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# An Integral Formula Related to Inner Isoptics

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ABSTRACT - An isoptic $C_\alpha$ of a strictly convex $C^2$-curve in the plane is the locus of all points from which $C$ is seen under the same fixed angle. The two supporting lines of $C$ through such a point determine a secant of $C$, and the envelope of all these secants is the inner isoptic of $C$ and $C_\alpha$. We describe an integral formula for inner isoptics in terms of quantities that naturally occur in this geometric configuration.

Let $C$ be an oval, i.e., a simple closed $C^2$-curve in the plane with positive curvature. We take a coordinate system with the origin $O$ in the interior of $C$. Let $p(t), t \in \mathbb{R}$, be the distance from $O$ to the supporting line $l(t)$ of $C$ perpendicular to the vector $e^{it} = \cos t + i \sin t$. It is well-known that $p(t)$ is of class $C^2$ and that the parametrization of $C$ is then given by $z(t) = p(t)e^{it} + p'(t)ie^{it}$, where $ie^{it} = -\sin t + i \cos t$. Note that $p(t)$, called the *support function* of $C$, is a periodic function on $\mathbb{R}$ with the period $2\pi$. We introduce the notations

$$q(t) := z(t) - z(t + \alpha),$$
$$b(t) := [q(t), e^{it}],$$
$$B(t) := [q(t), ie^{it}],$$

where $[\cdot, \cdot]$ means determinant. Let $C_\alpha$ be the locus of apices of a fixed angle $\pi - \alpha$, where $\alpha \in (0, \pi)$, formed by two supporting lines of $C$. Then $C_\alpha$ is called *$\alpha$-isoptic of $C$*; see Fig. 1. In this paper we want to describe geometric properties of so called inner isoptics which are derived from isoptics of the given curve $C$. To avoid possible confusion, we note that the notion of

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isoptic is sometimes also used with different meanings, e.g. in classical illumination geometry (see [6]) or in the theory of the light field (cf. [2]), for example with respect to so called isophotic families of area elements.

Regarding notations our paper is based on [4], where also the notion of inner isoptic was introduced; see also [5]. In particular, we use the symbols $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the Euclidean norm and the Euclidean dot product of the arguments, respectively.

We assume that the oval $C$ is parametrized as above, that the origin coincides with the Steiner point of $C$, and that the coordinate system is placed in such a way that the tangent line at $z(0)$ is perpendicular to the $x$-axis. Thus, in particular we have that $z(0) = (a, 0)$ for some $a > 0$ and $p'(0) = 0$. Then the distance from $O$ to the line $d(t)$ determined by the points $z(t)$ and $z(t+\alpha), \alpha \in (0, \pi)$, is given by

$$P(t) = -\frac{[z(t), q(t)]}{|q(t)|}.$$

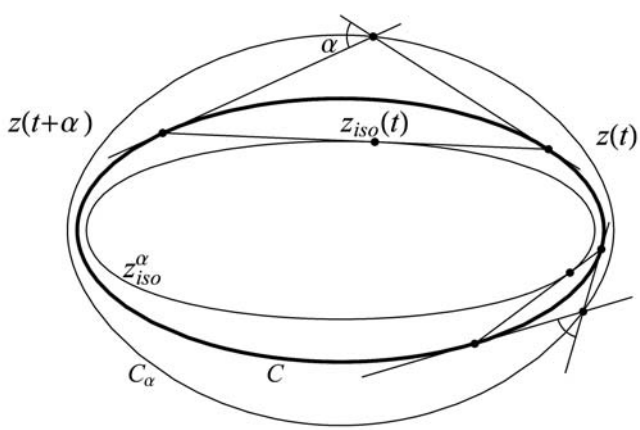


Fig. 1. Deriving an inner isoptic from $C_\alpha$.

