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On Riemann-Type Definition for the Wide Denjoy Integral

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ABSTRACT - We give variational and Riemann-type definitions for some Lusin-type Δ-continuous integrals being extensions of the wide Denjoy integral.

1. Introduction

Arnaud Denjoy [2, 3] considered transfinite sequences of expanding extensions of the Lebesgue integral and, as a result, obtained two integrals: D_*- and D-integral, both having the property of recovering an everywhere differentiable function from its derivative. (The D-integral has been independently introduced by Khintchine [10].) Nowadays these integrals are also referred to as *restricted* and *wide* Denjoy integrals (or Denjoy–*Perron* and Denjoy–*Khintchine* integrals), respectively. Nikolai Lusin [14] described the constructive integrals of Denjoy in terms of generalized primitives. This gave rise to a series of *descriptive* definitions of integrals. A descriptive definition of integral (*Lusin-type* definition) refers to some generalized absolute continuity property and uses differentiation (often as well generalized) as a link between an integral (primitive) and an integrand. Thus, Lusin-type definitions can be seen as extensions of the fundamental integral's definition due to Newton.

It turned out later on, that the restricted Denjoy integral allows also a Riemann-type definition; it is the definition proposed independently by Jaroslav Kurzweil [11] and Ralph Henstock [6, 7], and known as the generalized Riemann integral or *Kurzweil–Henstock integral*. The idea of defining wide Denjoy integral in terms of Riemann sums has been suggested by Henstock in 1968 [8]; see also [9]. What Henstock proposed was an integration with respect to a mix of so-called *composite* and ordinary

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differential bases. No correct/complete characterization has been given however, actually until nowadays.

In order to encompass integration of some generalized derivatives (as, for example, approximate derivatives) and integration in the wide Denjoy sense, several generalized continuous counterparts of Denjoy integral were considered. Among others, we should mention here works by Ridder, Kubota, Sarkhel and Kar, and Gordon (see the references in [20]). As a natural consequence of Henstock's concept, these integrals (as well as the wide Denjoy integral itself) were sought for Riemann-type definitions. There are three essential contributions related to this topic: [19, 13, 5] given in chronological order. The first work related was the work by Lee and Soedijono [19], where an attempt to provide a Riemann-type definition for the so-called Kubota integral (an approximately continuous variant of wide Denjoy integral) was made. That attempt was not successful as the authors used incorrect characterization of primitives for Kubota integral (see the paper by Ene [5] for detailed comments). Nevertheless, the definition offered does characterize an approximately continuous Lusin-type integral, however not the Kubota integral but the so-called $T_{\mathrm{ap}}D$-integral (defined in [18]). Their attitude was developed by Vasile Ene [5], who considered similar integrals for a wide class of local systems Δ (including the density local system system as a particular case). Ene defined with respect to Δ, integrals of Ward-, variational-, and Riemann-type, proved that the two former are equivalent and included in the latter (under some suitable, not very restrictive, assumptions on Δ), and that all of them cover the Δ-continuous analogue of wide Denjoy integral, but, despite being very close to, did not arrive to a complete connection between the three integrals. In the meantime, Lee and Lee [13] switched the quantifiers in the definition from [19] and thus obtained a correct Riemann-type definition of the Kubota integral.

The primary aim of this work is to summarize the so-far theory on Riemann-type definitions for wide Denjoy and generalized wide Denjoy integrals. We include some new results being a completion to the theory. It is intended to be relatively self-complete, thus several proofs published earlier are repeated (explicitly or inexplicitly) here.

2. Preliminaries

Most of, we keep our language of [20, 21, 22] and (when it makes no confusion) adopt some notations of [5]. $|E|$ will stand for the Lebesgue outer measure of a set $E \subset \mathbb{R}$. We will write $\langle a, b\rangle$ for the closed interval with endpoints in $a, b \in \mathbb{R}$; i.e., $\langle a, b\rangle = [a, b]$ if $a \le b$, $\langle a, b\rangle = [b, a]$ if $a > b$.

2.1 – *Divisions*

A *tagged interval* is a pair (I, x) where I is a nondegenerate compact subinterval of the real line and x is an endpoint of I. Any positive function δ defined on some $A \subset \mathbb{R}$ we call a *gauge*. We say tagged interval (I, x) is *δ-fine* if $I \subset (x - \delta(x), x + \delta(x))$. By a *division* we mean a finite collection of tagged intervals (I, x) in which intervals I are pairwise nonoverlapping. It is said to be *in* an interval $[a, b]$, if $I \subset [a, b]$ for all (I, x) from this division. A division is called a *partition* of $[a, b]$ if the union of intervals I is the whole $[a, b]$. If $f\colon [a, b] \to \mathbb{R}$ and $[c, d] = I \subset [a, b]$, by $f[I]$ we mean $f(d) - f(c)$. This shall not lead to a confusion with the image of I under f, which is to be denoted standardly as $f(I)$. If $\mathcal{P}$ is a division in $[a, b]$, then we denote

$$\sigma(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} f(x)|I|, \qquad f[\mathcal{P}] = \sum_{(I,x)\in\mathcal{P}} f[I].$$

A division is *δ-fine* if all its members are such.

2.2 – *Local systems*

By a *local system* [25] (or a *simple system of sets* [23, 24]) we mean a family $\varDelta = \{\varDelta(x)\}_{x\in\mathbb{R}}$ such that each $\varDelta(x)$ is a nonvoid collection of subsets of $\mathbb{R}$ with the following properties:

(i) $\{x\} \notin \varDelta(x)$;
(ii) if $S \in \varDelta(x)$, then $x \in S$;
(iii) if $S \in \varDelta(x)$ and $R \supset S$, then $R \in \varDelta(x)$;
(iv) if $S \in \varDelta(x)$ and $\delta > 0$, then $(x - \delta, x + \delta) \cap S \in \varDelta(x)$.

Every S belonging to $\varDelta(x)$ we call a *path* leading to x. A function $\mathcal{C}$ on $A \subset \mathbb{R}$ such that $\mathcal{C}(x) \in \varDelta(x)$ for each $x \in A$, we call *a $\varDelta$-choice on A*. Given $\mathcal{C}$, we say a tagged interval $(\langle x, y\rangle, x)$ is $\mathcal{C}$-fine if $y \in \mathcal{C}(x)$.

We say that a local system $\varDelta$ is *filtering down*, if for each $x \in \mathbb{R}$ and each two paths $S_1, S_2 \in \varDelta(x)$ one has $S_1 \cap S_2 \in \varDelta(x)$. We say that a local system $\varDelta$ is *bilateral* if $(x - \delta, x) \cap S \neq \emptyset$ and $(x, x + \delta) \cap S \neq \emptyset$ for each $x \in \mathbb{R}$, $S \in \varDelta(x)$, $\delta > 0$.

We say that $\varDelta$ satisfies the *intersection condition* (abbr. IC), if for every choice $\mathcal{C}$, there exists a gauge δ such that if

$$0 < y - x < \min\{\delta(x), \delta(y)\},$$

then

$$\mathcal{C}(x) \cap \mathcal{C}(y) \cap [x, y] \neq \emptyset.$$

As the most significant examples of local systems let us mention the local system $\varDelta_{\mathrm{e}}$ which consists of neighbourhoods in the Euclidean topology and the density local system $\varDelta_{\mathrm{ap}}$ defined as follows:

$$A \in \varDelta_{\mathrm{ap}}(x) \quad \Leftrightarrow \quad x \in A \text{ and some measurable } B \subset A \text{ has density one at } x.$$

Among others, the dyadic local system $\varDelta_{\mathrm{d}}$ [1], the proximal density local system $\varDelta_{\mathrm{pro}}$ [17], and the $\mathcal{I}$-density local system [15]. All of them are bilateral and (except $\varDelta_{\mathrm{pro}}$) satisfy IC.

Given $\varDelta$, we say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\varDelta$-continuous at $x \in \mathbb{R}$*, if for each $\varepsilon > 0$ there exists an $S \in \varDelta(x)$ such that

$$f(x) - \varepsilon < f(t) < f(x) + \varepsilon$$

for each $t \in S$. We say $f\colon \mathbb{R} \to \mathbb{R}$ is *$\varDelta$-continuous* if it is $\varDelta$-continuous at each $x \in \mathbb{R}$.

LEMMA 1 ([25]). *If a bilateral local system $\varDelta$ satisfies* IC, *then each $\varDelta$-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is Darboux Baire one.*

Despite the proximal density local system $\varDelta_{\mathrm{pro}}$ fails to have IC, each $\varDelta_{\mathrm{pro}}$-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is Darboux [17, Theorem 4.1] and Baire one.

We say that f is $\varDelta$-differentiable at x to a number g, if for each $\varepsilon > 0$ there exists an $S \in \varDelta(x)$ such that

$$g - \varepsilon < \frac{f(t) - f(x)}{t - x} < g + \varepsilon$$

for each $t \in S$, $t \neq x$. With the aid of filtering down property, one shows that the number g, if exists, is unique.

For functions defined on an $[a, b]$, at a and b the definitions of $\varDelta$-continuity and $\varDelta$-differentiability are to be understood 'relatively' to $[a, b]$.

