Lasry-Lions Regularization and a Lemma of Ilmanen

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Let $H$ be a Hilbert space. We define the following $\inf$ ($\sup$) convolution operators acting on bounded functions $u : H \to \mathbb{R}$:

$$T_t u(x) := \inf_y \left( u(y) + \frac{1}{t} \|y - x\|^2 \right)$$

and

$$\tilde{T}_t u(x) := \sup_y \left( u(y) - \frac{1}{t} \|y - x\|^2 \right).$$

We have the relation

$$T_t (\inf u) = -T_t (\sup u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \tilde{T}_t \circ \tilde{T}_s = \tilde{T}_{t+s}$$

for all $t \geq 0$ and $s \geq 0$, as can be checked by direct calculation. Note also that

$$\inf u \leq T_t u(x) \leq u(x) \leq \tilde{T}_t u(x) \leq \sup u$$

for each $t \geq 0$ and each $x \in H$. A function $u : H \to \mathbb{R}$ is called $k$-semi-concave, $k > 0$, if the function $x \mapsto u(x) - \|x\|^2/k$ is concave. We will occasionally consider semi-concave functions which take values in $[-\infty, +\infty)$. The function $u$ is called $k$-semi-convex if $-u$ is $k$-semi-concave. A function $u$ is $t$-semi-concave and upper semi-continuous if and only if it belongs to the image of the operator $T_t$, this follows from Lemma 1 and Lemma 3 below. A function $u$ is called semi-concave if it is $k$-semi-concave for some $k > 0$. A function $u$ is said $C^{1,1}$ if it is Frechet differentiable and if the gradient of $u$ is Lipschitz. Note that a continuous function $u : H \to \mathbb{R}$ is $C^{1,1}$.

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if and only if it is semi-concave and semi-convex, see Lemma 5. Let us recall two important results in that language:

**Theorem 1** (Lasry-Lions, [6]). Let $u$ be a bounded function. For $0 < s < t$, the function $T_s \circ T_t u$ is $C^{1,1}$ and, if $u$ is uniformly continuous, then it converges uniformly to $u$ when $t \to 0$.

**Theorem 2** (Imanen, [5]). Let $u \geq v$ be two bounded functions on $H$ such that $u$ and $-v$ are semi-concave. Then there exists a $C^{1,1}$ function $w$ such that $u \geq w \geq v$.

Our goal in the present paper is to “generalize” simultaneously both of these results as follows:

**Theorem 3.** The operator $R_t := T_t \circ T_{2t} \circ T_t$ has the following properties:

- Regularization: For each function $f : H \to \mathbb{R}$ and each $t > 0$, the function $R_t(f)$ is $C^{1,1}$ provided it is locally bounded. This holds for all $t > 0$ if $f$ is bounded.
- Approximation: If $f : H \to \mathbb{R}$ is uniformly continuous, then $R_t(f)$ is $C^{1,1}$ and converges uniformly to $f$ as $t \to 0$.
- Pinching: If there exists a $k$-semi-concave continuous function $u$ and a $k$-semi-convex continuous function $v$ such that $v \leq f \leq u$, then, for all $t \in [0,k]$, we have $u \geq R_t(f) \geq v$, and $R_t(f)$ is $C^{1,1}$.

Theorem 3 does not, properly speaking, generalize Theorem 5. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by $C^{1,1}$ functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator $R_t$ enjoys some nicer properties than the Lasry-Lions operators. For example, if $f$ is $C^{1,1}$, then it follows from the pinching property that $R_t f = f$ for $t$ small enough.

Theorem 2, can be proved using Theorem 3 by taking $w = R_t u$, for $t$ small enough. Note, in view of Lemma 3 bellow, that $R_t u = T_t \circ T_{t} u$ when $t$ is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of losing the simplicity of explicit expressions. Let $M$ be a paracompact manifold of dimension $n$, equipped once and for all with an atlas $(\phi_i, i \in \mathcal{A})$ composed of
charts \( \phi_i : B^n \to M \), where \( B^n \) is the open unit ball of radius one centered at the origin in \( \mathbb{R}^n \). We assume in addition that the image \( \phi_i(B^n) \) is a relatively compact open set, and that the sets \( \phi_i(B^n), i \in \mathcal{I} \) form a locally finite open covering of the manifold \( M \). Let us fix, once and for all, a partition of the unity \( g_i \) subordinated to the open covering \( \{ \phi_i(B^n), i \in \mathcal{I} \} \). It means that the function \( g_i \) is non-negative with support inside \( \phi_i(B^n) \), and that \( \sum g_i = 1 \) (note that this sum is finite at each point). Let us define the operator

