

Tropical Open Hurwitz Numbers

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ABSTRACT - We give a tropical interpretation of Hurwitz numbers extending the one discovered in [CJM]. In addition we treat a generalization of Hurwitz numbers for surfaces with boundary which we call open Hurwitz numbers.

Hurwitz numbers are defined as the (weighted) number of ramified coverings of a compact closed oriented surface S of a given genus having a given set of critical values with given ramification profiles. These numbers have a long history, and have connections to many areas of mathematics, among which we can mention algebraic geometry, topology, combinatorics, and representation theory (see [LZ04] for example).

Here we define a slight generalization of these numbers that we call *open Hurwitz numbers*. To do so, we fix not only points on S and ramification profiles, but also a collection of disjoint circles on S and the behavior of the coverings above each of these circles. Note that the total space of the ramified coverings we consider now is allowed to have boundary components.

We also define tropical open Hurwitz numbers, and establish a correspondence with their complex counterpart. This can simply be seen as a

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translation in the tropical language of the computation of open Hurwitz numbers by cutting S along a collection of circles. A decomposition of S into pairs of pants reduces the problem to the enumeration of ramified coverings of the sphere S^2 with 3 critical values. In the particular case where all ramification points are simple, except maybe two of them, we recover the tropical computation of double Hurwitz numbers in [CJM].

This note is motivated by the forthcoming paper [BBM] where the computation of genus 0 characteristic numbers of $\mathbb{C}P^2$ is reduced to enumeration of floor diagrams and computation of genus 0 open Hurwitz numbers.

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1. Open Hurwitz numbers

The data we need to define open Hurwitz numbers are

- S an oriented connected closed compact surface;
- \mathcal{L} a finite collection of disjoint smoothly embedded circles in S ; we denote by \mathring{S} the surface $S \setminus (\bigcup_{L \in \mathcal{L}} L)$;
- \mathcal{P} be a finite collection of points in \mathring{S} ;
- a number $\delta(S') \in \mathbb{Z}_{\geq 0}$ associated to each connected component S' of \mathring{S} ; to each circle $L \in \mathcal{L}$ which is in the closure of the connected components S' and S'' of \mathring{S} (note that we may have $S' = S''$), we associate the number $\gamma(L) = |\delta(S') - \delta(S'')|$;
- a partition $\mu(p)$ of $\delta(S')$ associated to each point $p \in \mathcal{P}$, where S' is the connected component of \mathring{S} containing p ;
- a partition $\mu(L)$ of $\gamma(L)$ associated to each circle $L \in \mathcal{L}$.

In this note we identify two continuous maps $f : S_1 \rightarrow S$ and $f' : S'_1 \rightarrow S$ if there exists a homeomorphism $\Phi : S_1 \rightarrow S'_1$ such that $f' \circ \Phi = f$.

Now let us denote by \mathcal{S} the set of all (equivalence class of) ramified coverings $f : S_1 \rightarrow S$ where

- S_1 is a connected compact oriented surface with boundary;
- $f(\partial S_1) \subset \bigcup_{L \in \mathcal{L}} L$;
- f is unramified over $S \setminus \mathcal{P}$;
- $f|_{f^{-1}(S')}$ has degree $\delta(S')$ for each connected component S' of \mathring{S} ;
- for each point $p \in \mathcal{P}$, if $\mu(p) = (\lambda_1, \dots, \lambda_k)$, then $f^{-1}(p)$ contains exactly k points, denoted by q_1, \dots, q_k , and f has ramification index λ_i at q_i ;

- for each circle $L \in \mathcal{L}$, if $\mu(L) = (\lambda_1, \dots, \lambda_k)$, then $f^{-1}(L)$ contains exactly k boundary components of S_1 , denoted by c_1, \dots, c_k , and $f|_{c_i} : c_i \rightarrow L$ is an unramified covering of degree λ_i .

Note that the Riemann-Hurwitz formula gives us

$$\chi(S_1) = \sum_{S'} \delta(S')(\chi(S') - |\mathcal{P} \cap S'|) + \sum_{p \in \mathcal{P}} l(\mu(p))$$

where $l(\mu(p))$ is the length of the partition $\mu(p)$ (i.e. its cardinality as a multi-set of natural numbers).

DEFINITION 1.1. *The open Hurwitz number $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ is defined as*

$$H_S^\delta(\mathcal{L}, \mathcal{P}, \mu) = \sum_{f \in \mathcal{S}} \frac{1}{|Aut(f)|}$$

where $Aut(f)$ is the set of automorphisms of f .

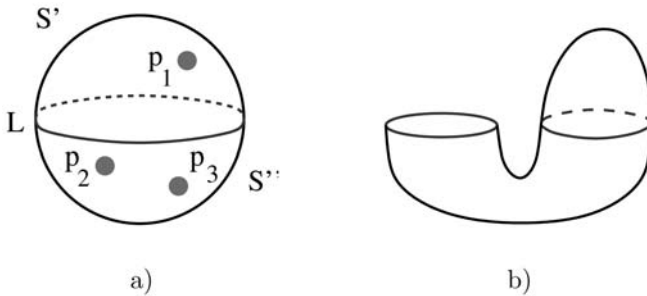


Fig. 1

EXAMPLE 1.2. Let S be the sphere, L be a circle in S , and p_1, p_2 , and p_3 three points distributed in S as depicted in figure 1a. Let us also denote by S' and S'' the two connected components of $S \setminus L$ according to figure 1a. We define $\mu(p_1) = \mu(p_2) = \mu(p_3) = (2)$. The table below lists some values of $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ easily computable by hand. Figure 1b depicts the only map to be taken into account in the second row of the table.

