 3.3], S has only one nonzero prime ideal, say M. Thus $p \in M$ and for every prime number $q \neq p$, we have $q \notin M$. Consequently, $\frac{1}{q} \in S$ and therefore $\mathbb{Z}_{(p)} \subseteq S$ which implies that $S \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ (note, $\mathbb{Z}_{(p)}$ is a maximal subring of $\mathbb{Q}$). We also observe that $c(S/M) = p$, for $p \in M$. Now we claim that ϕ is an injective homomorphism. To this end, we show that $P = ker\phi \neq 0$, leads us to a contradiction. Since S/P embeds in the domain S, we infer that P is a prime ideal and therefore we must have $P = M$. But $c(S/P) = p$ which is absurd, for $S/P \cong Im\phi \subseteq \mathbb{R}$. Next, we show that ϕ is order preserving. Let $a \in S$, $a > 0$ and therefore $a = x^2$, $x \in \mathbb{R}$. We claim that $x \in S$ and we are done, for in that case $\phi(a) = \phi(x)^2 > 0$. To see this, $x^2 = a$ implies that x is integral over S and therefore if $x \notin S$, then $\mathbb{R} = S[x]$ is integral over S. Consequently, S is a field which is impossible. Hence we must have $x \in S$ and $\phi(a) > 0$ which means that ϕ is order preserving. Finally, let $\phi(a) = b$, where $a, b \in S$, $a \neq b$ and seek a contradiction. We may assume that $a < b$ (note, $\phi(-a) = -b$). Since $\mathbb{Z}_{(p)}$ is not a cyclic group, we infer that $\mathbb{Z}_{(p)}$ is dense in $\mathbb{R}$ and therefore there exists $q \in \mathbb{Z}_{(p)}$ with $a < q < b$. Hence $\phi(a) < \phi(q) < \phi(b)$. But trivially, ϕ is the identity on $\mathbb{Z}_{(p)}$ which means that $b < q < \phi(b)$, a contradiction.

Case(II): $\mathbb{Q} \subseteq S$. First, let us get rid of the case when S might be a field (note, we shall see shortly in the next remark that S can never be a field, but let us ignore this for a moment). For every $\alpha \in \mathbb{R} \setminus S$, we have $\mathbb{R} = S[\alpha]$ and $\mathbb{R}$ is algebraic over S (note, $\alpha^{-1} \in S[\alpha]$). Hence we may naturally extend ϕ to $\phi^* : \mathbb{R} \longrightarrow \mathbb{R}$ by defining $\phi^*|_S = \phi$ and $\phi^*(\alpha) = \alpha$. But it is well-known and easy to prove that ϕ^* is the identity (in fact, we have already proved this in the last part of the proof of the previous case) and we are done. Hence we may suppose that S is not a field. We now claim that ϕ is also order preserving in this case. It suffices to show that $a > 0$ implies $\phi(a) > 0$ for all $a \in S$. We have $a = x^2$, where $x \in \mathbb{R}$. We now claim that $x \in S$, for similar to the previous case, $x \notin S$ implies that S is a field which is absurd. Hence $\phi(a) = \phi(x)^2 \geq 0$. Now let $q \in \mathbb{Q}$ be an element with $0 < q < a$, then $0 \leq \phi(q) \leq \phi(a)$. But trivially ϕ is the identity on $\mathbb{Q}$, Hence $0 < q \leq \phi(a)$, i.e., $\phi(a) > 0$. Finally, by applying the final part of the proof of the previous case and replacing $\mathbb{Z}_{(p)}$ by $\mathbb{Q}$ we are done. □

We give the following remark for the sake of the reader.

Remark 2.11. We should clarify our ignoring, intentionally, an important fact concerning the maximal subring S of $\mathbb{R}$ in the previous proof. Let us first recall an amazing and well-known theorem of Artin-Schreier, which says, whenever a field F is not algebraically closed but its algebraic closure, say L, is a finite extension of F, then F must be real closed and $L = F(i)$, where $i^2 = -1$, see [11, P. 316, Theorem 17]. It follows easily that no maximal subring of a real closed field (note, the algebraic closure of a real close field is a finite extension over it), let alone $\mathbb{R}$, can be a field (note, if S is a maximal subring of $\mathbb{R}$ which is a field, then clearly $\mathbb{R}$ is a finite extension of S, see [2, Theorem 3.3] and therefore $\mathbb{C}$, the field of complex numbers, is a finite extension of S). We can also easily infer from the Artin-Schreier Theorem, that no maximal subring of an algebraically closed field with a nonzero characteristic, can be a field. Hence in the proof of Proposition 2.10, although we know that, S is actually not a field, but we prefer not to make use of this in the proof, on purpose, in order, to avoid invoking the sophisticated theorem of Artin-Schreier, to just get rid of the trivial case when S might be a field.