2.3 – *Composite $\varDelta$-gauges*

A sequence of sets $\{E_i\}_{i=1}^{\infty} = \{E_i\}_i$ with $E = \bigcup_{i=1}^{\infty} E_i$ we call an *E-form*. If, moreover, all E_i are closed (measurable) we say the E-form is closed (measurable). Given two E-forms $\{E_i\}_i$ and $\{F_j\}_j$, we write $\{E_i\}_i \succ \{F_j\}_j$ if for each i there is a j such that $E_i \subset F_j$. Beware that the superset F_j need not be unique, as F_j's are not assumed to be disjoint. We write $\mathrm{Is}\{E_i\}_i$ for the set of all x such that for some i, $x \in E_i$ and x is isolated from either side of E_i.

Consider a local system $\varDelta$, a $\varDelta$-choice $\mathcal{C}$ defined on a set $A \subset [a,b]$, and an $[a,b]$-form $\{E_i\}_i$. On each E_i define a gauge δ_i. We call the sequence $\{\delta_i\}_i$ *related* to $\{E_i\}_i$. Consider a tagged interval $(\langle x,y\rangle, x)$. We say it is $\{\delta_i\}_i$*-fine* if for some i, $x,y \in E_i$ and it is δ_i-fine. We say $(\langle x,y\rangle, x)$ is $(\mathcal{C}, \{\delta_i\}_i)$*-fine* if it is either $\{\delta_i\}_i$-fine or $\mathcal{C}$-fine. We say a division $\mathcal{P}$ is $\{\delta_i\}_i$-fine, $\mathcal{C}$-fine, or $(\mathcal{C}, \{\delta_i\}_i)$-fine if all its members are such. We shall refer to pairs $(\mathcal{C}, \{\delta_i\}_i)$ as to *composite $\varDelta$-gauges* and (as for $\{\delta_i\}_i$) call them related to $\{E_i\}_i$.

The proof of the theorem below is a slight modification of the result due to Henstock [8, Exercise 43.9] who proved it for $\varDelta = \varDelta_{\mathrm{e}}$.

THEOREM 2 ([5, Lemma 4.2]). *Let an $[a,b]$-form $\{E_i\}_i$ be closed and a local system $\varDelta$ be bilateral. For each interval $[a,b]$, each $\varDelta$-choice $\mathcal{C}$ on $\mathrm{Is}\{E_i\}_i$, and each sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$ there exists a $(\mathcal{C}, \{\delta_i\}_i)$-fine partition of $[a,b]$.*

Notice that for the above result the assumption of $\{E_i\}_i$ being closed is essential. Indeed, note that for the $[0,\pi]$-form $\{\mathbb{Q} \cap [0,\pi], [0,\pi] \setminus \mathbb{Q}\}$ and any $\{\delta_1, \delta_2\}$ there is no $\{\delta_1, \delta_2\}$-fine partition of $[0,\pi]$.

2.4 – *Lusin-type integrals*

Let $F\colon E \to \mathbb{R}$. If a subset $A \subset E$ is nonvoid, then we set $\omega_F(A) = \sup F(A) - \inf F(A)$. We will say that F satisfies the *condition $\mathcal{N}$*, if $|F(N)| = 0$ for each $N \subset E$, $|N| = 0$. The F is said to be an *AC-function*, if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for any pairwise non-overlapping intervals $[a_1,b_1], \ldots, [a_n,b_n]$, with both endpoints in E,

$$\sum_{i=1}^{n} (b_i - a_i) < \eta \quad \Rightarrow \quad \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

The F is said to be a *VB-function*, if there is a number $M > 0$ such that for any pairwise nonoverlapping intervals $[a_1, b_1], \dots, [a_n, b_n]$, with both endpoints in E,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < M.$$

The lower bound for all such M's we call the *variation* of F.

A function F on E is said to be an *ACG*- and a *VBG-function*, if there exists an E-form $\{E_i\}_i$ such that for each i, $F \restriction E_i$ is an *AC*- and a *VB*-function respectively. The F is said to be respectively an [*ACG*]- and a [*VBG*]-*function* if, moreover, the form $\{E_i\}_i$ above is assumed to be closed.

First of all we recall wide Denjoy integral, then four its Lusin-type Δ-continuous counterparts.

Definition 3. *We call a function $f\colon [a, b] \to \mathbb{R}$, Denjoy integrable in the wide sense (abbr. D-integrable), if there exists a continuous ACG-function $F\colon [a, b] \to \mathbb{R}$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a, b]$. The integral of f is defined as $F(b) - F(a)$.*

Here and on, $F'_{\mathrm{ap}}(x)$ denotes the approximate derivative (i.e., Δ_{ap}-derivative) of F at x.

Consider the following four classes of functions defined on some $[a, b]$:

- $\mathcal{L}_1$: [*ACG*]-functions,
- $\mathcal{L}_2$: [*VBG*]-functions satisfying $\mathcal{N}$,
- $\mathcal{L}_3$: measurable *ACG*-functions,
- $\mathcal{L}_4$: measurable *VBG*-functions satisfying $\mathcal{N}$.

Clearly, $\mathcal{L}_1 \subset \mathcal{L}_2, \mathcal{L}_3 \subset \mathcal{L}_4$. For each i, the class $\mathcal{L}_i$ is a linear space. For $i = 1, 3$ it is evident, while for $i = 2$ it was justified by Sarkhel and Kar [18, Corollary 3.1.1 and Theorem 3.6], for $i = 4$ by Ene [4, Corollary 2]. It is well known [16, Chapter VII, (4.3)] that each member of $\mathcal{L}_i$ is approximately differentiable almost everywhere.

Let $\mathcal{F}$ be a linear space of Baire one Darboux functions defined on $[a, b]$.

Definition 4. *We call a function $f\colon [a, b] \to \mathbb{R}$, $\mathcal{F}_i$-integrable, $i = 1, 2, 3, 4$, if there exists a function $F \in \mathcal{F}_i = \mathcal{L}_i \cap \mathcal{F}$, on $[a, b]$, such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a, b]$. The $\mathcal{F}_i$-integral of f is defined as $F(b) - F(a)$.*

The $\mathcal{F}_i$-integral is uniquely defined since $\mathcal{L}_i$ is a linear space and since the lemma below holds.

LEMMA 5 ([12, Theorem 1]). *Assume that an $F\colon \mathbb{R} \to \mathbb{R}$ satisfies $\mathcal{N}$ and is Baire one Darboux. If $F'(x) \geq 0$ at almost every point $x \in \mathbb{R}$ at which F is differentiable (in the usual sense), then F is nondecreasing.*

From now on, we assume the local system $\varDelta$ considered is filtering down and such that every $\varDelta$-continuous function $f\colon [a,b] \to \mathbb{R}$ is Darboux Baire one. This can follow from bilaterality and IC (Lemma 1). We confine then our attention to $\mathcal{F}$ being the class of all $\varDelta$-continuous functions. We set $\mathcal{F}_i^{\varDelta}$ instead of $\mathcal{F}_i$ in this case.

Due to the Banach–Zarecki theorem [16, Chapter VII, (6.8)], the conditions defining classes $\mathcal{L}_i$, $i = 1,2,3,4$, are equivalent among continuous functions, and so $\mathcal{F}_i^{\varDelta}$-integrals, $i = 1,2,3,4$, for $\varDelta = \varDelta_{\mathrm{e}}$ are exactly the wide Denjoy integral. In [20] the author gave a complete chart of connections between $\mathcal{F}_i^{\varDelta}$-integrals for $\varDelta = \varDelta_{\mathrm{ap}}$. For some results in general case see [21].

3. Riemann-type definitions

We seek for Riemann-type definitions for all $\mathcal{F}_i^{\varDelta}$-integrals. First we make a review of definitions existing [5, 13, 19] in the literature.

DEFINITION 6 (AH-integral of [19] if $\varDelta = \varDelta_{\mathrm{ap}}$). *We say a function $f\colon [a,b] \to \mathbb{R}$ is $LS_{\varDelta}$-integrable to a number $\boldsymbol{I}$, if to each $\varepsilon > 0$ we can find a closed $[a,b]$-form $\{E_i\}_i$ such that for any closed $[a,b]$-form $\{D_j\}_j \succ \{E_i\}_i$ there is a $\varDelta$-choice $\mathcal{C}$ on $\mathrm{Is}\{D_j\}_j$ and a sequence of gauges $\{\delta_j\}_j$ related to $\{D_j\}_j$ such that*

$$|\sigma(\pi, f) - \boldsymbol{I}| < \varepsilon \tag{1}$$

holds for each $(\mathcal{C}, \{\delta_j\}_j)$-fine partition π of $[a,b]$.

DEFINITION 7 (AH-integral of [13] if $\varDelta = \varDelta_{\mathrm{ap}}$). *We say a function $f\colon [a,b] \to \mathbb{R}$ is $LL_{\varDelta}$-integrable to a number $\boldsymbol{I}$, if there is a closed $[a,b]$-form $\{E_i\}_i$ such that to each $\varepsilon > 0$ and to each closed $[a,b]$-form $\{D_j\}_j \succ \{E_i\}_i$, a $\varDelta$-choice $\mathcal{C}$ on $\mathrm{Is}\{D_j\}_j$ and a sequence of gauges $\{\delta_j\}_j$ related to $\{D_j\}_j$ can be found so that* (1) *holds for each $(\mathcal{C}, \{\delta_j\}_j)$-fine partition π of $[a,b]$.*

By Theorem 2 both definitions are correct; i.e., the number $\boldsymbol{I}$ is unique. Following Vasile Ene [5], they can be reformulated removing the refinement $\succ$ so that the essential ingredient is better seen.