$\delta(S)$	$\delta(S')$	$\delta(S'')$	\mathcal{L}	$\mu(\mathcal{L})$	\mathcal{P}	$H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$
2			\emptyset		$\{p_2, p_3\}$	$\frac{1}{2}$
	1	2	$\{L\}$	(1)	$\{p_2, p_3\}$	1
	1	2	$\{L\}$	(1)	$\{p_1, p_3\}$	0
	0	2	$\{L\}$	(1, 1)	$\{p_2, p_3\}$	$\frac{1}{2}$

EXAMPLE 1.3. If $\mu(p) = (1, \dots, 1)$, then it is clear that $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu) = H_S^\delta(\mathcal{L}, \mathcal{P}', \mu)$ where $\mathcal{P}' = \mathcal{P} \setminus \{p\}$.

EXAMPLE 1.4. Suppose that $L \in \mathcal{L}$ bounds a disk D which contains no point of \mathcal{P} , and that $\mu(L) = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \neq 1$. Choose a point $p \in D$, define $\mathcal{L}' = \mathcal{L} \setminus \{L\}$, $\mathcal{P}' = \mathcal{P} \cup \{p\}$, and extend μ at the point p by $\mu(p) = (1, \dots, 1, \lambda_1, \dots, \lambda_k)$ where the number of 1 we add is equal to $\delta(D)$. Then $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu) = H_S^{\delta'}(\mathcal{L}', \mathcal{P}', \mu)$. Here δ' is obtained from δ by increasing it by $\gamma(L)$ over D .

We have to assume that $\lambda_i \neq 1$ for all i in $\{1, \dots, k\}$ to get this identity. Otherwise, new automorphisms of coverings might appear (e.g. the first two rows of example 1.2).

Note that the open Hurwitz number $H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$ is a topological invariant that depends only on the topological type of the triple $(S, \mathring{S}, \mathcal{P})$, and the functions δ and μ .

In the special case where \mathcal{L} is empty, we recover the usual Hurwitz numbers. In particular δ is just a positive integer number, the degree of the maps we are counting and that we denote by d . We simply denote Hurwitz numbers by $H_S^d(\mathcal{P}, \mu)$.

The problem of computing $H_S^d(\mathcal{P}, \mu)$ is equivalent to counting the number of some group morphisms from the fundamental group of a punctured surface to the symmetric group \mathfrak{S}_δ . Hence, Hurwitz numbers are theoretically computed by Frobenius's Formula (see for example [LZ04, Appendix, Theorems A.1.9 and A.1.10]).

EXAMPLE 1.5. If $\mathcal{P} = \{p_1, p_2\}$ is a set of two points on the sphere S^2 with $\mu(p_1) = \mu(p_2) = (d)$, then

$$H_{S^2}^d(\mathcal{P}, \mu) = \frac{1}{d}$$

EXAMPLE 1.6. If $\mathcal{P} = \{p_1, p_2, p_3\}$ is a set of three points on the sphere S^2 with $\mu(p_1) = (\lambda_1, \lambda_2)$, $\mu(p_2) = (d)$, and $\mu(p_3) = (2, 1, \dots, 1)$, then

$$H_{S^2}^d(\mathcal{P}, \mu) = \frac{1}{|\text{Aut}(\mu(p_1))|}$$

To end this section, let us mention the following nice closed formula due to Hurwitz.

PROPOSITION 1.7 (Hurwitz). *If $\mu(p) = (2, 1, \dots, 1)$ for all p in \mathcal{P} except for one point p_0 for which we have $\mu(p_0) = (\lambda_1, \dots, \lambda_k)$, then*

$$H_{S^2}^d(\mathcal{P}, \mu) = \frac{d^{k-3}(d+k-2)!}{|Aut(\mu(p_0))|} \prod_{i=1}^k \frac{\lambda_i^{\lambda_i}}{\lambda_i!}$$

2. Tropical open Hurwitz numbers

2.1 – Tropical curves with boundary

Given a finite graph C (i.e. C has a finite number of edges and vertices) we denote by $Vert(C)$ the set of its vertices, by $Vert^0(C)$ the set of its vertices which are not 1-valent, and by $Edge(C)$ the set of its edges.

DEFINITION 2.1. *An irreducible tropical curve C with boundary is a connected finite graph with $Edge(C) \neq \emptyset$ such that*

- $C \setminus Vert^\infty(C)$ is a complete metric graph for some set of 1-valent vertices $Vert^\infty(C)$ of C ;
- the vertices of $Vert^0(C)$ have non-negative integer weights, i.e. C is equipped with a map

$$\begin{array}{ccc} Vert^0(C) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ v & \longmapsto & g_v \end{array}$$

- any 2-valent vertex v of C satisfies $g_v \geq 1$.