Now, we digress for a moment to record the following interesting fact.

Proposition 2.12. *Let R be a domain. Then there is no principal maximal ideal in the following rings.*

(1) $R[x]$, *where R is an infinite domain whose set of units has smaller cardinality than $|R|$.*
(2) $R[x_1, x_2, \ldots, x_n]$, $n \geq 2$.

Proof. First we prove (1). Let $M = (f(x))$ be a maximal ideal in $R[x]$ and seek a contradiction. Clearly, the set A of all elements $a \in R$ such that $f(a)$ is a unit, has smaller cardinality than $|R|$ (note, if the set of units is finite then A is finite too, i.e., $|A| < |R|$, if the set of units is infinite and has cardinality α, then $|A| \leq \alpha < |R|$). Thus there is $b \in R$ such that $f(b) = c \neq 0$ is not a unit (note, $f(x) = 0$ cannot have a root in R, for otherwise $R[x]/M \cong R$, which is absurd; we should also emphasize that we may assume $c \neq 0$ by just using the fact that $|A| < |R|$). Now we note that $(f(x)) \subsetneq (c, f(x)) \neq R[x]$, for if $1 = cg(x) + h(x)f(x)$, then $1 = c(g(b) + h(b))$, a contradiction. Hence $(f(x))$ is contained in a proper ideal $(c, f(x))$ which is a contradiction.

Finally we prove (2). This part must be well-known (at least when R is a field), but we give a quick proof. It suffices to prove it for $n = 2$. Hence let (f) be a maximal ideal in $R[x_1, x_2]$ and obtain a contradiction. Clearly, f is not a constant element in R. Put $f = f_0 + f_1 x_2 + \cdots + f_m x_2^m$, where $f_i \in R[x_1]$, $i = 0, 1, \ldots, m$. It is clear that $f \notin R[x_1]$, $f \notin R[x_2]$ and $f_0 \neq 0$. Now we note that $f = F(x_2) \in R[x_1][x_2]$ and there is some k such that $F(x_1^k) \notin R$, for otherwise for all $r \neq s$ we have $x_1^r - x_1^s | F(x_1^r) - F(x_1^s)$ which is absurd. Finally, we claim that $(f) \subsetneq (F(x_1^k), f) \neq R[x_1, x_2]$, which is the desired contradiction. To this end, it is evident that $(f) \neq (F(x_1^k), f)$. Hence we must show that $(F(x_1^k), f) \neq R[x_1, x_2]$. If $(F(x_1^k), f) = R[x_1, x_2]$, then

$1 = F(x_1^k)g + fh$ implies that $1 = F(x_1^k)(g(x_1, x_1^k) + h(x_1, x_1^k))$, where $g, h \in R[x_1, x_2]$. Clearly, the latter equality is absurd. $\square$

REMARK 2.13. We recall that a domain R is a G-domain if and only if there is a principal maximal ideal M in $R[x]$ with $M \cap R = (0)$, see the proof of Theorem 24 in [12]. This and part (2) of the previous result immediately gives another proof of the well-known fact that $R[x]$ is never G-domain.

Part (1) of Proposition 2.12, immediately yields the following corollary.

COROLLARY 2.14. *If R is an infinite G-domain and $U(R)$ is the set of units of R, then $|U(R)| = |R|$.*

REMARK 2.15. It is interesting to notice that if R is a domain whose Jacobson radical J is nonzero, then $|U(R)| = |R|$ (note, $|J| = |R|$ and $1 - x$ is a unit for all $x \in J$). This immediately gives another quick proof to the previous corollary.

In [5], it is proved that if R is a finite maximal subring of a ring T, then T must be finite too. This is a trivial consequence of the fact that whenever R is a maximal subring of T, then R is Artinian if and only if T is Artinian and integral over R, see [3, the proof of Theorem 2.9] or [4, Theorem 3.8].

COROLLARY 2.16. *If R is a maximal subring of an infinite G-domain T, then $|U(R)| = |R|$.*

PROOF. We note that T is algebraic over R, for if $t \in T$, then either $t^2 \in R$ or $t \in R[t^2]$. Clearly, T is also finitely generated as a ring over R (note, for each $t \in T \setminus R$, we have $R[t] = T$). Now, by [12, Theorem 22], R is a G-domain and by the preceding comment it must be infinite too and therefore by Corollary 2.14, we are done. $\square$

REMARK 2.17. By the proof of the above corollary, and [12, Theorem 22], in fact if R is a maximal subring of an integral domain T, then R is a G-domain if and only if T is a G-domain. In particular, if R is a maximal subring of a ring T (not necessarily an integral domain), then an ideal Q in T is a G-ideal if and only if $Q \cap R$ is a G-ideal in R.