DEFINITION 6′. *We say a function* $f\colon [a,b] \to \mathbb{R}$ *is* LS_Δ*-integrable to a number* $\boldsymbol{I}$*, if to each* $\varepsilon > 0$ *we can find a closed* $[a,b]$*-form* $\{E_i\}_i$ *such that for any countable* $A \supset \operatorname{Is}\{E_i\}_i$ *there are a* Δ*-choice* $\mathcal{C}$ *on* A *and a sequence of gauges* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *such that* (1) *holds for each* $\big(\mathcal{C}, \{\delta_i\}_i\big)$*-fine partition* π *of* $[a,b]$.

A PROOF THAT DEFINITIONS 6 AND 6′ ARE EQUIVALENT. ($\Leftarrow$) Assume f is integrable in the sense of Definition 6′. For $\varepsilon > 0$ pick accordingly $[a,b]$-form $\{E_i\}_i$ and for any countable $A \supset \operatorname{Is}\{E_i\}_i$ fix suitable Δ-choice $\mathcal{C}_A$ on A and gauges δ_i^A on E_i. Then consider any $\{D_j\}_j \succ \{E_i\}_i$ and define a suitable composite Δ-gauge by taking $\mathcal{C}_{\operatorname{Is}\{D_j\}_j}$ and putting $\delta'_j(y) = \delta_i^A(y)$ at $y \in D_j$ for any i with $D_j \subset E_i$ $\big(i$ being the same for all $y \in D_j\big)$, $A = \operatorname{Is}\{D_j\}_j$. Each $\big(\mathcal{C}_{\operatorname{Is}\{D_j\}_j}, \{\delta'_j\}_j\big)$-fine partition is then $\big(\mathcal{C}_{\operatorname{Is}\{D_j\}_j}, \{\delta_i^A\}_i\big)$-fine.

($\Rightarrow$) Straightforward. Let f be integrable in the sense of Definition 6. For $\varepsilon > 0$ pick a suitable $\{E_i\}_i$. Then consider any countable $A \supset \operatorname{Is}\{E_i\}_i$ and refine $\{E_i\}_i$ by adding $\bigcup_{x\in A} \{\{x\}\}$ to $\{E_i\}_i$. The composite Δ-gauge found for the refined $[a,b]$-form suits Definition 6′ for the set A. $\square$

A similar justification shows that instead of Definition 7 the following one can be used.

DEFINITION 7′. *An* $f\colon [a,b] \to \mathbb{R}$ *is* LL_Δ*-integrable to a number* $\boldsymbol{I}$*, if there is a closed* $[a,b]$*-form* $\{E_i\}_i$ *with the following property: for each* $\varepsilon > 0$ *and each countable* $A \supset \operatorname{Is}\{E_i\}_i$ *we are able to pick a* Δ*-choice* $\mathcal{C}$ *on* A *and a sequence of gauges* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *such that* (1) *holds for each* $\big(\mathcal{C}, \{\delta_i\}_i\big)$*-fine partition* π *of* $[a,b]$.

If the condition from Definition 7 or 7′ holds, we say f is integrable *using* $\{E_i\}_i$. Vasile Ene used a bit different condition than that of Definition 6′. In his version the sequence $\{\delta_i\}_i$ does not depend on A.

DEFINITION 8 ($[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$- or $[\mathcal{S}\mathcal{R}]$-integral of [5]). *We say a function* $f\colon [a,b] \to \mathbb{R}$ *is* E_Δ*-integrable to a number* $\boldsymbol{I}$*, if to each* $\varepsilon > 0$ *we can find a closed* $[a,b]$*-form* $\{E_i\}_i$ *and a sequence of gauges* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *such that for any countable* $A \supset \operatorname{Is}\{E_i\}_i$ *there is a* Δ*-choice* $\mathcal{C}$ *on* A *such that* (1) *holds for each* $\big(\mathcal{C}, \{\delta_i\}_i\big)$*-fine partition* π *of* $[a,b]$.

Clearly, if an f is E_Δ-integrable then it is LS_Δ-integrable. In fact, the converse is also true; i.e., the dependence of $\{\delta_i\}_i$ on A is not essential. We shall prove it later on.

3.1 – *A Saks–Henstock lemma issue*

Lee and Soedijono [19] made a serious use of Saks–Henstock type lemma for their integral (LS_Δ-integral in our terms), however did not prove it leaving the reader only with a claim that it is easy to do. Since it is apparently impossible to prove a Saks–Henstock lemma in the most routine way which refers to the so-called filtering property of integration base, this has met some criticism in [5]. We show here that Saks–Henstock lemmas for the above defined integrals (LS_Δ-integral in particular) are indeed easy to prove, however proofs require a bit of different argument that makes use of a Cauchy criterion for LS_Δ-integrability. This idea of proving Saks–Henstock lemma is not original and can be found e.g. in the proof of [23, Lemma 4.4, chapter 3]. It refers to the argument used to show integrability on a subinterval.

In the next two lemmas, as we consider integration on a figure rather than on a single interval, for the sake of completeness we give full proofs despite they pattern single interval case proofs from [5]. A definition of integrability on a figure; i.e., a union of finitely many compact intervals, is the same as for an interval.

LEMMA 9. *A function f defined on a figure $S = \bigcup_k I_k$, where I_k are nonoverlapping intervals, is E_Δ-integrable if and only if for each $\varepsilon > 0$ there are a closed S-form $\{E_i\}_i$ and a related sequence of gauges $\{\delta_i\}_i$ with the following property: for each $A \supset \mathrm{Is}\{E_i\}_i$ there is a Δ-choice $\mathcal{C}_A$ on A such that if partitions π_1 and π_2 of S are respectively $(\mathcal{C}_{A_1}, \{\delta_i\}_i)$- and $(\mathcal{C}_{A_2}, \{\delta_i\}_i)$-fine, $A_1, A_2 \supset \mathrm{Is}\{E_i\}_i$, then*

$$|\sigma(\pi_1, f) - \sigma(\pi_2, f)| < \varepsilon.$$

PROOF. We repeat the proof of [5, Lemma 7.2]. Only the sufficiency requires a proof. For each $n \in \mathbb{N}$ we pick an S-form $\{E_i^{(n)}\}_i$ and a related sequence of gauges $\{\delta_i^{(n)}\}_i$ suitable for $\varepsilon_n = 1/n$ in the sense of the Cauchy-like condition described in the lemma. Let $\mathcal{C}^n$ be Δ-choices on $\mathrm{Is}\{E_i^{(n)}\}_i$ suitable for ε_n, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ fix some $(\mathcal{C}^n, \{\delta_i^{(n)}\}_i)$-fine partition

π^n of S. Let $m<n$. Take $\varDelta$-choices $\mathcal{B}^m$ and $\mathcal{B}^n$ both on $\mathrm{Is}\{E_i^{(m)}\cap E_j^{(n)}\}_{i,j}$ suitable for respectively ε_m and ε_n. We define another composite $\varDelta$-gauge, now related to the S-form $\{E_i^{(m)}\cap E_j^{(n)}\}_{i,j}$:

$$\mathcal{B}(x)=\mathcal{B}^m(x)\cap\mathcal{B}^n(x)\qquad\text{for } x\in\mathrm{Is}\{E_i^{(m)}\cap E_j^{(n)}\}_{i,j},$$
$$\delta_{i,j}(x)=\min\{\delta_i^{(m)}(x),\delta_j^{(n)}(x)\}\qquad\text{for } x\in E_i^{(m)}\cap E_j^{(n)}.$$

There exists a $(\mathcal{B},\{\delta_{i,j}\}_{i,j})$-fine partition of S, say π. It is both $(\mathcal{B}^m,\{\delta_i^{(m)}\}_i)$-fine and $(\mathcal{B}^n,\{\delta_i^{(n)}\}_i)$-fine, so

$$|\sigma(\pi^m,f)-\sigma(\pi^n,f)|\le|\sigma(\pi^m,f)-\sigma(\pi,f)|+|\sigma(\pi,f)-\sigma(\pi^n,f)|<\varepsilon_m+\varepsilon_n.$$

Therefore $\sigma(\pi^n,f)$, $n\in\mathbb{N}$, is a Cauchy sequence. Denote $\boldsymbol{I}=\lim_{n\to\infty}\sigma(\pi^n,f)$. Let now $\varepsilon>0$. There is n with $\varepsilon_n<\varepsilon$ and there is m with $|\boldsymbol{I}-\sigma(\pi^k,f)|<\varepsilon$ for all $k\ge m$. Let $n_0=\max\{m,n\}$. Consider any countable $A\supset\mathrm{Is}\{E_j^{(n_0)}\}_j$ and the $\varDelta$-choice $\mathcal{C}_A$ found for A and ε_{n_0} according to the Cauchy condition and take any $(\mathcal{C}_A,\{\delta_i^{(n_0)}\}_i)$-fine partition π'. We have

$$|\sigma(\pi',f)-\boldsymbol{I}|\le|\sigma(\pi',f)-\sigma(\pi^{n_0},f)|+|\sigma(\pi^{n_0},f)-\boldsymbol{I}|<\varepsilon_{n_0}+\varepsilon<2\varepsilon.$$

It means f is $E_\varDelta$-integrable. □

Notice that for the above proof, a weaker property of $\varDelta$ than being filtering down could have been assumed: at each x, for every two $S_1,S_2\in\varDelta(x)$, x is a bilateral accumulation point of $S_1\cap S_2$; i.e., the property that pointwise intersection $\varDelta\cap\varDelta$ is also a bilateral local system.