If v is an element of $Vert^0(C)$, the integer g_v is called the genus of v . The genus of C is defined as

$$g(C) = b_1(C) + \sum_{v \in Vert^0(C)} g_v$$

where $b_1(C)$ denotes the first Betti number of C . When $g(C) = b_1(C)$, we say that the curve C is explicit.

A boundary component of C is a 1-valent vertex which not in $Vert^\infty(C)$. An element of $Vert^\infty$ is called a leaf of C , and its adjacent edge is called an end of C .

By definition, the leaves of C are at infinite distance from all the other points of C . A tropical curve without any boundary component is said to be closed. Our definition of tropical curves with boundary extends the defi-

nition of tropical curves with stops introduced by Nishinou in [Nis]. We denote by ∂C the set of the boundary components of C , and by $\text{Edge}^0(C)$ the set of its edges which are not adjacent to a 1-valent vertex.

Since $O_1(\mathbb{R}) = GL_1(\mathbb{Z})$, the data of the metric on C is equivalent to the data of a \mathbb{Z} -affine structure on each edge of C , i.e. the data of a lattice $M_p \simeq \mathbb{Z}$ in each tangent line of such an edge at the point p . In order to avoid unnecessary formal complications, we treat points on edges of a tropical curve as 2-valent vertices of genus 0 in the next definition.

DEFINITION 2.2. *A continuous map $h : C_1 \rightarrow C$ is a (non-proper) tropical morphism between the two tropical curves C_1 and C if*

- $h^{-1}(\partial C) \subset \partial C_1$;
- for any edge e of C_1 , the set $h(e)$ is contained either in an edge of C or in a vertex in $\text{Vert}^0(C)$; moreover the restriction $h|_e$ is a dilatation by some integer $w_{h,e} \geq 0$ (i.e. $dh_p(M_p) = w_{h,e}M_{h(p)}$ for any $p \in e$ in the first case, and $w_{h,e}$ is obviously 0 in the second case);
- for any vertex v in $\text{Vert}^0(C_1)$, if we denote by e_1, \dots, e_k the edges of C adjacent to $h(v)$, and by $e'_{i,1}, \dots, e'_{i,l_i}$ the edges of C_1 adjacent to v such that $h(e'_{i,j}) \subset e_i$, then one has the balancing condition

$$(1) \quad \forall i, j, \quad \sum_{l=1}^{l_i} w_{h,e'_{i,l}} = \sum_{l=1}^{l_j} w_{h,e'_{j,l}}$$

This number is called the local degree of h at v , and is denoted by $d_{h,v}$;

- for any vertex v in $\text{Vert}^0(C_1)$, if l (resp. k) denotes the number of edges e of C (resp. of C_1 with $w_{f,e} > 0$) adjacent to $h(v)$ (resp. to v) and $k > 0$ then one has the Riemann-Hurwitz condition

$$(2) \quad k - d_{h,v}(2g_{h(v)} + l - 2) + 2g_v - 2 \geq 0$$

This number is denoted by $r_{h,v}$.

The morphism h is called proper if $h^{-1}(\partial C) = \partial C_1$.

Note that the definition implies that $h^{-1}(\text{Vert}^\infty(C)) \subset \text{Vert}^\infty(C_1)$ for a tropical morphism $h : C_1 \rightarrow C$. The Riemann-Hurwitz condition in the previous definition comes from the classical Riemann-Hurwitz Theorem: if S_1 is a genus g_v oriented surface with k punctures, S is a genus $g_{h(v)}$ oriented surface with l punctures, and $f : S_1 \rightarrow S$ is a ramified covering of degree $d_{h,v}$, then the left hand side of inequality (2) is the sum of the ramification index of all points of S_1 . In particular, it is non-negative.

The integer $w_{h,e}$ is called the *weight of the edge e with respect to h* . When no confusion is possible, we will speak about the weight of an edge, without referring to the morphism h . If $w_{h,e} = 0$, we say that the morphism h *contracts* the edge e . The morphism h is called *minimal* if $h^{-1}(\text{Vert}^\infty(C)) = \text{Vert}^\infty(C_1)$, i.e. h does not contract any end.

Two tropical morphisms $h : C_1 \rightarrow C$ and $h' : C_2 \rightarrow C$ are said to be of the same *combinatorial type* if there exists a homeomorphism of graphs $\phi : C_1 \rightarrow C_2$ (i.e. we forget about the metric on C_1 and C_2) such that $h = h' \circ \phi$, $g(v) = g(\phi(v))$ for any vertex v of C_1 and $w_{h,e} = w_{h',\phi(e)}$ for all $e \in \text{Edge}(C_1)$.

EXAMPLE 2.3. We depicted in figure 2a a tropical morphism from a rational tropical curve with five leaves and one boundary component to a rational curve with four ends. Three edges have weight 2 with respect to h . In the picture of a tropical morphism $h : C_1 \rightarrow C$, we do not precise the lengths of edges of C_1 and C since the length of edges of C and the weights of edges of C_1 determine the length of edges of C_1 .