Then we can determine the envelope of the lines $d(t)$ (see Fig. 1), obtaining

$$z^\alpha_{\text{iso}}(t) = z_{\text{iso}}(t) = P(t)E^{it} + P_{\text{prim}}(t)iE^{it},$$

where

$$E^{it} = \left\{-\frac{[z(0), q]}{|z(0)|\cdot|q|}, \frac{\langle z(0), q\rangle}{|z(0)|\cdot|q|}\right\}(t) = J\left(\frac{q}{|q|}(t)\right),$$

with $J$ denoting the positive rotation about $\frac{\pi}{2}$,

$$iE^{it} = \left\{ -\frac{\langle z(0), q\rangle}{|z(0)| \cdot |q|} \,,\quad -\frac{\langle z(0), q\rangle}{|z(0)| \cdot |q|} \right\}(t) = -\frac{q}{|q|}(t)\,,$$

and

$$P_{\text{prim}}(t) = -\frac{[z', q] \cdot |q|^2 + \langle q, z\rangle \cdot [q, q']}{|q| \cdot [q, q']}(t)\,. \tag{1}$$

Note that the function $P$ for the inner isoptic $z^{\alpha}_{\text{iso}}$ is a counterpart of a support function of an oval. Therefore the term $P_{\text{prim}}$ is used to have also a similar formula for its parametrization. Moreover, it is in fact a derivative of $P$ with respect to a some naturally arising parameter as explained in formula (3.10) in [4]. We call this envelope the *inner isoptic associated to $C_\alpha$ and $C$*; see also Fig. 2.

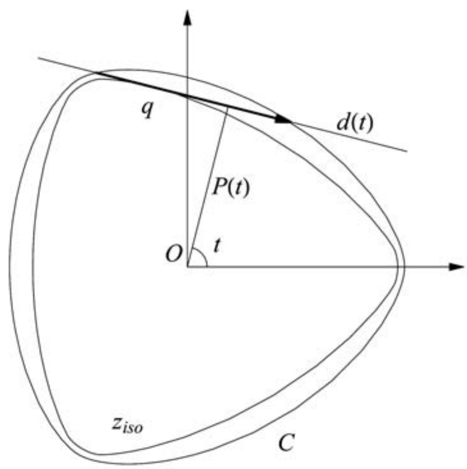


Fig. 2. An inner isoptic as envelope of the vectors $q$.

Dropping the point $t$, we get

$$z'_{\text{iso}} = \frac{\partial z_{\text{iso}}}{\partial t} = P'E^{it} + PiE^{it}\frac{[q, q']}{|q|^2} + P_{\text{prim}|t}\, iE^{it} - P_{\text{prim}}E^{it}\frac{[q, q']}{|q|^2}\,.$$

LEMMA 1. *With the above notations we have*

$$P'\frac{|q|^2}{[q, q']} = P_{\text{prim}}\,.$$

Hence we get

$$z'_{\text{iso}} = \left( P\frac{[q,q']}{|q|^2} + P_{\text{prim}|t} \right) iE^{it} .$$

We define

$$P\frac{[q,q']}{|q|^2} + P_{\text{prim}|t} =: K .$$

Thus $\|z'_{\text{iso}}\| = |K|$, but for small values of $\alpha$ (cf. [4]) the number $K$ is positive.

LEMMA 2. *The curvature of* $z_{\text{iso}}$ *is given by*

$$\kappa_{\text{iso}} = \frac{1}{|K|} \cdot \frac{[q,q']}{|q|^2} .$$

PROOF. From the above considerations we have

$$z'_{\text{iso}} = KiE^{it}.$$

Then we obtain

$$z''_{\text{iso}} = K'iE^{it} - KE^{it}\frac{[q,q']}{|q|^2} ,$$

and next

$$\kappa_{\text{iso}} = \frac{[z'_{\text{iso}}, z''_{\text{iso}}]}{|z'_{\text{iso}}|^3} = -\frac{[iE^{it}, E^{it}] \cdot [q,q']}{|K| \cdot |q|^2} = \frac{[q,q']}{|K| \cdot |q|^2} ,$$

since $[E^{it}, iE^{it}] = \left[ J\left(\frac{q}{|q|}\right), \frac{q}{|q|} \right] = \left\langle \frac{q}{|q|}, \frac{q}{|q|} \right\rangle = 1.$ □

Let us write

$$\alpha_0 = \sup\left\{ \alpha > 0 \colon z^{\beta}_{\text{iso}} \text{ is convex for every } \beta \in (0, \alpha) \right\} ,$$

and let the angle $\alpha_0$ be called the *limit angle* for inner isoptics.