We will need the following enhancement of Lemma 9, which is a consequence of the proof.

Corollary A. *Assume that for each $\varepsilon>0$ there is a closed S-form $\{E_i^\varepsilon\}_i$ and a related sequence of gauges $\{\delta_i^\varepsilon\}_i$ such that for each countable $A\supset\mathrm{Is}\{E_i^\varepsilon\}_i$ there is a $\varDelta$-choice $\mathcal{C}_A$ on A such that for any partitions π_1 and π_2 of S being respectively $(\mathcal{C}_{A_1},\{\delta_i^\varepsilon\}_i)$- and $(\mathcal{C}_{A_2},\{\delta_i^\varepsilon\}_i)$-fine, $A_1,A_2\supset\mathrm{Is}\{E_i^\varepsilon\}_i$,*

$$|\sigma(\pi_1,f)-\sigma(\pi_2,f)|<\varepsilon.$$

Then, f is $E_\varDelta$-integrable on S. Moreover, if π is a $(\mathcal{C}_A,\{\delta_i^\varepsilon\}_i)$-fine partition of S, $A\supset\mathrm{Is}\{E_i^\varepsilon\}_i$,

$$\left|\sigma(\pi,f)-\int_S f\right|<2\varepsilon.$$

LEMMA 10. *If $f\colon [a,b] \to \mathbb{R}$ is E_Δ-integrable, then it is E_Δ-integrable on each figure $S \subset [a,b]$.*

PROOF. We repeat the proof of [5, Lemma 7.3(ii)]. Denote $S = \bigcup_{j=1}^{n} J_j$, where the intervals J_j are pairwise disjoint. Fix an $\varepsilon > 0$ and take an $[a,b]$-form $\{E_i\}_i$ and a related sequence of gauges $\{\delta_i\}_i$ found for ε and f on $[a,b]$ in the sense described in Lemma 9. Consider any countable subsets $A, B \supset \mathrm{Is}\{E_i\}_i \cap S$ of S. Extend A and B to A' and B' respectively by adding $\mathrm{Is}\{E_i\}_i \setminus S$. Take choices $\mathcal{C}_{A'}$ and $\mathcal{C}_{B'}$ on A' and B' according to the Cauchy condition for ε. Denote $\mathcal{C}'_A = \mathcal{C}_{A'} \restriction S$, $\mathcal{C}'_B = \mathcal{C}_{B'} \restriction S$ and consider now a $(\mathcal{C}'_A, \{\delta_i\}_i)$-fine partition π_1 of S and a $(\mathcal{C}'_B, \{\delta_i\}_i)$-fine partition π_2 of S. Fix moreover a $(\mathcal{C}_{\mathrm{Is}\{E_i\}_i}, \{\delta_i\}_i)$-fine partition ρ of $[a,b] \setminus \operatorname{int} S$. The unions $\pi_1 \cup \rho$ and $\pi_2 \cup \rho$ are respectively $(\mathcal{C}_{A'}, \{\delta_i\}_i)$-fine and $(\mathcal{C}_{B'}, \{\delta_i\}_i)$-fine partitions of $[a,b]$. Thus, by the assumption,

$$|\sigma(\pi_1, f) - \sigma(\pi_2, f)| = |\sigma(\pi_1 \cup \rho, f) - \sigma(\pi_2 \cup \rho, f)| < \varepsilon.$$

It means, for f on S we have the Cauchy criterion of E_Δ-integrability fulfilled. □

From the above reasoning follows immediately

COROLLARY B. *Let, for a given $\varepsilon > 0$, an $[a,b]$-form $\{E_i^\varepsilon\}_i$, a related sequence of gauges $\{\delta_i^\varepsilon\}_i$, choices $\mathcal{C}_A$, $A \supset \mathrm{Is}\{E_i^\varepsilon\}_i$ countable, be as in Corollary A for $S = [a,b]$. Let $\mathcal{P} = \{(J_j, y_j)\}_j$ be a $(\mathcal{C}_{A_1}, \{\delta_i^\varepsilon\}_i)$-fine division, $\mathcal{P}' = \{(J'_j, y'_j)\}_j$ a $(\mathcal{C}_{A_2}, \{\delta_i^\varepsilon\}_i)$-fine division with $\bigcup_j J_j = \bigcup_j J'_j \subset [a,b]$. Then $|\sigma(\mathcal{P}, f) - \sigma(\mathcal{P}', f)| < \varepsilon$.*

Lemma 10 allows to define the *indefinite integral* F of f by $F(x) = \int_a^x f$, $x \in [a,b]$. We shall write $F = \int f$.

Thanks to Corollaries A and B we are able to conclude with a Saks–Henstock lemma for E_Δ-integral.

LEMMA 11 (Saks–Henstock lemma for E_Δ-integral). *Let $f\colon [a,b] \to \mathbb{R}$ be E_Δ-integrable, $\varepsilon > 0$. Let an $[a,b]$-form $\{E_i\}_i$ and a related sequence of gauges $\{\delta_i\}_i$ correspond to ε, let a Δ-choice $\mathcal{C}_A$ correspond to a fixed countable $A \supset \mathrm{Is}\{E_i\}_i$, in the sense of Definition 8. Then for each $(\mathcal{C}_A, \{\delta_i\}_i)$-fine division $\mathcal{P}$ with all intervals in $[a,b]$ the* *inequalities*

$$(2) \qquad |\sigma(\mathcal{P},f)-F[\mathcal{P}]|<4\varepsilon, \qquad \sum_{(I,x)\in\mathcal{P}} |f(x)|I|-F[I]|<8\varepsilon$$

hold, where $F=\int f$.

Proof. We will prove only the first inequality, as the second is its straightforward consequence. Let $\mathcal{P}=\{(J_j,y_j)\}_j$ be a $(\mathcal{C}_A,\{\delta_i\}_i)$-fine division, $\mathcal{P}'=\{(J'_j,y'_j)\}_j$ a $(\mathcal{C}_B,\{\delta_i\}_i)$-fine division with $\bigcup_j J_j=\bigcup_j J'_j=S\subset[a,b]$. If $\mathcal{P}$ and $\mathcal{P}'$ were *partitions of* $[a,b]$, then clearly $|\sigma(\mathcal{P},f)-\sigma(\mathcal{P}',f)|\leqslant \left|\sigma(\mathcal{P},f)-\int_a^b f\right|+\left|\int_a^b f-\sigma(\mathcal{P}',f)\right|<2\varepsilon$. Thus, by Corollary B, $|\sigma(\mathcal{P},f)-\sigma(\mathcal{P}',f)|<2\varepsilon$, even if $\mathcal{P}$ and $\mathcal{P}'$ are not partitions of $[a,b]$. So, with Corollary A we have that $|\sigma(\mathcal{P},f)-F[\mathcal{P}]|<4\varepsilon$. □

Ene [5, Lemma 7.4] proved the statement of Lemma 11 only for $\{\delta_i\}_i$-fine divisions $\mathcal{P}$. Reasoning similarly as above, equipped with Cauchy criteria for LS_Δ- and LL_Δ- integrability, one obtains

Lemma 12 (Saks–Henstock lemma for LS_Δ-integral). *Let* $f\colon[a,b]\to\mathbb{R}$ *be* LS_Δ*-integrable,* $\varepsilon>0$*. Let an* $[a,b]$*-form* $\{E_i\}_i$ *correspond to* ε*, let a* Δ*-choice* $\mathcal{C}$ *and a sequence of gauges* $\{\delta_i\}_i$ *correspond to a countable* $A\supset \mathrm{Is}\{E_i\}_i$ *(in the sense of Definition* 6′*). Then for each* $(\mathcal{C},\{\delta_i\}_i)$*-fine division* $\mathcal{P}$ *in* $[a,b]$ *the inequalities* (2) *hold.*

Lemma 13 (Saks–Henstock lemma for LL_Δ-integral). *Let* $f\colon[a,b]\to\mathbb{R}$ *be* LL_Δ*-integrable using* $\{E_i\}_i$, $\varepsilon>0$*. Let a sequence of gauges* $\{\delta_i\}_i$*, related to* $\{E_i\}_i$*, and a* Δ*-choice* $\mathcal{C}$ *on* A *correspond to a countable* $A\supset\mathrm{Is}\{E_i\}_i$ *(in the sense of Definition* 7′*). Then for each* $(\mathcal{C},\{\delta_i\}_i)$*-fine division* $\mathcal{P}$ *the inequalities* (2) *hold.*

Using Lemma 12 it is easy to prove that indefinite integrals in the LS_Δ, and so in E_Δ and LL_Δ, sense are Δ-continuous. Indeed, let $\varepsilon>0$, $x\in[a,b]$. Consider the set $A=\mathrm{Is}\{E_i\}_i\cup\{x\}$ and take its corresponding Δ-choice $\mathcal{C}$. If $y\in\mathcal{C}(x)\in\Delta(x)$, then according to Lemma 12,

$$|F(y)-F(x)|\le|f(x)(y-x)-F(y)+F(x)|+|f(x)|\,|y-x|<4\varepsilon+|f(x)|\,|y-x|.$$

As $\mathcal{C}(x)$ could have been found arbitrarily close to x (condition (iv)), it follows F is Δ-continuous at x.