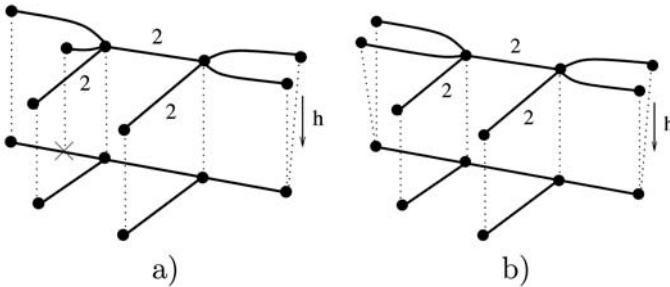


Fig. 2. Representation of two tropical morphisms.

The sum of all local degrees of elements in $h^{-1}(v)$ is a locally constant function on $C \setminus h(\partial C_1)$; if C' is a connected component of $C \setminus h(\partial C_1)$ then this sum over a point of C' is called the *degree* of h over C' .

Note that for the morphism from Figure 2a we have two connected components of $C \setminus h(\partial C_1)$ as the boundary of C_1 consists of a single point (the only 1-valent vertex whose image is inside an edge of C). The degree over the components of $C \setminus h(\partial C_1)$ are 1 and 2.

DEFINITION 2.4. Let $h : C_1 \rightarrow C$ be a tropical morphism.

A subset E of C_1 such that $h(E)$ is a point of C is called a *ramification component* of h if E is a connected component of $h^{-1}(h(E))$, and contains

either an edge in $\text{Edge}^0(C_1)$, or a vertex $v \in \text{Vert}^0(C_1)$ with $r_{h,v} > 0$, or a vertex $v \in \text{Vert}^\infty(C_1)$ adjacent to an end e with $w_{h,e} > 1$ (note that in such case $h(e)$ cannot be a point as otherwise $w_{h,e} = 0$).

If $p \in C$ is such that $h^{-1}(p)$ does not contain any ramification component of h , we say that h is unramified over p .

Let $v = (\lambda_1, \dots, \lambda_l)$ be an unordered l -tuple of positive integer numbers. We say that the map h has ramification profile v over the leaf v of C if $h^{-1}(v) = \{v_1, \dots, v_l\}$ where v_i is a leaf of C_1 adjacent to an end of weight λ_i .

As in section 1, we identify two tropical morphisms $h : C_1 \rightarrow C$ and $h' : C'_1 \rightarrow C$ if there exist two tropical isomorphisms $\Phi : C_1 \rightarrow C'_1$ and $\phi : C \rightarrow C$ such that ϕ restricts to the identity map on \mathcal{Q} and \mathcal{R} , and $h' \circ \Phi = \phi \circ h$. An automorphism ϕ of a tropical morphism $h : C_1 \rightarrow C$ is a tropical isomorphism $\phi : C_1 \rightarrow C_1$ such that $h \circ \phi = h$.

2.2 – Definition of tropical open Hurwitz numbers

Similarly to section 1, we start with the following data

- C a closed explicit tropical curve with $\text{Vert}^0(C) \neq \emptyset$;
- \mathcal{R} a finite collection of points in $\overset{\circ}{C} \setminus \text{Vert}(C)$ such that any connected component of the set $C \setminus \mathcal{R}$, denoted $\overset{\circ}{C}$, contains a vertex of C ;
- \mathcal{Q} be a finite collection of points in $\text{Vert}^\infty(C)$;
- a number $\delta(C') \in \mathbb{Z}_{\geq 0}$ associated to each connected component C' of $\overset{\circ}{C}$; to each point $q \in \mathcal{R}$ which is in the closure of the connected components C' and C'' of $\overset{\circ}{C}$, we associate the number $\gamma(q) = |\delta(C') - \delta(C'')|$;
- a partition $\nu(q)$ of $\delta(\overset{\circ}{C})$ associated to each point $q \in \mathcal{Q}$, where C' is the connected component of $\overset{\circ}{C}$ containing q ;
- a partition $\nu(q)$ of $\gamma(q)$ associated to each point $q \in \mathcal{R}$.

We denote by \mathcal{S}^\top the set of all minimal tropical morphisms $h : C_1 \rightarrow C$ such that

- C_1 is a tropical curve with boundary;
- $h(\partial C_1) \subset \mathcal{R}$;
- h is unramified over $C \setminus \mathcal{Q}$;
- $h|_{h^{-1}(C')}$ has degree $\delta(C')$ for each connected component C' of $\overset{\circ}{C}$;
- for each point $q \in \mathcal{Q}$, the map h has ramification profile $\nu(q)$ over q ;
- for each point $q \in \mathcal{R}$, if $\nu(q) = (\lambda_1, \dots, \lambda_k)$, the set $h^{-1}(q)$ contains exactly k boundary components of C_1 , denoted by c_1, \dots, c_k , and c_i is adjacent to an edge of C_1 of weight λ_i .