\[
G_t(u) := \sum_i \left[ R_{i_0}((g_i u) \circ \phi_i) \right] \circ \phi_i^{-1},
\]

where \( a_i, i \in \mathcal{I} \) are positive real numbers. In this expression, we consider each of the terms \( R_{i_0}((g_i u) \circ \phi_i) \circ \phi_i^{-1} \) as defined on the whole manifold \( M \) with the value 0 outside of \( \phi_i(B^n) \). The sum is then locally finite hence well-defined. We say that a function \( u : M \to \mathbb{R} \) is locally semi-concave if, for each \( i \in \mathcal{I} \), there exists a constant \( b_i \) such that the function \( u \circ \phi_i - ||\cdot||^2 / b_i \) is concave on \( B^n \).

**Theorem 4.** Let \( u \geq v \) be two continuous functions on \( M \) such that \( u \) and \( -v \) are locally semi-concave. Then, the real numbers \( a_i \), can be chosen such that, for each \( t \in [0, 1] \) and each function \( f \) satisfying \( u \geq f \geq v \), we have:

- The function \( G_t(f) \) is locally \( C^{1,1} \).
- If \( f \) is continuous, then \( G_t(f) \) converges locally uniformly to \( f \) as \( t \to 0 \).
- \( u \geq G_t(f) \geq v \).

We will give some properties, most of which are well-known, of the operators \( T_t \) and \( T_i \) in Section 1, and derive the proof of the main results in Section 2.

**Notes and Acknowledgements.** Theorem 2 appears in Ilmanen’s paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Cardialaguget, Albert Fathi and Maxime Zavidovique. Albert Fathi and Maxime Zavidovique also recently wrote a detailed proof of
Theorem 1, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of $C^{1,1}$ subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, it seems that the main result of [1] (the existence of $C^{1,1}$ subsolutions in the continuous case) can’t be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

1. The operators $T_t$ and $\bar{T}_t$ on Hilbert spaces.

The proofs of the theorems follow from standard properties of the operators $T_t$ and $\bar{T}_t$ that we now recall in details.

**Lemma 1.** For each function $u : H \to \mathbb{R}$, the function $T_t u$ (which takes values in $[-\infty, +\infty]$), is $t$-semi-concave and upper semi-continuous. The function $\bar{T}_t u$ (which takes values in $(-\infty, +\infty]$), is $t$-semi-convex and lower semi-continuous. Moreover, if $u$ is $k$-semi-concave, then for each $t < k$ the function $\bar{T}_t u$ is $(k - t)$-semi-concave. Similarly, if $u$ is $k$-semi-convex, then for each $t < k$ the function $T_t u$ is $(k - t)$-semi-convex.

**Proof.** We shall prove the statements concerning $T_t$. We have

$$T_t u(x) - \|x\|^2/t = \inf_y (u(y) + \|y - x\|^2/t - \|x\|^2/t) = \inf_y (u(y) + \|y\|^2/t - 2x \cdot y/t),$$

this function is concave and upper semi-continuous as an infimum of continuous linear functions. On the other hand, we have

$$T_t u(x) + \|x\|^2/l = \inf_y (u(y) + \|y - x\|^2/t + \|x\|^2/l).$$

Setting $f(x, y) := u(y) + \|y - x\|^2/t + \|x\|^2/l$, the function $\inf_y f(x, y)$ is a convex function of $x$ if $f$ is a convex function of $(x, y)$. This is true if $u$ is $k$-semi-convex, $t < k$, and $l = k - t$ because we have the expression

$$f(x, y) = u(y) + \|y - x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \sqrt{\frac{l}{kt}} - \sqrt{\frac{k}{l}}.$$

$\square$
Given a uniformly continuous function \( u : H \rightarrow \mathbb{R} \), we define its modulus of continuity \( \rho(r) : [0, \infty) \rightarrow [0, \infty) \) by the expression
\[
\rho(r) = \sup_{x \in H} u(x + re) - u(x),
\]
where the supremum is taken on all \( x \in H \) and all \( e \) in the unit ball of \( H \). The function \( \rho \) is non-decreasing, it satisfies \( \rho(r + r') \leq \rho(r) + \rho(r') \), and it converges to zero in zero (this last fact is equivalent to the uniform continuity of \( u \)). We say that a function \( \rho : [0, \infty) \rightarrow [0, \infty) \) is a modulus of continuity if it satisfies these properties. Given a modulus of continuity \( \rho(r) \), we say that a function \( u \) is \( \rho \)-continuous if \( |u(y) - u(x)| \leq \rho(\|y - x\|) \) for all \( x \) and \( y \) in \( H \).