4. Relations between $\mathcal{F}_i^{\Delta}$- and other type integrals

First we recall all that is known in this direction, making precise of what has been said in the introduction. We begin with the following result of Lee and Lee for the case of density local system.

THEOREM 14 ([13, Theorems 3.1&3.2]). *For $\Delta = \Delta_{\mathrm{ap}}$, $\mathcal{F}_1^{\Delta}$- and LL_{Δ}-integrals are equivalent.*

Let us remark that, despite being true, the proof of the above given in [13] is not clear at one point, namely when it is claimed that each indefinite LL_{Δ}-integral is $[ACG]$ (or, equivalently, that it satisfies $\mathcal{N}$); we mean here [13, Lemma 3.5] left without a proof. We fill this gap in Lemma 25.

In an earlier paper [19], introducing the LS_{Δ}-integral for $\Delta = \Delta_{\mathrm{ap}}$, Soedijono and Lee claimed it is equivalent to $\mathcal{F}_1^{\Delta_{\mathrm{ap}}}$-integral (so-called *Kubota integral*). They proved that (1) *each Kubota integrable function is $LS_{\Delta_{\mathrm{ap}}}$-integrable* [19, Theorem 4.1] and claimed to prove [19, Theorem 4.2] that (2) *each indefinite $LS_{\Delta_{\mathrm{ap}}}$-integral is an $[ACG]$-function.* The proof of (1) can be, without much effort, adapted to a proof that each $\mathcal{F}_2^{\Delta_{\mathrm{ap}}}$-integrable function is $LS_{\Delta_{\mathrm{ap}}}$-integrable; we shall do it for general Δ (Theorem 18). The proof of (2) contains serious gaps (e.g., the authors were not aware of the fact that in case of approximately continuous, not continuous, functions the Banach–Zarecki theorem is not true, i.e., that an approximately continuous $[VBG]$-function with $\mathcal{N}$ need not be $[ACG]$), so that what can be found proven is only that each indefinite $LS_{\Delta_{\mathrm{ap}}}$-integral is a VBG-function. We strengthen this result (as well as [5, Lemma 7.6]) in

LEMMA 15. *Let $f\colon [a,b] \to \mathbb{R}$ be LS_{Δ}-integrable. Then $F = \int f$ is $[VBG]$.*

PROOF. Let $\{E_i\}_i$ be a closed $[a,b]$-form appropriate for $\varepsilon = 1$ in the sense of Definition 6′. We intend to show F is $[VBG]$ on each E_i. Consider an arbitrary closed subset $D \subset E_i$. Take a sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$, chosen for $A = \mathrm{Is}\{E_i\}_i$. By the Baire category theorem there is a portion $I \cap D \neq \emptyset$ of D, I an open interval, such that $|f| \leq n$ and $\delta_i \geq 1/n$ for some $n \in \mathbb{N}$, both on the same dense subset E of $I \cap D$. We can assume that $|I| < 1/n$. Consider any collection $\{[a_j, b_j]\}_j$ of nonoverlapping intervals with both endpoints in $I \cap D$. For each j there is $c_j \in E \cap I$ arbitrarily close to $[a_j, b_j]$. It is possible to split $\{[a_j, b_j]\}_j$ into two families of disjoint intervals, so we can assume that intervals $\langle a_j, c_j \rangle$ for distinct j do not overlap (and the same for $\langle b_j, c_j\rangle$). Thus both $\{(\langle a_j, c_j\rangle, c_j)\}_j$ and $\{(\langle b_j, c_j\rangle, c_j)\}_j$ are $\{\delta_i\}_i$-fine divisions. From Lemma 12,

$$\begin{aligned}\sum_j |F(b_j) - F(a_j)| \le \sum_j |F(b_j) - F(c_j)| + \sum_j |F(c_j) - F(a_j)| \le \\ \le \sum_j |f(c_j)(b_j - c_j) - F(b_j) + F(c_j)| + \\ + \sum_j |f(c_j)(c_j - a_j) - F(c_j) + F(a_j)| + \\ + \sum_j |f(c_j)| \cdot (|b_j - c_j| + |c_j - a_j|) \le 16 + 2n(b - a).\end{aligned}$$

This proves F is *VB* on $I \cap D$. We have shown that each closed subset of E_i contains a portion on which F is *VB*. Then by the use of Cantor–Baire stationary principle we get F is $[VBG]$ on E_i and thus on $[a, b]$. □

The following obvious remark will allow us not to repeat the same argument several times later on.

Remark 16. *Let $\varepsilon > 0$, $f\colon [a, b] \to \mathbb{R}$ be any, $F\colon [a, b] \to \mathbb{R}$ be Δ-continuous, $A \subset [a, b]$ countable. Then there is a Δ-choice $\mathcal{C}$ on A such that for any $\mathcal{C}$-fine division $\mathcal{P}$,*

$$\sum_{(I,x)\in\mathcal{P}} |f(x)|I|| + \sum_{(I,x)\in\mathcal{P}} |F[I]| < \varepsilon.$$

Lemma 17 is well known from the theory of Kurzweil–Henstock integral.

Lemma 17. *Let $F\colon [a, b] \to \mathbb{R}$ be absolutely continuous, $F' = f$ almost everywhere on $[a, b]$. Then, given $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that for each δ-fine division $\mathcal{P}$,*

$$\sum_{(I,x)\in\mathcal{P}} |F[I] - f(x)|I|| < \varepsilon.$$

Theorem 18. *Let $f\colon [a, b] \to \mathbb{R}$ be $\mathcal{F}_2^{\Delta}$-integrable. Then it is E_Δ-integrable with the same integral.*

Proof. Let $F = \int f$. By assumption, F is $[VBG]$, Δ-continuous, and satisfies $\mathcal{N}$. Let $\{D_n\}_n$ be a closed $[a, b]$-form such that F is *VB* on each D_n. Let $\varepsilon > 0$. Fix an n. Let $\{x_i^{(n)}\}_i \subset D_n$ be the set of discontinuities of the restriction $F{\restriction}D_n$. Denote by $\omega_n(x)$ the oscillation of $F{\restriction}D_n$ at $x \in D_n$; i.e., the upper limit of $|F(y) - F(z)|$ as $(y,z) \to (x,x)$, $y, z \in D_n$. There is $I = I(n)$ such that

$$\sum_{i=I+1}^{\infty} \omega_n\bigl(x_i^{(n)}\bigr) < \frac{\varepsilon}{2^n}. \tag{3}$$

Let a closed $[a,b]$-form $\{E_k\}_k \succ \{D_n\}_n$ contain $\bigl\{x_1^{(n)}\bigr\}, \dots, \bigl\{x_I^{(n)}\bigr\}$, $n \in \mathbb{N}$, and be such that no open interval with endpoints in $E_k \subset D_n$ contains any of the points $x_1^{(n)}, \dots, x_I^{(n)}$. We define a composite Δ-gauge related to $\{E_k\}_k$.

Since E_k is closed, we can decompose $F \restriction E_k$ as $G + H$, $G, H\colon E_k \to \mathbb{R}$, where G is absolutely continuous, H a singular function (i.e., with $H'(x) = 0$ a.e. on E_k). This is the classical Lebesgue decomposition theorem. From Lemma 17, as $G'(x) = f(x)$ at almost every $x \in E_k$, there is a gauge δ_k on E_k with the property that

$$\sum_i |G[I_i] - f(x_i)|I_i|| < \frac{\varepsilon}{2^k}$$

for every δ_k-fine division $\{(I_i, x_i)\}_i$ with all I_i having both endpoints in E_k. Notice that, as H satisfies $\mathcal{N}$, it is a pure jump function, and so, for the same $\{(I_i, x_i)\}_i$,

$$\sum_i |H[I_i]| \le 2 \sum_m \omega_H(y_m),$$

where $\{y_m\}_m$ is the set of all discontinuities of $F \restriction E_k$ (or H) and $\omega_H(y_m)$ denotes the oscillation of H at y_m. Each y_m is among $x_{I(n)+1}^{(n)}, \dots$, where $E_k \subset D_n$.

The definition of $\{\delta_k\}_k$ does not depend on A. For any countable $A \supset \operatorname{Is}\{E_k\}_k$ we define $\mathcal{C}$ according to Remark 16.

Now, consider a $\bigl(\mathcal{C}, \{\delta_k\}_k\bigr)$-fine partition π of $[a,b]$. Denote $\pi_k = \{(I,x) \in \pi : (I,x) \text{ is } \delta_k\text{-fine}\}$ and agree that each (I,x) belongs to at most one π_k. We get

$$\sum_{(I,x)\in\pi_k} |F[I] - f(x)|I|| \le \sum_{(I,x)\in\pi_k} |G[I] - f(x)|I|| + \sum_{(I,x)\in\pi_k} |H[I]| < \\ < \frac{\varepsilon}{2^k} + 2\sum_m \omega_H(y_m).$$

Therefore, by (3),

$$\sum_k \sum_{(I,x)\in\pi_k} |F[I] - f(x)|I|| < \sum_k \frac{\varepsilon}{2^k} + 2\sum_n \sum_{i=I+1}^{\infty} \omega\bigl(x_i^{(n)}\bigr) < 3\varepsilon. \tag{4}$$

If $(I, x) \in \pi_k$ for no k, then it is $\mathcal{C}$-fine. So, by the definition of $\mathcal{C}$,

$$\sum_{(I,x)\notin \bigcup_k \pi_k} |F[I] - f(x)|I|| < \varepsilon. \tag{5}$$

Thus, taking into account (4) and (5), we get

$$\left| F(b) - F(a) - \sum_{(I,x)\in\pi} f(x)|I| \right| < 4\varepsilon. \qquad \square$$

4.1 – *Variational definitions*

Thanks to Saks–Henstock lemmas, each of the integrals considered in Section 3 allows an equivalent variational definition.