Note that the fact that h is minimal and unramified over $C \setminus \mathcal{Q}$ implies that $h(\text{Vert}^0(C_1)) \subset \text{Vert}^0(C)$ and that h does not contract any edge of C_1 . In particular the set \mathcal{S}^\top is finite. Moreover the length of edges of C_1 are completely determined by the combinatorial type of h . In other words, the following lemma holds.

LEMMA 2.5. *Two distinct elements of \mathcal{S}^\top have distinct combinatorial types.*

As usual in tropical geometry, a tropical morphism $h : C_1 \rightarrow C$ in \mathcal{S}^\top should be counted with some multiplicity. Given v a vertex in $\text{Vert}^0(C_1)$ such that $h(v)$ is adjacent to the edges e_1, \dots, e_{k_v} of C , we choose a configuration $\mathcal{P}' = \{p'_1, \dots, p'_{k_v}\}$ of k_v points on the sphere S^2 , and we define $\mu'(p'_i)$ as the partition of $d_{h,v}$ defined by h at v above the edge e_i (cf the balancing condition (1)).

DEFINITION 2.6. *The multiplicity of $h : C_1 \rightarrow C$ is defined as*

$$m(h) = \frac{1}{|\text{Aut}(h)|} \prod_{e \in \text{Edge}^0(C_1)} w_{h,e} \prod_{v \in \text{Vert}^0(C_1)} \left(\prod_{i=1}^{k_v} |\text{Aut}(\mu'(p'_i))| \right) H_{S^2}^{d_{h,v}}(\mathcal{P}', \mu')$$

The tropical open Hurwitz number $\mathbb{T}H_C^\delta(\mathcal{R}, \mathcal{Q}, \nu)$ is defined as

$$\mathbb{T}H_C^\delta(\mathcal{R}, \mathcal{Q}, \nu) = \sum_{h \in \mathcal{S}^\top} m(h)$$

As in section 1, if $\mathcal{R} = \emptyset$ then δ is a number denoted by d , and we denote by $\mathbb{T}H_C^d(\mathcal{Q}, \nu)$ the corresponding tropical (closed) Hurwitz number.

EXAMPLE 2.7. Let $h : C_1 \rightarrow C$ be the tropical morphism depicted in figure 2a. It is the tropical analog of the map considered in figure 1. Let q_1 be the image of the boundary component of C_1 , and q_2 and q_3 be the leaves of C which are image of a leaf of C_1 adjacent to an edge of weight 2. We denote by C' (resp. C'') the connected component of $C \setminus \{q_1\}$ which does not contain (resp. contains) q_2 and q_3 , and we define $\delta(C') = 1$, $\delta(C'') = 2$, $\nu(q_1) = (1)$, and $\nu(q_2) = \nu(q_3) = (2)$. To compute $\mathbb{T}H_C^\delta(\mathcal{R}, \mathcal{Q}, \nu)$, the morphism of figure 2a is the only one to consider and it has multiplicity 1 so $\mathbb{T}H_C^\delta(\mathcal{R}, \mathcal{Q}, \nu) = 1$ (see the second row of the table in example 1.2).

EXAMPLE 2.8. Let C be a closed rational curve with four leaves. We set $\mathcal{Q} = \text{Vert}^\infty(C)$ and $\nu(q) = (3)$ for $q \in \mathcal{Q}$. Then according to figure 3, we have $\mathbb{T}H_C^3(\mathcal{Q}, \nu) = 1$.

Indeed, for the first morphism of Figure 3 we have the (classical) Hurwitz numbers at the inner vertices equal to $\frac{1}{3}$ each. In the same time the group of local automorphism at each vertex is the symmetric group \mathfrak{S}_3 , so its order is 6. Finally the group of automorphisms of the morphism itself is also \mathfrak{S}_3 . Using Definition 2.6 we get $\frac{1}{6} \left(\frac{6}{3}\right)^2 = \frac{2}{3}$ as the multiplicity.

For the second morphism we have local Hurwitz numbers at the inner vertices equal to $\frac{1}{3}$ again. There are no local or global automorphisms, but there is an inner edge of weight 3. Thus we get $3 \left(\frac{1}{3}\right)^2 = \frac{1}{3}$ as the multiplicity.

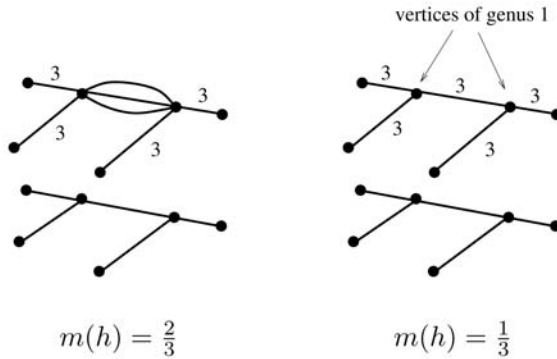


Fig. 3

EXAMPLE 2.9. Let C be a genus 2 explicit tropical curve with two leaves and whose combinatorial type is as in figure 4. We set $\mathcal{Q} = \text{Vert}^\infty(C)$ and $\nu(q) = (2)$ for $q \in \mathcal{Q}$. Then according to figure 4, we have $\mathbb{T}H_C^2(\mathcal{Q}, \nu) = 8$.