EXAMPLES:

a) Since one can show that $|z'_{\text{iso}}| = 0$ iff the curvature of the corresponding usual $\alpha$-isoptic vanishes, in case of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get

$$\cos\alpha_0 = \frac{b^2}{b^2 - a^2} ,$$

and for $1 < \frac{a}{b} < \sqrt{2}$ all inner isoptics are convex with $K > 0$ everywhere. Note that usual isoptics are convex under the same condition.

b) This is no longer true for curves with $p(t) = a + b\cos 3t$, where $a > 8b$. And inner isoptics can also have cusps (i.e., points with $K = 0$) when their generating isoptics are still convex.

Recall that

$$\begin{aligned} z_{\mathrm{iso}} &= PE^{it} + P_{\mathrm{prim}} iE^{it} \\ &= PJ\left(\frac{q}{|q|}\right) + P_{\mathrm{prim}}\left(-\frac{q}{|q|}\right) \\ &= \left[\frac{q}{|q|}, z\right] J\left(\frac{q}{|q|}\right) + \left(\frac{[z', q]\cdot|q|}{[q, q']} + \frac{\langle q, z\rangle}{|q|}\right)\frac{1}{|q|} \\ &= z\left\langle \frac{q}{|q|}, \frac{q}{|q|}\right\rangle - \frac{q}{|q|}\langle z, q\rangle + \left(\frac{[z', q]\cdot|q|}{[q, q']} + \frac{\langle q, z\rangle}{|q|}\right)\frac{q}{|q|} \\ &= z + \frac{[z', q]}{[q, q']} q\,, \end{aligned}$$

where we used the formula

$$v\langle w, w\rangle - w\langle v, w\rangle = [w, v]\, J(w)\,. \tag{2}$$

Note that

$$z_{\mathrm{iso}} - z = \frac{[z', q]}{[q, q']}\cdot q\,.$$

Thus $\overrightarrow{zz_{\mathrm{iso}}} = -\frac{[z', q]}{[q, q']}\cdot q$. Moreover, we have $\frac{[z', q]}{[q, q']} < 0$.

Thus, if $z_{\mathrm{iso}}$ is convex, then $t_1 = -\frac{[z', q]}{[q, q']}\cdot|q|$ and $t_2 = -\frac{[q, z'(t+\alpha)]}{[q, q']}\cdot|q|$ (see Fig. 3 for the quantities $t_1$, $t_2$).

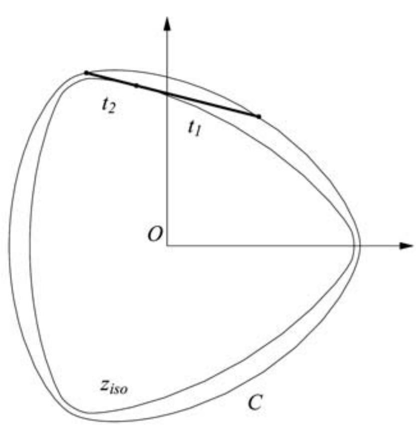

Fig. 3. Geometric illustration of the numbers $t_1$ and $t_2$.

For a given oval $C$ the interval $(0, \alpha_0)$ yields only inner isoptics that are convex. Having this, the inner isoptics fill an “annulus” (see Fig. 4), and we can parametrize this “annulus” $CC_\alpha$ minus some set of measure zero (i.e., minus a graph of the curve $\alpha \mapsto z^{\alpha}_{\text{iso}}(0)$) by means of $z^{\alpha}_{\text{iso}}(t)$.