**Lemma 2.** If \( u : H \rightarrow \mathbb{R} \) is uniformly continuous, then the functions \( T_t u \) and \( T_{1t} u \) converge uniformly to \( u \) when \( t \rightarrow 0 \). Moreover, given a modulus of continuity \( \rho \), there exists a non-decreasing function \( \epsilon(t) : [0, \infty) \rightarrow [0, \infty) \) satisfying \( \lim_{t \to 0} \epsilon(t) = 0 \) and such that, for each \( \rho \)-continuous bounded function \( u \), we have:

- \( T_t u \) and \( T_{1t} u \) are \( \rho \)-continuous for each \( t \geq 0 \).
- \( u - \epsilon(t) \leq T_t u(x) \leq u \) and \( u \leq T_{1t} u \leq u + \epsilon(t) \) for each \( t \geq 0 \).

**Proof.** Let us fix \( y \in H \), and set \( v(x) = u(x + y) \). We have
\[
u(x) - \rho(\|y\|) \leq v(x) \leq u(x) + \rho(\|y\|).\]
Applying the operator \( T_t \) gives
\[
T_t u(x) - \rho(y) \leq T_t v(x) \leq T_t u(x) + \rho(y).
\]
On the other hand, we have
\[
T_t v(x) = \inf_{z} (u(z + y) + \|z - x\|^2/t) = \inf_{z} (u(z) + \|z - (x + y)\|^2/t) = T_t u(x + y),
\]
so that
\[
T_t u(x) - \rho(\|y\|) \leq T_{1t} u(x + y) \leq T_t u(x) + \rho(\|y\|).
\]
We have proved that \( T_{1t} u \) is \( \rho \)-continuous if \( u \) is, the proof for \( T_t u \) is the same.

In order to study the convergence, let us set \( \epsilon(t) = \sup_{r > 0} (\rho(r) - r^2/t) \). We have
\[
\epsilon(t) = \sup_{r > 0} (\rho(r\sqrt{t}) - r^2) \leq \sup_{r > 0} ((r + 1)\rho(\sqrt{t}) - r^2) = \rho(\sqrt{t}) + \rho^2(\sqrt{t})/4.
\]
We conclude that \( \lim_{t \to 0} \epsilon(t) = 0 \). We now come back to the operator \( T_t \), and observe that
\[
u(y) - \|y - x\|^2/t \geq u(x) - \rho(\|y - x\|) + \|y - x\|^2/t \geq u(x) - \epsilon(t)
\]
for each \( x \) and \( y \), so that
\[
u - \epsilon(t) \leq T_t u \leq u.
\]
**Lemma 3.** For each function \( u : H \rightarrow (-\infty, +\infty] \), we have \( T_t \circ T_t(u) \leq u \) and the equality \( T_t \circ T_t(u) = u \) holds if and only if \( u \) is t-semi-convex and lower semi-continuous. Similarly, given a function \( v : H \rightarrow (-\infty, +\infty) \), we have \( T_t \circ T_t(v) \geq v \), with equality if and only if \( v \) is t-semi-concave and upper semi-continuous.

**Proof.** Let us write explicitly
\[
T_t \circ T_t u(x) = \sup_y \inf_z \left( u(z) + \|z - y\|^2/t - \|y - x\|^2/t \right).
\]

Taking \( z = x \), we obtain the estimate \( T_t \circ T_t u(x) \leq \sup_y u(z) = u(z) \). Let us now write
\[
T_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z \left( u(z) + \|z\|^2/t + (2y/t) \cdot (x - z) \right)
\]
which by an obvious change of variable leads to
\[
T_t \circ T_t u(x) + \|x\|^2/t = \sup_y \inf_z \left( u(z) + \|z\|^2/t + y \cdot (x - z) \right).
\]

We recognize here that the function \( T_t \circ T_t u(x) + \|x\|^2/t \) is the Legendre bidual of the function \( u(x) + \|x\|^2/t \). It is well-know that a function is equal to its Legendre bidual if and only if it is convex and lower semi-continuous. \( \Box \)