EXAMPLE 2.10. Suppose that $q \in \mathcal{R}$ is on an end with an adjacent leaf q' not in \mathcal{Q} , and that $\nu(q) = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \neq 1$. Define $\mathcal{R}' = \mathcal{R} \setminus \{q\}$, $\mathcal{Q}' = \mathcal{Q} \cup \{q'\}$, and $\nu(q') = (1, \dots, 1, \lambda_1, \dots, \lambda_k)$ where the number of 1 we add is equal to $\delta((qq'))$. Then $H_\delta^\mathbb{T}(C, \mathcal{R}, \mathcal{Q}, \nu) = H_\delta^\mathbb{T}(C, \mathcal{R}', \mathcal{Q}', \nu)$, where δ' is obtained from δ by increasing it by $\gamma(q)$ over (qq') . This is the tropical counterpart of the identity described in example 1.4. Note that as in example 1.4, we have to assume that $\lambda_i \neq 1$ for all i in $\{1, \dots, k\}$ otherwise new automorphisms of coverings might appear (e.g. Figures 2a and b which are the tropical analogs of the morphisms corresponding to the first two rows of example 1.2).

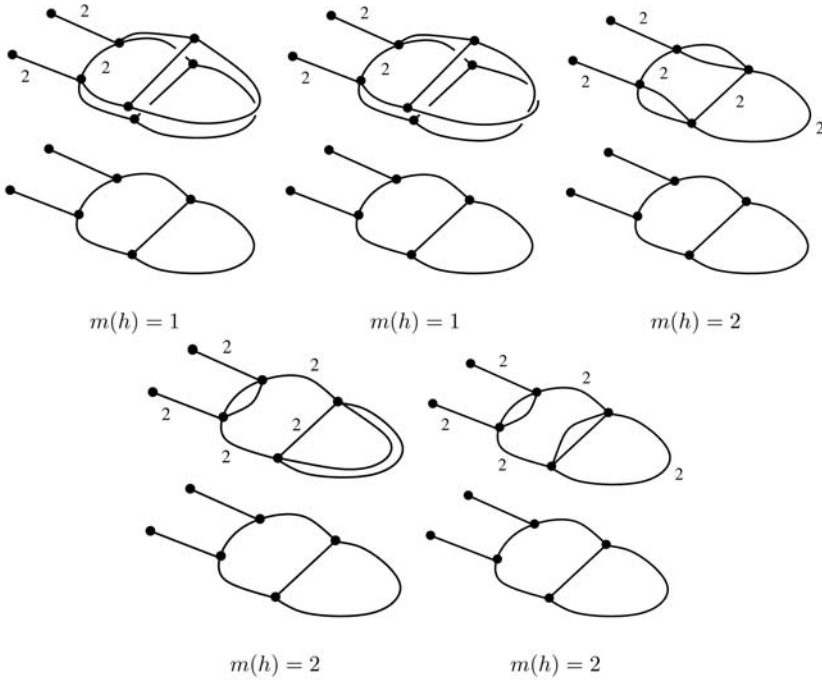


Fig. 4

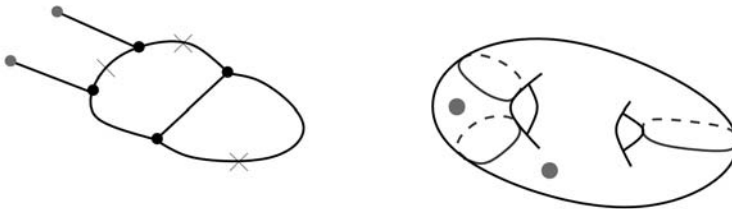


Fig. 5. A tropical curve C is depicted on the left and the corresponding surface S on the right. The two leaves of C are elements of \mathcal{Q} and correspond to elements of \mathcal{P} (depicted by dots) on S while crosses on C represent points of \mathcal{R} and correspond to the circles of \mathcal{L} pictured on S .

Let us relate these tropical open Hurwitz numbers to the open Hurwitz numbers we defined in section 1. Let C be a tropical curve as in definition 2.6 with the data introduced at the beginning of this subsection. Let S be an oriented connected compact closed surface whose genus is the genus of C . We choose a collection $\mathcal{L} = \{L_q\}_{q \in \mathcal{R}}$ of disjoint smoothly embedded

circles in S such that there is a natural correspondence $C' \rightarrow S'_{C'}$ between the connected components of C and S which preserves incidence relations and such that $b_1(C') = g(C'_S)$. For each point $q \in \mathcal{Q}$, we choose a point $p_q \in S'_{C'}$ where C' is the connected component of C containing q , such that $p_q \neq p_{q'}$ for $q \neq q'$ (see figure 5 for an example). Finally we define $\mathcal{P} = \cup_{q \in \mathcal{Q}} \{p_q\}$, $\delta(S'_{C'}) = \delta(C')$, $\mu(L_q) = v(q)$, and $\mu(p_q) = v(q)$.