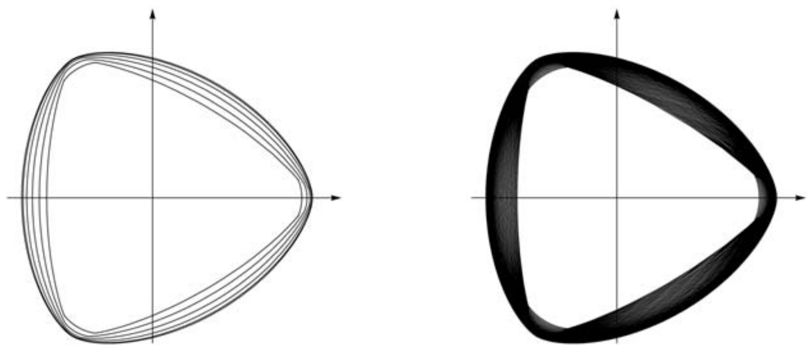

Fig. 4. Several inner isoptics and the whole annulus $CC_\alpha$.

Let

$$F(t, \alpha) = z^{\alpha}_{\text{iso}}(t)\,, \quad \alpha \in (0, \alpha_0)\,,\ t \in (0, 2\pi),$$

and calculate the Jacobian matrix

$$\frac{\partial(F)}{\partial(t, \alpha)} = \left[\frac{\partial z_{\text{iso}}}{\partial t}, \frac{\partial z_{\text{iso}}}{\partial \alpha}\right].$$

We have

$$\frac{\partial z_{\text{iso}}}{\partial t} = -K \frac{q}{|q|}\,.$$

Let us calculate

$$\frac{\partial z_{\text{iso}}}{\partial \alpha} = \frac{\partial P}{\partial \alpha} \cdot J\left(\frac{q}{|q|}\right) + P \cdot J\left(\left(\frac{q}{|q|}\right)_{|\alpha}\right) + \frac{\partial P_{\text{prim}}}{\partial \alpha} \cdot \left(\frac{-q}{|q|}\right) + P_{\text{prim}}\left(\frac{-q}{|q|}\right)_{|\alpha}.$$

To this aim we have by (2)

$$\left(\frac{q}{|q|}\right)_{|\alpha} = \frac{q_{|\alpha}\langle q, q\rangle - q\langle q, q_{|\alpha}\rangle}{|q|^3} = \frac{[q, q_{|\alpha}]}{|q|^2}\left(\frac{q}{|q|}\right),$$

but $q(t) = z(t) - z(t + \alpha)$, and so

$$q_{|\alpha} = -z'(t + \alpha)\,.$$

Hence

$$\left(\frac{q}{|q|}\right)_{|\alpha} = -\frac{[q, z'(t+\alpha)]}{|q|^2} J\left(\frac{q}{|q|}\right) = -\frac{[q, z'(t+\alpha)]}{|q|^2} E^{it}$$

and

$$(iE^{it})_{|\alpha} = \left(-\frac{q}{|q|}\right)_{|\alpha} = +\frac{[q, z'(t+\alpha)]}{|q|^2} E^{it} \,,$$

yielding

$$(E^{it})_{\alpha} = J\left(\left(\frac{q}{|q|}\right)_{|\alpha}\right) = \frac{[q, z'(t+\alpha)]}{|q|^2} \frac{q}{|q|} = -\frac{[q, z'(t+\alpha)]}{|q|^2} iE^{it} \,.$$

Also we have

$$\frac{\partial P}{\partial \alpha} = -\left[z, \left(\frac{q}{|q|}\right)_{|\alpha}\right] = \frac{[q, z'(t+\alpha)]}{|q|^2} \left\langle z, \frac{q}{|q|} \right\rangle ,$$