DEFINITION 19. *We say a function $f\colon [a, b] \to \mathbb{R}$ is variationally LS_Δ-integrable if there is a function $F\colon [a, b] \to \mathbb{R}$ (an indefinite integral of f) such that to each $\varepsilon > 0$ we can find a closed $[a, b]$-form $\{E_i\}_i$ such that for any countable $A \subset [a, b]$ there is a Δ-choice $\mathcal{C}$ on A and a sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$ such that*

$$\sum_{(I,x)\in\mathcal{P}} |f(x)|I| - F[I]| < \varepsilon \tag{6}$$

holds for each $\left(\mathcal{C}, \{\delta_i\}_i\right)$-fine division $\mathcal{P}$ in $[a, b]$.

We say an f is variationally E_Δ-integrable if there is a function $F\colon [a, b] \to \mathbb{R}$ (an indefinite integral of f) such that to each $\varepsilon > 0$ we can find a closed $[a, b]$-form $\{E_i\}_i$ and a related sequence of gauges $\{\delta_i\}_i$ with the property that for each countable $A \subset [a, b]$ there is a Δ-choice $\mathcal{C}$ on A such that (6) *holds for each $\left(\mathcal{C}, \{\delta_i\}_i\right)$-fine division $\mathcal{P}$ in $[a, b]$.*

We say an f is variationally LL_Δ-integrable if there is a function $F\colon [a, b] \to \mathbb{R}$ (an indefinite integral of f) and a closed $[a, b]$-form $\{E_i\}_i$ such that to each $\varepsilon > 0$ and to each countable $A \subset [a, b]$ there is a Δ-choice $\mathcal{C}$ and a sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$ such that (6) *holds for each $\left(\mathcal{C}, \{\delta_i\}_i\right)$-fine division $\mathcal{P}$ in $[a, b]$.*

Ene introduced a variational version of E_Δ-integral [5, section 9] and called it $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$*-integral*. The difference between our definition (variational E_Δ-integral) and that of Ene is that in the latter F is *a priori* Δ-continuous, whereas for variational equivalence only $\{\delta_i\}_i$-fine divisions are relevant. Here, we do not assume F to be Δ-continuous, however we are able to prove it since we included not only $\{\delta_i\}_i$-fine, but also $\mathcal{C}$-fine intervals (I, x) in $\mathcal{P}$ for (6). Therefore our definition is stronger than that of Ene. The opposite statement is easy to prove, as contribution of $\mathcal{C}$-fine intervals is negligible for $\sum\limits_{(I,x)\in\mathcal{P}} |f(x)||I|$ and for $\sum\limits_{(I,x)\in\mathcal{P}} |F[I]|$ in case of $\varDelta$-continuous F (Remark 16). Hence our variational $E_\varDelta$-integral and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integral of [5] are equivalent.

From definitions and Saks–Henstock lemmas 11-13, follows immediately

COROLLARY 20. *A function $f\colon [a,b] \to \mathbb{R}$ is $LL_\varDelta$-integrable iff it is variationally $LL_\varDelta$-integrable. An f is $LS_\varDelta$-integrable iff it is variationally $LS_\varDelta$-integrable. An f is $E_\varDelta$-integrable iff it is variationally $E_\varDelta$-integrable.*

As a result, all the three variational integrals are properly defined, i.e., the value $\int\limits_a^b f = F(b) - F(a)$ is unique.

Yet before accomplishing the proofs for characterizations of $\mathcal{F}_1^\varDelta$- and $\mathcal{F}_2^\varDelta$-integrals, we introduce a generalization of Definition 19. It is to provide variational characterizations for the $\mathcal{F}_4^\varDelta$-integral. The generalization is based on passing from closed $[a,b]$-forms to $[a,b]$-forms that need not be closed.

DEFINITION 21. *We say a function $f\colon [a,b] \to \mathbb{R}$ is weakly variationally $LS_\varDelta$-integrable if there is a function $F\colon [a,b] \to \mathbb{R}$ (an indefinite integral of f) such that to each $\varepsilon > 0$ we can find a measurable $[a,b]$-form $\{E_i\}_i$ such that for any countable set $A \subset [a,b]$ there is a $\varDelta$-choice $\mathcal{C}$ on A and a sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$ such that* (6) *holds for each $(\mathcal{C}, \{\delta_i\}_i)$-fine division $\mathcal{P}$ in $[a,b]$.*

We say a function $f\colon [a,b] \to \mathbb{R}$ is weakly variationally $E_\varDelta$-integrable if there is a function $F\colon [a,b] \to \mathbb{R}$ (an indefinite integral of f) such that to each $\varepsilon > 0$ we can find a measurable $[a,b]$-form $\{E_i\}_i$ and a related sequence of gauges $\{\delta_i\}_i$, such that for any countable $A \subset [a,b]$ there is a $\varDelta$-choice $\mathcal{C}$ on A such that (6) *holds for each $(\mathcal{C}, \{\delta_i\}_i)$-fine division $\mathcal{P}$ in $[a,b]$.*

We say a function $f\colon [a,b] \to \mathbb{R}$ is weakly variationally $LL_\varDelta$-integrable if there is a function $F\colon [a,b] \to \mathbb{R}$ (an indefinite integral of f) and a measurable $[a,b]$-form $\{E_i\}_i$ such that to each $\varepsilon > 0$ and each countable $A \subset [a,b]$ there is a $\varDelta$-choice $\mathcal{C}$ on A and a sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$ such that (6) *holds for each $(\mathcal{C}, \{\delta_i\}_i)$-fine division $\mathcal{P}$ in $[a,b]$.*

We conjecture that the assumption of measurability of $\{E_i\}_i$ is not essential, however as the definitions given suit our purpose, we do not follow this question. The corollary below is a consequence of Corollary 20.

Corollary 22. *LS_Δ-integrability implies weak variational LS_Δ-integrability, E_Δ-integrability implies weak variational E_Δ-integrability, LL_Δ-integrability implies weak variational LL_Δ-integrability.*

We aim to prove, in some contrary to the case with closed $[a,b]$-forms, that all three integrals from Definition 21 yield the same integral, namely the $\mathcal{F}^4_\Delta$-integral. First of all, we prove some properties of indefinite integrals in the weak variational LS_Δ sense and show that each such integral is a primitive for $\mathcal{F}^4_\Delta$-integral. As a result, uniqueness of the above definition shall follow. We start with a result analogous to Lemma 15.

Lemma 23. *Let $f\colon[a,b]\to\mathbb{R}$ be weakly variationally LS_Δ-integrable. Then $F=\int f$ is VBG and Δ-continuous.*

Proof. Let $\{E_i\}_i$ be an $[a,b]$-form appropriate for $\varepsilon=1$ in the sense of weak variational LS_Δ-integrability of f. Take a sequence of gauges $\{\delta_i\}_i$ suitable for $\varepsilon=1$, $A=\emptyset$, and $\{E_i\}_i$. Given i, set $E_{i,k}=\{x\in E_i:\delta_i(x)>(b-a)/k, |f(x)|\le k\}$, $k\in\mathbb{N}$. It is straightforward (see the proof of Lemma 15) that for each collection $\{[a_j,b_j]\}_j$ of nonoverlapping intervals with both endpoints in

$$E_{i,k}\cap\left[m\,\frac{b-a}{k},(m+1)\,\frac{b-a}{k}\right],\qquad m=0,\dots,k-1,$$

the inequality

$$\sum_j|F(b_j)-F(a_j)|\le 1+b-a$$

holds. This means F is *VB* on $E_{i,k}\cap\left[m\,\frac{b-a}{k},(m+1)\,\frac{b-a}{k}\right]$ and so *VBG* on $[a,b]$. Δ-continuity of F can be justified similarly as for LS_Δ-integral. □

Lemma 24 [Lemma 5.2]. *Let $F\colon E\to\mathbb{R}$ be VB, $|E|=0$, $|F(E)|>0$. Then there exists a set $E_0\subset E$ such that $|F(E_0)|>0$ and $F\restriction E_0$ is strictly monotone.*

LEMMA 25. *Assume* $f\colon [a,b] \to \mathbb{R}$ *is weakly variationally* LS_Δ*-integrable. Then* $F = \int f$ *satisfies* $\mathcal{N}$.