We remind the reader that whenever R is a maximal subring of a ring T, then either T is integral over R or R is integrally closed in T. Also one can easily see that if R is a maximal subring of T, then the conductor ideal $(R:T)$ is a prime ideal in R, see [8] or [17, Theorem 1 and Theorem 7], see also [4, Remark 3.1]. Moreover, in [8] or [17] it is proved that if R is a maximal subring of a ring T, then T is integral over R if and only if $(R:T) \in Max(R)$; it is also shown that if R is integrally closed in T, then $(R:T) \in Spec(T)$. The following is now in order.

PROPOSITION 2.18. *Let R be a maximal subring of a ring T. Then the following statements hold.*

(1) *If R is a Hilbert ring then so is T.*

(2) *If T is integral over R, then R is a Hilbert ring if and only if T is a Hilbert ring.*

(3) *If T is not integral over R and $(R:T) \in Max(T)$, then R is never a Hilbert ring.*

PROOF. We remind the reader that, whenever R is a Hilbert ring, then the rings of polynomials in finitely many variables over R and epimorphic images of R are also Hilbert, see [12]. This proves (1) (note, for any $x \in T \setminus R$, we have $R[x] = T$). Therefore for (2), it suffices to show that R is Hilbert whenever T is Hilbert. Let P be a G-ideal of R, since T is integral over R, there exists a prime ideal Q of T such that $Q \cap R = P$. Now, we have two cases, if $Q \not\subseteq R$, then $R + Q = T$. Hence we infer that $R/P \cong T/Q$. Therefore Q is also a G-ideal of T. Thus Q is a maximal ideal of T, which implies that P is a maximal ideal in R, too. Now, assume that $Q \subseteq R$. Therefore $Q = P$ and R/P is a maximal subring of T/P. Inasmuch as R/P is a G-domain, we infer that T/P is also a G-domain, by Remark 2.17. Hence P is a maximal ideal of T and since T/P is integral over R/P, we infer that R/P is a field too, and we are done. For (3), we note that by Remark 2.17, $M = (R:T)$ is a G-ideal of R. Now, since T is not integral over R, M is not a maximal ideal in R, by the above comment. Hence we are done. □

Finally, we conclude this article with the following short section on submaximal ideals.

3. Submaximal Ideals

Let us for the sake of brevity, in this section, call a ring R submaximal, if R has a maximal subring and call an ideal I of R, a submaximal ideal if R/I is a submaximal ring (note, $I \neq R$). It is manifest that whenever R

contains a field which is either of zero characteristic or uncountable, then every ideal of R is submaximal. We also note that if R is a Noetherian ring with $|R| > 2^{\aleph_0}$, then the nilradical of R is submaximal, by [4, Theorem 2.9]. It is easy to see that a ring R is submaximal if and only if there exist a proper subring S of R and $x \in R \setminus S$ such that $S[x] = R$, see [2, Theorem 2.5].

The following is interesting.

LEMMA 3.1. *Let R be a ring and I be an ideal of $R[x]$. Then I is submaximal if and only if at least one of the following conditions holds,*

(1) *For any $r \in R$, $x - r \notin I$.*

(2) *There exists an element $r \in R$ such that $x - r \in I$ and $I \cap R$ is submaximal in R. In particular in this case $I = (I \cap R, x - r)$.*

PROOF. Let $\phi : R[x] \to \dfrac{R[x]}{I}$ be the natural epimorphism and put $S = \phi(R)$. We consider two cases.

(1) Suppose for each $r \in R$, we have $x - r \notin I$ which is equivalent to $x + I \notin S$. Hence $\dfrac{R[x]}{I} = S[x + I]$ and we are done by the above comment.

(2) Suppose that $x - r \in I$ for some $r \in R$ or equivalently, $S = \dfrac{R[x]}{I}$. But clearly, $\dfrac{R[x]}{I} = S \cong \dfrac{R}{I \cap R}$ implies that I is submaximal in $R[x]$ if and only if $I \cap R$ is submaximal in R. It remains to be shown that $I = (x - r, I \cap R)$. To this end, it is evident that $(x - r, I \cap R) \subseteq I$. Now let $f(x) \in I$ and put $f(x) = (x - r)g(x) + b$, where $g(x) \in R[x]$ and $b \in R$. Consequently, $f(x) \in (x - r, I \cap R)$ and we are done. $\square$

COROLLARY 3.2. *The ring T is submaximal if and only if there exist a ring R and a ring epimorphism $\phi : R[x] \to T$ such that either $Ker(\phi) \cap \{x - r \mid r \in R\} = \emptyset$ or $R \cap Ker(\phi)$ is a submaximal ideal in R.*

We recall that if P is a G-ideal of a ring R (i.e., R/P is a G-domain), then $P = R \cap M$, for some maximal ideal M of $R[x]$. If P is not maximal, then by the proof of [12, Theorem 24], $R[x]/M$ is the quotient field of R/P.