where we used the formula

$$\langle w, v\rangle = [w,\, J(v)] \,.$$

Hence

$$\begin{aligned}
\left[\frac{\partial z_{\mathrm{iso}}}{\partial t}, \frac{\partial z_{\mathrm{iso}}}{\partial \alpha}\right] &= \left[KiE^{it}, \frac{[q, z'(t+\alpha)]}{|q|^2} \left(\left\langle z, \frac{q}{|q|} \right\rangle + P_{\mathrm{prim}}\right) E^{it}\right], \\
&= -K \frac{[q, z'(t+\alpha)]}{|q|^2} \left(\left\langle z, \frac{q}{|q|} \right\rangle + P_{\mathrm{prim}}\right),
\end{aligned}$$

but by formula (1) we get

$$\begin{aligned}
\left[\frac{\partial z_{\mathrm{iso}}}{\partial t}, \frac{\partial z_{\mathrm{iso}}}{\partial \alpha}\right] &= K \frac{[q, z'(t+\alpha)]}{|q|^2} \cdot \frac{[z', q]}{[q, q']} \cdot |q| \\
&= K \frac{[q, z'(t+\alpha)][z', q]}{|q|[q, q']} = \frac{K}{|q|} \cdot \frac{[q, q']}{|q|^2} \cdot t_1 \cdot t_2 > 0 \,.
\end{aligned}$$

Moreover, we have

$$\frac{\partial(F)}{\partial(t, \alpha)} = \frac{1}{\kappa_{\mathrm{iso}}} \cdot \frac{[q, z'(t+\alpha)][z', q]}{|q|^3} \,,$$

where $\kappa_{\mathrm{iso}}$ is the curvature of $z^{\alpha}_{\mathrm{iso}}(t)$.

Since

$$z'(t+\alpha) = R(t+\alpha)ie^{i(t+\alpha)}\,,\ z'(t) = R(t)ie^{it}\,,$$

then

$$\begin{aligned}\frac{\partial(F)}{\partial(t,\alpha)} &= \frac{R(t)\cdot R(t+\alpha)}{\kappa_{\text{iso}}}\cdot\frac{[q, ie^{it}(\cos\alpha + i\sin\alpha)][ie^{it}, q]}{|q|^3}\\ &= \frac{R\cdot R\alpha}{\kappa_{\text{iso}}}\cdot\frac{(-\mu)\cdot\lambda\cdot\sin^2\alpha}{|q|^3}\,,\end{aligned}$$

where $R = R(t)$ is the curvature radius of $C$ at $z(t), R_\alpha = R(t+\alpha)$ is the curvature radius of $C$ at $z(t+\alpha)$, $\lambda$ is the length of the segment $\overline{z_\alpha(t)z(t)}$ on the tangent line of $C$ at $z(t)$ (forming an angle $\pi-\alpha$ with the tangent line at $z(t+\alpha)$ and apex $z_\alpha(t)$), and $-\mu$ is then the length of the second tangential segment $\overline{z_\alpha(t)z(t+\alpha)}$ bounding this angle; see Fig. 5.

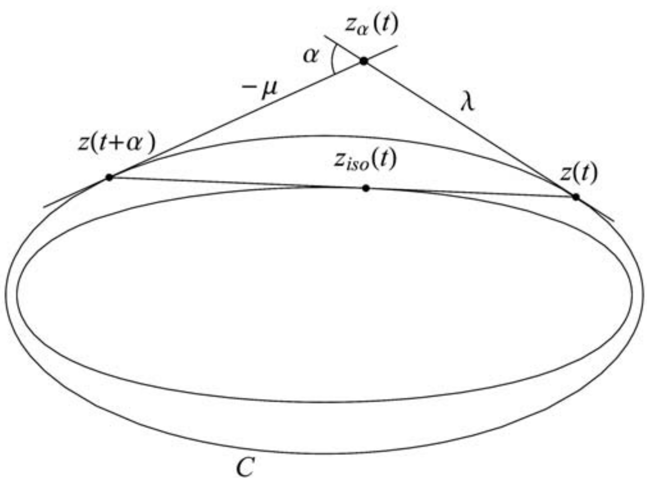


Fig. 5. Notations for $\lambda$ and $\mu$.

Recall the sine theorem for isoptics, namely

$$\frac{|q|}{\sin\alpha} = \frac{\lambda}{\sin\alpha_1} = \frac{-\mu}{\sin\alpha_2}\,;$$

see [1] and Fig. 6.