PROOF. We follow the pattern of the proof of [5, Theorem 5.1(ii)]. We know F is *VBG* (Lemma 23). Let $\{E_i\}_i$ be an $[a,b]$-form such that F is *VB* and continuous with respect to each E_i. Suppose F does not fulfil $\mathcal{N}$. Then, there is an i such that $|F(Z)| > 0$ for some nullset $Z \subset E_i$. By Lemma 24 we can assume $F \restriction Z$ is strictly monotone. Moreover, we can assume $f = 0$ on Z. For $\varepsilon = |F(Z)|/2$ there is a Z-form $\{Z_k\}_k$ and a related sequence of gauges $\{\delta_k\}_k$ such that

$$\sum_{(\langle x,y\rangle,x)\in\mathcal{P}} |F(x) - F(x)| = \sum_{(\langle x,y\rangle,x)\in\mathcal{P}} |F(x) - F(y) - f(x)(x-y)| < \varepsilon$$

holds for each $\{\delta_k\}_k$-fine division $\mathcal{P}$. As $F \restriction Z$ is continuous and it is possible to exclude from Z all points that are isolated in any Z_k, we can claim that the collection of intervals $\langle F(x), F(y)\rangle$ where $(\langle x,y\rangle, x)$ is δ_k-fine, $k \in \mathbb{N}$, forms a Vitali cover of the image $F(Z)$. By the Vitali covering lemma, we can pick a finite collection $\{\langle F(x_j), F(y_j)\rangle\}_j$ of such intervals with the property that

$$\sum_j |F(y_j) - F(x_j)| \geq \frac{|F(Z)|}{2}.$$

The division $\{(\langle x_j, y_j\rangle, x_j)\}_j$ is $\{\delta_k\}_k$-fine, so there should be

$$\varepsilon = \frac{|F(Z)|}{2} > \sum_j |F(y_j) - F(x_j)| \geq \frac{|F(Z)|}{2},$$

a contradiction. □

COROLLARY 26. $\mathcal{F}_2^\Delta$-, LS_Δ-, *and* E_Δ*-integrals are equivalent.*

PROOF. It follows from Theorem 18, Lemmas 15 and 25, Corollary 22, the remark after Definition 8, and the fact that indefinite LS_Δ-integrals $F = \int f$ are almost everywhere approximately differentiable to f (see the subsection 4.3 below). □

This corollary formulates an answer to the query posed by Ene [5, p. 91]. It turns out all integrals defined in [5] are equivalents of $\mathcal{F}_2^\Delta$-integral (strong $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral in the language therein).

4.2 – *Characterization for $\mathcal{F}_1^{\Delta}$-integral*

Lemma 27. *Let f be variationally LL_{Δ}-integrable using an $[a,b]$-form $\{E_i\}_i$. Then if $F=\int f$, $F\restriction E_i$ is continuous for each i.*

Proof. For an $\varepsilon>0$ take an appropriate sequence of gauges $\{\delta_i\}_i$ related to $\{E_i\}_i$. Let $x\in E_i$. By definition, for $y\in E_i$ with $|x-y|<\delta_i(x)$ and $|x-y|<\varepsilon/(|f(x)|+1)$ we get

$$|F(x)-F(y)|\le|F(x)-F(y)-f(x)(x-y)|+|f(x)||x-y|<\varepsilon+\varepsilon.$$

This means the restriction $F\restriction E_i$ is continuous. □

Theorem 28. *An $f\colon[a,b]\to\mathbb{R}$ is LL_{Δ}-integrable if and only if it is $\mathcal{F}_1^{\Delta}$-integrable. Moreover, both integrals coincide.*

Proof. ($\Rightarrow$) Since f is LL_{Δ}-integrable, it is LS_{Δ}-integrable, and so weakly variationally LS_{Δ}-integrable. By Lemmas 15 and 25, $F=\int f$ is a $[VBG]$-function with condition $\mathcal{N}$. By Lemma 27 and the Banach–Zarecki theorem [16, Chapter VII, (6.8)], F is an $[ACG]$-function. As will be shown in the next subsection, F is a.e. approximately differentiable to f, hence f is $\mathcal{F}_1^{\Delta}$-integrable with the indefinite integral F.

($\Leftarrow$) This is a simplified version of the proof of Theorem 18 (we skip functions H here). Let f be $\mathcal{F}_1^{\Delta}$-integrable. Then $F=\int f$ is a Δ-continuous $[ACG]$-function. Let $\{E_i\}_i$ be a closed $[a,b]$-form such that F is AC on E_i, $i\in\mathbb{N}$. Since for each i, $(F\restriction E_i)'(x)=F'_{\mathrm{ap}}(x)=f(x)$ at almost every $x\in E_i$, by Lemma 17 there is a gauge δ_i on E_i such that $2^i\sum_{(\langle x,y\rangle,x)\in\mathcal{P}}|F(y)-F(x)-f(x)(y-x)|<\varepsilon$ holds for each δ_i-fine division $\mathcal{P}$ with all intervals $\langle x,y\rangle$ having both endpoints in E_i. Choose a countable $A\supset\mathrm{Is}\{E_i\}_i$, $A\subset[a,b]$, and define $\mathcal{C}$ according to Remark 16. For any $(\mathcal{C},\{\delta_i\}_i)$-fine partition π of $[a,b]$ we have thus

$$\sum_{(\langle x,y\rangle,x)\in\pi}|F(y)-F(x)-f(x)(y-x)|\le\sum_i\frac{\varepsilon}{2^i}+\varepsilon=2\varepsilon.$$

So, f is LL_{Δ}-integrable to $F(b)-F(a)$. □

From the ($\Leftarrow$) part of the above proof it follows that in Definition 7′, $\{\delta_i\}_i$ need not depend on A. Theorem 28 is a generalization of Theorem 14.

4.3 – *Characterizations for $\mathcal{F}_4^{\Delta}$-integral*

For the $\mathcal{F}_4^{\Delta}$-integral it would be natural to expect a characterization similar to that of Corollary 26, however with the integral defined with $[a,b]$-forms that need not be closed. For non-closed $[a,b]$-forms composite Δ-gauges miss the partitioning property (the remark after Theorem 2), so we must restrict ourselves to variational description only.

LEMMA 29 ([4, Theorem 2] and [22, Lemma 3.3]). *Let $F\colon E \to \mathbb{R}$ be a VB-function satisfying $\mathcal{N}$ and let $F = F^+ - F^-$ be the Jordan decomposition of F. Then both F^+ and F^- also satisfy $\mathcal{N}$.*

THEOREM 30. *Let $f\colon [a,b] \to \mathbb{R}$ be $\mathcal{F}_4^{\Delta}$-integrable. Then it is weakly variationally LL_{Δ}-integrable with the same integral.*

PROOF. Let $F = \int f$. Then F is *VBG*, Δ-continuous, and satisfies $\mathcal{N}$. By assumption, F is Baire one and so measurable. Thus there exists a measurable $[a,b]$-form $\{E_i\}_i$ such that each $F{\upharpoonright}E_i$ is *VB* [16, Chapter VII, (4.2)]. We can assume that each $F \upharpoonright E_i$ is moreover continuous. We will show f is weakly variationally LL_{Δ}-integrable using $\{E_i\}_i$. Take a countable set $A \subset [a,b]$ and $\varepsilon > 0$. Each restriction $F \upharpoonright E_i$ is almost everywhere differentiable to f. Denote $E'_i = \{x \in E_i : (F \upharpoonright E_i)'(x) = f(x)\}$. We will define gauges related to $\{E_i\}_i$ and suitable for ε. For $x \in E_i$ we have two cases: (1) $x \in E'_i$, (2) $x \in E_i \setminus E'_i$. In the case (1) we define $\delta_i(x) > 0$ so that $|F(y) - F(x) - f(x)(y-x)| \le \varepsilon|x-y|$ for all $y \in (x - \delta_i(x), x + \delta_i(x)) \cap E_i$. We pass to the case (2). By Lemma 29, $F \upharpoonright E_i = F_i^+ - F_i^-$ where both $F_i^+, F_i^- : E_i \to \mathbb{R}$ are nondecreasing and satisfy $\mathcal{N}$. The images $P_i^+ = F_i^+(E_i \setminus E'_i)$ and $P_i^- = F_i^-(E_i \setminus E'_i)$ are nullsets. There are open sets $O_i^+ \supset P_i^+$, $O_i^- \supset P_i^-$ with $2^i\,|O_i^+| < \varepsilon$ and $2^i\,|O_i^-| < \varepsilon$. The preimages $(F_i^+)^{-1}(O_i^+)$, $(F_i^-)^{-1}(O_i^-)$ are open in E_i. At $x \in E_i \setminus E'_i$ we define $\delta_i(x) > 0$ as to have $E_i \cap (x - \delta_i(x), x + \delta_i(x)) \subset (F_i^+)^{-1}(O_i^+) \cap (F_i^-)^{-1}(O_i^-)$. Finally, for a countable A we define a Δ-choice $\mathcal{C}$ according to Remark 16. We have accomplished the construction of composite Δ-gauge $\big(\mathcal{C}, \{\delta_i\}_i\big)$.