The following is now immediate.

COROLLARY 3.3. *Let M be a maximal ideal in the ring $R[x]$. Then the following statements are equivalent.*

(1) *M is not submaximal in $R[x]$.*

(2) *$M = (x - r, m)$ for some $r \in R$, where $m = M \cap R \in Max(R)$ is not submaximal in R.*

PROOF. We note that whenever I is an ideal of $R[x]$ with $x - r \in I$ for some $r \in R$, then $I = (x - r, I \cap R)$. This and the previous lemma complete the proof if we show that when M is not submaximal, then $m = M \cap R$ is maximal in R. To see this, we note that m is a G-ideal and if it is not maximal, then $\frac{R[x]}{M} = \frac{R}{m}[t]$, for some $t \in \frac{R[x]}{M} \setminus \frac{R}{m}$ (noting that $\frac{R[x]}{M}$ is the quotient field of $\frac{R}{m}$). Consequently, by the observation in the introduction of this section, we infer that $\frac{R[x]}{M}$ is submaximal, which is absurd. Thus m must be maximal. □

COROLLARY 3.4. *Let R be a non-submaximal ring and I be an ideal in $R[x]$. Then I is submaximal in $R[x]$ if and only if $I \cap \{x - r \mid r \in R\} = \emptyset$.*

COROLLARY 3.5. *Let R be a ring and I be an ideal in $R[x_1, \ldots, x_n]$. Then I is not submaximal if and only if $I = (I \cap R, x_1 - r_1, \ldots, x_n - r_n)$, where $r_i \in R$ and $I \cap R$ is not submaximal in R.*

In view of our main result in Section 1, the next corollary characterizes non-submaximal ideals in the rings of polynomials in finitely many variables over a field K.

COROLLARY 3.6. *Let K be a field and I be an ideal in $R = K[x_1, \ldots, x_n]$. Then I is not submaximal if and only if I is a maximal ideal of the form $I = (x_1 - a_1, \ldots, x_n - a_n)$ and K is non-submaximal.*

Using the above corollary and the Hilbert's Nullstellensatz, we next present a fact which can be considered as a slight generalization of Hilbert's Nullstellensatz.

REMARK 3.7. Let K be an algebraically closed field and I be an ideal in $R = K[x_1, \ldots, x_n]$. If I is a non-submaximal ideal of R, then I is a maximal ideal of the form $I = (x_1 - a_1, \ldots, x_n - a_n)$ and $K = \bar{F}_p$, for some prime number p (note, in this case there is no need for K to be algebraically closed to get the latter form) and if I is a maximal ideal of R which is submaximal, then I has the same form and K is a submaximal field. Consequently, in this case each proper ideal of R is submaximal, too. We should also emphasize that in the former case proper non-maximal ideals are submaximal. Hence, when $K = \bar{F}_p$ and I is a proper non-maximal ideal in R, then there exist maximal subrings S of R such that $I \subseteq S$ and $S/(S \cap M) \cong \bar{F}_p$ for all maximal ideals M of R.

Finally, in the following proposition, we characterize rings in which every maximal ideal of $R[x]$ is non-submaximal, which is a generalization of Corollary 3.6.

PROPOSITION 3.8. *Let R be a ring with nonzero characteristic, say n. Then the following conditions are equivalent.*

(1) *Every maximal ideal of $R[x]$ is non-submaximal*

(2) *R is a Hilbert ring and for each maximal ideal m of R, R/m is an algebraically closed field with nonzero characteristic.*

In particular, if the number of prime divisors of n is k and $|Max(R)| > k$, then R is submaximal.

PROOF. If (1) holds, then by the proof of Corollary 3.3, R is a Hilbert ring; and if M is a maximal ideal in $R[x]$, then $M = (M \cap R, x - r)$, where $r \in R$. This shows that $R/(R \cap M)$ is algebraically closed field and we are done (note, if m is a maximal ideal in R, and $p(x)$ is an irreducible polynomial in $(R/m)[x]$, then $M = (m, q(x))$, where $q(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ with $p(x) = \sum_{i=0}^{n} \bar{a}_i x^i$, $\bar{a}_i = a_i + m$, is a maximal ideal of $R[x]$ and $M \cap R = m$, hence $R[x]/M \cong \dfrac{(R/m)[x]}{(p(x))}$ and since the latter field is non-submaximal we must have $\deg(p(x)) = 1$). The converse is similar, by applying Corollary 3.3. For the final part, since if m is a maximal ideal of R, then we have $R/m \cong \bar{F}_p$ for some prime number $p|n$, we infer that there exist distinct maximal ideals m_1 and m_2 such that $R/m_1 \cong R/m_2$, i.e., $m_1 \cap m_2$ is submaximal, by [2, Theorem 2.2], and we are done. □

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