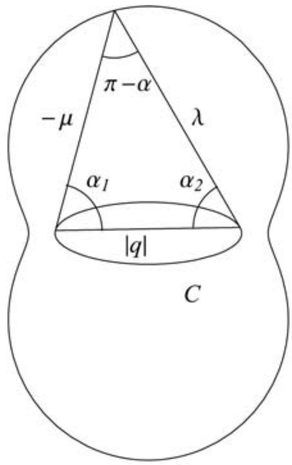


Fig. 6. Notations used for the sine theorem.

Hence, we have

$$\frac{\partial F}{\partial(t,\alpha)} = \frac{R\cdot R_\alpha}{\kappa_{\text{iso}}\cdot|q|}\cdot(-\mu)\cdot\lambda\cdot\frac{\sin\alpha}{|q|}\cdot\frac{\sin\alpha}{|q|}$$

$$(3)\qquad = \frac{R\cdot R_\alpha}{\kappa_{\text{iso}}\cdot|q|}\cdot\sin\alpha_1\cdot\sin\alpha_2$$

$$= \frac{1}{|q|}\cdot R\cdot R_\alpha\cdot R_{\text{iso}}\cdot\sin\alpha_1\cdot\sin\alpha_2\,.$$

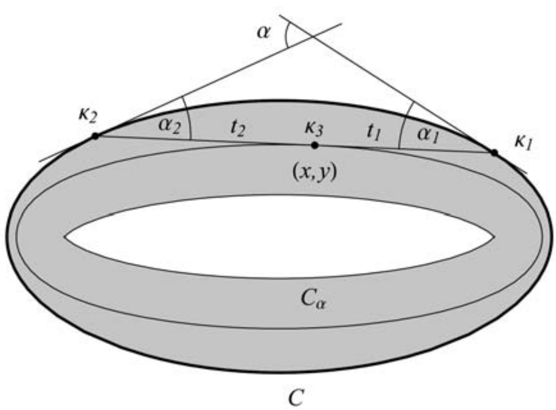


Fig. 7. Notations for $\alpha_1, \alpha_2, \kappa_1, \kappa_2, \kappa_3$.

Note that for a given point $(x, y)$ inside the curve $C$ it would be interesting to know the angle $\alpha$ generating an inner isoptic containing this point. We will consider this problem in a forthcoming paper, but we know already the following: in case of an ellipse one has to consider all chords passing trough this point, looking for $\alpha$ giving the maximum of the angle $\pi - \alpha$ between the tangents at the endpoints of the corresponding chord. We conjecture that this is true for all ovals.

Our derivations above yield the following theorem.

THEOREM. *We have*

$$\iint_{CC_{\alpha_0}} \frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3}{\sin \alpha_1 \cdot \sin \alpha_2} \cdot (t_1 + t_2) \cdot dxdy = 2\pi \cdot \alpha_0 \,,$$

*where $\kappa_i$ are curvatures at suitable points, the $\alpha_i$ present the adequate angles, and $t_i$ are determined as shown in* Fig. 7.

PROOF. Using the formula (3), we have

$$\iint_{CC_{\alpha_0}} \frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3}{\sin \alpha_1 \cdot \sin \alpha_2} dxdy$$

$$= \int_0^{2\pi} \int_0^{\alpha_0} \frac{\kappa(t) \cdot \kappa(t+\alpha)\kappa_{\text{iso}}(t)(t_1 + t_2)}{\sin \alpha_1 \cdot \sin \alpha_2} \cdot \frac{\sin \alpha_1 \cdot \sin \alpha_2}{\kappa(t) \cdot \kappa(t+\alpha) \cdot \kappa_{\text{iso}}(t) \cdot |q|} d\alpha dt$$

$$= \int_0^{2\pi} \int_0^{\alpha_0} d\alpha dt = 2\pi\alpha_0 \,,$$

since $|q| = t_1 + t_2$. □

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