THEOREM 2.11. *For any $\delta, \mathcal{R}, \mathcal{Q}$, and v , one has*

$$\mathbb{T}H_C^\delta(\mathcal{R}, \mathcal{Q}, v) = H_S^\delta(\mathcal{L}, \mathcal{P}, \mu)$$

PROOF. The proof is quite straightforward: there exists a natural surjection $\phi : S \rightarrow S^\top$, and for any element $h : C_1 \rightarrow C$ we have

$$m(h) = \sum_{f \in \phi^{-1}(h)} \frac{1}{|Aut(f)|}$$

Note that according to examples 1.4 and 2.10, we may suppose that $\text{Vert}^\infty(C) = \mathcal{Q} = \mathcal{P} = \emptyset$.

Let us construct the map ϕ . For this we choose a configuration \mathcal{R}' of points lying in the edges of C such that each edge in $\text{Edge}^0(C)$ contains exactly one point of $\mathcal{R} \cup \mathcal{R}'$, and we define $C_0 = C \setminus (\mathcal{R} \cup \mathcal{R}')$. For each point q in \mathcal{R}' , we choose a smoothly embedded circle L_q in S such that $L_q \cap L_{q'} = \emptyset$ if $q \neq q' \in \mathcal{R} \cup \mathcal{R}'$, such that there is a natural correspondence $C' \rightarrow S'_{C'}$ between the connected components of C_0 and $\mathring{S}_0 = S \setminus (\mathcal{L} \cup_{q \in \mathcal{R}'} L_q)$ which preserves incidence relations, and such that all connected components of \mathring{S}_0 have genus 0. Note that C is a realisation of the dual graph of the decomposition of S induced by $(\mathcal{L} \cup_{q \in \mathcal{R}'} L_q)$, and that a connected component of \mathring{S}_0 cannot be a disk since $\text{Vert}^\infty(C) = \emptyset$.

Let $f : S_1 \rightarrow S$ be an element of \mathcal{S} , and let us denote $\mathring{S}_1 = f^{-1}(\mathring{S}_0)$. We construct a graph C_1 in the following way

- to each connected component S'_1 of \mathring{S}_1 corresponds a vertex $v_{S'_1}$ in $\text{Vert}^0(C_1)$; we set $g_{v_{S'_1}}$ to be equal to the genus of S'_1 ;
- to each circle L in $f^{-1}(\cup_{q \in \mathcal{R} \cup \mathcal{R}'} L_q) \setminus \partial S_1$ adjacent to the connected components S'_1 and S''_1 of \mathring{S}_1 corresponds an edge e_L in $\text{Edge}^0(C_1)$ joining $v_{S'_1}$ and $v_{S''_1}$;
- to each connected component L of ∂S_1 adjacent to the connected component S'_1 of \mathring{S}_1 correspond a boundary point v_L in ∂C_1 and an edge e_L joining v_L to $v_{S'_1}$.

For each circle L in $f^{-1}(\bigcup_{q \in \mathcal{R} \cup \mathcal{R}'} L_q)$, we denote by $w(e_L)$ the degree of the unramified covering $f|_L : L \rightarrow f(L)$. There exists a unique tropical structure on C_1 and a unique tropical morphism $h : C_1 \rightarrow C$ such that

$$h(v_{S'_1}) = v \Leftrightarrow f(S'_1) = S'_v, \quad h(e_L) \subset e \Leftrightarrow f(L) = L_q \text{ where } q = (\mathcal{R} \cup \mathcal{R}') \cap e,$$

and

$$w_{h,e_L} = w(e_L) \quad \forall e_L \in \text{Edge}(C_1)$$

We define $\phi(f) = h$.

Now it remains us to prove that $m(h) = \sum_{f \in \phi^{-1}(h)} \frac{1}{|Aut(f)|}$ for any $h : C_1 \rightarrow C$ in S^T . For this, we reconstruct elements in $\phi^{-1}(h)$ step by step, taking care of automorphisms.

The tropical curve C_1 induces the structure of a reducible tropical curve on the topological closure \mathring{C}_1 of $C_1 \setminus h^{-1}(\mathcal{R} \cup \mathcal{R}')$. Note that to any vertex v in $\text{Vert}^0(C_1)$ corresponds a connected component of \mathring{C}_1 . The tropical morphism $h : C_1 \rightarrow C$ induces a reducible tropical morphism $\mathring{h} : \mathring{C}_1 \rightarrow C$. For each vertex v in $\text{Vert}^0(C_1)$ adjacent to the edges e_1, \dots, e_{k_v} , we choose an unramified covering $f_v : S'_{1,v} \rightarrow S'_{f(v)}$ of degree $d_{h,v}$ where $S'_{1,v}$ is a surface with boundary components L_1, \dots, L_{k_v} of genus g_v , and $f_v|_{L_i}$ is a degree w_{f,e_i} unramified covering from L_i to L_q where $q = (\mathcal{R} \cup \mathcal{R}') \cap e_i$. Let us set $\mathring{S}_1 = \bigcup_{v \in \text{Vert}^0(C_1)} S'_{1,v}$, and let $\mathring{f} : \mathring{S}_1 \rightarrow S$ be the map defined by $\mathring{f}|_{S'_{1,v}} = f_v$ for any v in $\text{Vert}^0(C_1)$. The union of all maps \mathring{f} constructed in this way (up to equivalence) is a finite set A_0 , and we clearly have (in the notation of Definition 2.6)

$$\sum_{\mathring{f} \in A_0} \frac{1}{|Aut(\mathring{f})|} = \frac{1}{|Aut(\mathring{h})|} \prod_{v \in \text{Vert}^0(C_1)} \left(\prod_{i=1}^{k_v} |Aut(\mu'(p'_i))| \right) H_{S^2}^{d_{h,v}}(\mathcal{P}', \mu').$$