Consider a $\big(\mathcal{C}, \{\delta_i\}_i\big)$-fine division π in $[a,b]$. Set

$$\pi_{\mathcal{C}} = \big\{(\langle x,y\rangle, x) \in \pi : (\langle x,y\rangle, x) \text{ is } \mathcal{C}\text{-fine}\big\},$$
$$\pi_i = \big\{(\langle x,y\rangle, x) \in \pi : x \in E_i \setminus E'_i\big\}, \quad \pi'_i = \big\{(\langle x,y\rangle, x) \in \pi : x \in E'_i\big\}, \quad i \in \mathbb{N}.$$

As usual, we agree that each member of π belongs to only one of the above divisions. We have

$$\sum_{(\langle x,y\rangle,x)\in\pi_C} |F(x)-F(y)-f(x)(x-y)|<\varepsilon$$

and

$$\sum_i \sum_{(\langle x,y\rangle,x)\in\pi_i'} |F(x)-F(y)-f(x)(x-y)| \le \varepsilon(b-a),$$

so in order to accomplish the proof it is enough to show the variational equivalence condition holds for $\bigcup_i \pi_i$. We have

$$\sum_{(\langle x,y\rangle,x)\in\pi_i} |F(x)-F(y)-f(x)(x-y)|\le \sum_{(\langle x,y\rangle,x)\in\pi_i} |F(x)-F(y)|+\sum_{(\langle x,y\rangle,x)\in\pi_i} |f(x)(x-y)|.$$

Let $(\langle x,y\rangle,x)\in\pi_i$. As $x\in E_i\setminus E_i'$, by definition of δ_i, $\langle x,y\rangle\cap E_i\subset (F_i^+)^{-1}(O_i^+)\cap(F_i^-)^{-1}(O_i^-)$ and hence $\langle F_i^+(x),F_i^+(y)\rangle\subset O_i^+$, $\langle F_i^-(y),F_i^-(x)\rangle\subset O_i^-$. For this reason, as the intervals $\langle F_i^+(y),F_i^+(x)\rangle$, $(\langle x,y\rangle,x)\in\pi_i$, and the intervals $\langle F_i^-(x),F_i^-(y)\rangle$, $(\langle x,y\rangle,x)\in\pi_i$, are pairwise nonoverlapping, we have

$$\begin{aligned}\sum_{(\langle x,y\rangle,x)\in\pi_i} |F(x)-F(y)| &\le \sum_{(\langle x,y\rangle,x)\in\pi_i} |F_i^+(x)-F_i^+(y)|+\sum_{(\langle x,y\rangle,x)\in\pi_i} |F_i^-(x)-F_i^-(y)|\\ &\le |O_i^+|+|O_i^-|<2\cdot\frac{\varepsilon}{2^i}.\end{aligned}$$

By minimizing δ_i in accordance with the value of f, we can claim that

$$\sum_{(\langle x,y\rangle,x)\in\pi_i} |f(x)(x-y)|<\frac{\varepsilon}{2^i}.$$

So altogether,

$$\begin{aligned}\sum_{(\langle x,y\rangle,x)\in\bigcup_i\pi_i} |F(x)-F(y)-f(x)(x-y)| &= \sum_i \sum_{(\langle x,y\rangle,x)\in\pi_i} |F(x)-F(y)-f(x)(x-y)|\\ &< 3\sum_i \frac{\varepsilon}{2^i}=3\varepsilon.\end{aligned}$$

This finishes the proof. □

Since in the proof above $\{\delta_i\}_i$ did not depend on A, we get

COROLLARY 31. *Let $f\colon[a,b]\to\mathbb{R}$ be $\mathcal{F}_4^{\varDelta}$-integrable. Then it is weakly variationally $E_{\varDelta}$-integrable with the same integral.*

THEOREM 32. *The weak variational LS_Δ-integral is properly defined; i.e., each two indefinite integrals of the same function differ by an additive constant.*

PROOF. Let f be weakly variationally LS_Δ-integrable and let F and G be two indefinite integrals for f. From Lemmas 23 and 25 we know both F and G are VBG-functions that satisfy $\mathcal{N}$. Moreover, they are both Δ-continuous, so Baire one and Darboux. As $\mathcal{L}_4$ is a linear space and Δ is filtering down, $F - G$ has all these properties as well. So, in order to show $F - G$ is constant it is enough to prove that $F'_{\text{ap}}(x) = G'_{\text{ap}}(x) = f(x)$ at almost every $x \in [a, b]$. Suppose, $f(x)$ is not $F'_{\text{ap}}(x)$ almost everywhere. Then the set $Z = \{x \in [a, b] \colon F'_{\text{ap}}(x) \neq f(x)\}$ has $|Z| > 0$. For each $x \in Z$ there is a $\gamma(x) > 0$ with the property: in each $O \in \Delta_{\text{ap}}(x)$ there is a $y \in O$ such that $|f(x)(y - x) - F(y) + F(x)| > \gamma(x)|y - x|$. Denote by Z_n the set of $x \in Z$ with $\gamma(x) > 1/n$, $n \in \mathbb{N}$. For some n we have $\varepsilon = |Z_n| > 0$. Take a measurable $[a, b]$-form $\{E_i\}_i$ suitable for $\varepsilon/2n$ and f according to the definition of weak variational LS_Δ-integrability. We can assume F is VB on each E_i [16, Chapter VII, (4.2)]. Take a related sequence of gauges $\{\delta_i\}_i$. By the Lebesgue density theorem, for almost every $x \in E_i$, $i \in \mathbb{N}$, the set $E_i \in \Delta_{\text{ap}}(x)$. Thus for almost every $x \in Z_n \cap E_i$, $i \in \mathbb{N}$, in each neighborhood of x there is a $y \in E_i$ such that $|f(x)(y - x) - F(y) + F(x)| > |y - x|/n$. Such intervals $\langle x, y\rangle$ form a Vitali cover of Z_n, whence there are nonoverlapping $\langle x_k, y_k\rangle$, $x_k \in E_{i_k}$, $y_k \in E_{i_k} \cap (x_k - \delta_{i_k}(x_k), x_k + \delta_{i_k}(x_k))$, such that $\sum_k |y_k - x_k| > \varepsilon/2$. It stems from the above that

$$\sum_k |f(x_k)(y_k - x_k) - F(y_k) + F(x_k)| > \gamma(x_k) \sum_k |y_k - x_k| > \frac{\varepsilon}{2n},$$

a contradiction since $\{(\langle x_k, y_k\rangle, x_k)\}_k$ is a $\{\delta_i\}_i$-fine division in $[a, b]$. So $F'_{\text{ap}}(x) = f(x)$ almost everywhere and $G'_{\text{ap}}(x) = f(x)$ almost everywhere as well. □

COROLLARY 33. *Let $f \colon [a, b] \to \mathbb{R}$ be weakly variationally LS_Δ-integrable, $F = \int f$. Then $F'_{\text{ap}} = f$ almost everywhere.*

COROLLARY 34. *$\mathcal{F}_4^\Delta$-, weak variational LS_Δ-, weak variational E_Δ-, and weak variational LL_Δ-integrals are equivalent.*

PROOF. It follows from Lemmas 23, 25, Theorem 30, and Corollaries 31 and 33. □

There are two matters that make our results less concluding in a sense. First of all, there is left open the problem if it is possible to define $\mathcal{F}_4^{\Delta}$-integral in the Riemann manner. We have noted that the Riemann-type definition with *all* $[a, b]$-forms lacks sense as we do not have partitioning property for such a rich 'base'. Perhaps, however, we can restrict ourselves to a subclass of the class of all $[a, b]$-forms with the resulting 'base' having partitioning property and the integral being equivalent to $\mathcal{F}_4^{\Delta}$-integral. The second question we leave without any answer is a possible Riemann- or at least variational-type definition for $\mathcal{F}_3^{\Delta}$-integral.

The chart below shows all relationships between the integrals considered. All inclusions (implications) are proper (in general). The sign '=' means the integrals are equivalent. For $\mathcal{F}_2^{\Delta}$- and $\mathcal{F}_3^{\Delta}$-integrals there is no inclusion relation [21].

$$\begin{array}{ccccc}
 & \subset & E_\Delta = vE_\Delta = LS_\Delta = vLS_\Delta = \mathcal{F}_2^{\Delta} & \subset & \\
LL_\Delta = vLL_\Delta = \mathcal{F}_1^{\Delta} & & & & vwLL_\Delta = wvE_\Delta = wvLS_\Delta = \mathcal{F}_4^{\Delta} \\
 & \subset & \mathcal{F}_3^{\Delta} & \subset &
\end{array}$$

5. Wide Denjoy integral

We end up with a corollary related to the wide Denjoy integral. As all our considerations work in the case when $\Delta = \Delta_{\mathrm{e}}$ is the local system of neighborhoods in the natural topology, we can provide the reader with a vast collection of alternative definitions for wide Denjoy integral.

COROLLARY 35. *For any $f \colon [a, b] \to \mathbb{R}$ the following statements are equivalent:*

(i) *f is integrable in the wide Denjoy sense,*
(ii) *f is $E_{\Delta_{\mathrm{e}}}$-integrable,*
(iii) *f is $LS_{\Delta_{\mathrm{e}}}$-integrable,*
(iv) *f is $LL_{\Delta_{\mathrm{e}}}$-integrable,*
(v) *f is variationally $E_{\Delta_{\mathrm{e}}}$-integrable,*
(vi) *f is variationally $LS_{\Delta_{\mathrm{e}}}$-integrable,*
(vii) *f is variationally $LL_{\Delta_{\mathrm{e}}}$-integrable,*
(viii) *f is weakly variationally $E_{\Delta_{\mathrm{e}}}$-integrable,*
(ix) *f is weakly variationally $LS_{\Delta_{\mathrm{e}}}$-integrable,*
(x) *f is weakly variationally $LL_{\Delta_{\mathrm{e}}}$-integrable,*

All is up to the fact that $\mathcal{F}_i^{\Delta_e}$-integrals, $i = 1, 2, 4$, are equivalent to the wide Denjoy integral. The equivalence (i)$\Leftrightarrow$(v) and the implication (i)$\Rightarrow$(ii) were given by Ene [5, Theorem 11.1]. It seems, none of other characterizations have been stated explicitly so far, however we are in no position to claim priority here.

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