To get an element $f : S_1 \rightarrow S$ of S , it remains to glue all the coverings f_v according to the edges in $\text{Edge}^0(C_1)$. Suppose we already have performed these gluings according to s edges in $\text{Edge}^0(C_1)$ and obtained a set A_s of coverings of S , and that we want now to glue elements of A_s along the edge e in $\text{Edge}^0(C_1)$. Since we can identify in exactly d different ways two degree d unramified coverings of the circle by the circle, we have $w_{h,e}$ different ways to perform this gluing given an element $\tilde{f} : \tilde{S}_1 \rightarrow S$ of A_s . We denote by $\tilde{f}_1, \dots, \tilde{f}_{w_{h,e}}$ the coverings of S constructed in this way. Let $i = 1, \dots, w_{h,e}$. Any automorphism $\psi \in Aut(\tilde{f})$ fixing globally the two boundary components of \tilde{S}_1 corre-

sponding to e extends to an equivalence of coverings $\psi_i : \tilde{f}_i \rightarrow \tilde{f}_j$ for some j . The homeomorphism ψ_i is in $Aut(\tilde{f}_i)$ if $i = j$, and identifies the two coverings \tilde{f}_i et \tilde{f}_j otherwise. Hence at the end of this gluing procedure, we obtain a set $A_{|Edge^0(C_1)|}$ and we have

$$\sum_{f \in A_{|Edge^0(C_1)|}} \frac{1}{|Aut(f)|} = m(h).$$

According to Lemma 2.5, the set $A_{|Edge^0(C_1)|}$ is exactly $\phi^{-1}(h)$ so the Theorem is proved. □

We can allow points with ramification profile $\nu(q) = (2, 1, \dots, 1)$ in $C \setminus Vert^\infty(C)$, and recover in this way results from [CJM]. Suppose that C is trivalent, and that there exists $q \in \mathcal{Q}$ with $\nu(q) = (2, 1, \dots, 1)$. Let e be the end of C adjacent to q , and v be the other vertex adjacent to e . We denote by e_1, \dots, e_k the edges of C_1 in $h^{-1}(e)$, and by $\{v_1, \dots, v_l\} = h^{-1}(v)$. By assumption on $\nu(q)$, one of the edges e_j , say e_1 , has weight 2 while the other edges e_j have weight 1. Suppose that e_1 is adjacent to v_1 . The Riemann-Hurwitz condition implies that all vertices v_j are $(d_{h,v_j} + 2)$ -valent vertices of C_1 , so the vertex v_1 (resp. $v_j, j \geq 2$) has exactly 3 (resp. 2) adjacent edges not mapped to e . Now according to Examples 1.3, 1.5, and 1.6 we have

$$\prod_{j=1}^l \left(\prod_{i=1}^3 |Aut(\mu'(p'_i))| \right) \mathbb{T}H_C^{d_{h,v_j}}(\mathcal{P}, \mu') = (d_{h,v_1} - 2)! \prod_{j=2}^l \frac{d_{h,v_j}!}{d_{h,v_j}}$$

The contraction $\pi : C \rightarrow C_0$ of e induces a contraction $\pi_1 : C_1 \rightarrow C_2$ and a tropical morphism $h' : C_2 \rightarrow C_0$ such that $\pi \circ h = h' \circ \pi_1$. Moreover, all points in $h'^{-1}(\pi(q))$ lie in the interior of edges of C_2 , except one point which is a trivalent vertex of C_2 (see figure 6). We also have

$$|Aut(h)| = |Aut(h')|(d_{h,v_1} - 2)! \prod_{j=2}^l d_{h,v_j}!$$

So if C is rational, $\mathcal{L} = \emptyset$, and all points $p \in \mathcal{P}$ satisfy $\nu(q) = (2, 1, \dots, 1)$ except maybe two of them, by contracting all the points $q \in \mathcal{Q}$ with $\nu(q) = (2, 1, \dots, 1)$ we find ourselves in the situation discussed in [CJM]. In particular, we recover the same multiplicities in all cases except for example 1.5 (in this special case, the tropical computation from [CJM] is wrong).

EXAMPLE 2.12. The result of this sequence of contractions in the case of the tropical morphism depicted in figure 2 a) is depicted in figure 6.

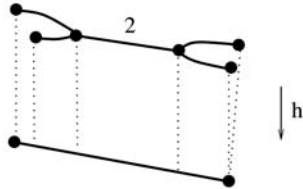


Fig. 6

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