2. Proof of the main results.

**Proof of Theorem 3.** For each function \( f \) and each \( t > 0 \), the function \( T_t \circ T_2 \circ T_t f \) is both t-semi-concave and t-semi-convex. It is t-semi-convex by Lemma 1, and it is semi-concave because \( T_2(T_t f) \) is 2t-semi-concave by Lemma 1, which implies, still by Lemma 1, that \( T_t \circ T_2 \circ T_t f \) is t-semi-concave. As a consequence, Lemma 5 below implies that the function \( R_t f \) is \( C^{1,1} \) provided it is locally bounded. The function \( R_t(f) \) is bounded if \( f \) is bounded, hence its is \( C^{1,1} \) in this case.

In the case where \( f \) is uniformly continuous, Lemma 2 implies that
\[
|f - c(2t)| \leq R_t(f) \leq f + 2c(t).
\]

As a consequence, \( R_t(f) \) is converging uniformly to \( f \), and it is locally bounded hence \( C^{1,1} \).

We now consider two continuous functions \( u \) and \( v \) such that \( u \) and \( -v \) are \( k \) semi-concave, and such that \( v \leq u \). We claim that
\[
|f - c(2t)| \leq R_t(f) \leq f + 2c(t).
\]

As a consequence, \( R_t(f) \) is converging uniformly to \( f \), and it is locally bounded hence \( C^{1,1} \).
for $t \leq k$. This claim implies that $u \geq T_t \circ T_{2t} \circ T_t f \geq v$ when $u \geq f \geq v$ and $t \leq k$. Let us now prove the claim concerning $T_t \circ T_n$, the other part being similar. Since $v$ is $k$-semi-convex and continuous, we have $T_t \circ T_tv = v$ for $t \leq k$, by Lemma 3. Then,

$$u \geq f \geq T_t \circ T_tv \geq T_tv = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators $T_t$ and $T_t$ are order-preserving.

We have proved that $v \leq R_t(f) \leq u$ if $v \leq f \leq u$ and $t \leq k$. For $t \in ]0, k]$, the function $R_t(f)$ is thus locally bounded hence $C^{1,1}$. \hfill \Box

**Proof of Theorem 4.** Let $a_i$ be chosen such that the functions $(g,u) \circ \phi_i$ and $-(g,v) \circ \phi_i$ are $a_i$-semi-concave on $\mathbb{R}^n$ (when extended by 0 outside of $B^n$). The existence of real numbers $a_i$ with this property follows from Lemma 4 below. Given $u \geq f \geq v$, we can apply Theorem 3 for each $i$ to the functions

$$(g,u) \circ \phi_i \geq (g,v) \circ \phi_i \geq (g,v) \circ \phi_i$$

extended by zero outside of $B^n$. We conclude that, for $t \in ]0, 1]$, the function $R_{tn}((g,f) \circ \phi_i)$ is $C^{1,1}$ and satisfies

$$(g,u) \circ \phi_i \geq R_{tn}((g,f) \circ \phi_i) \geq (g,v) \circ \phi_i.$$  

As a consequence, the function

$$[R_{tn}((g,f) \circ \phi_i)] \circ \phi_i^{-1},$$

extended as a function on $M$ equal to 0 outside of $\phi_i(B^n)$, is $C^{1,1}$. The function $G_t(f)$ is thus locally a finite sum of $C^{1,1}$ functions hence it is locally $C^{1,1}$. Moreover, we have

$$u = \sum_i g_i u \geq G_t(f) \geq \sum_i g_i v = v.$$ \hfill \Box

We have used:

**Lemma 4.** Let $u : B^n \rightarrow \mathbb{R}$ be a bounded function such that $u - \|x\|^2/a$ is concave, for some $a > 0$. For each compactly supported non-negative $C^2$ function $g : B^n \rightarrow \mathbb{R}$, the product $gu$ (extended by zero outside of $B^n$) is semi-concave on $\mathbb{R}^n$.

**Proof.** Since $u$ is bounded, we can assume that $u \geq 0$ on $B^n$. Let $K \subset B^n$ be a compact subset of the open ball $B^n$ which contains the support
of $g$ in its interior. Since the function $u - \| \cdot \|^2 / a$ is concave on $B_1$ it admits super-differentials at each point. As a consequence, for each $x \in B^n$, there exists a linear form $l_x$ such that

$$0 \leq u(y) \leq u(x) + l_x(y-x) + \|y-x\|^2 / a$$

for each $y \in B^1$. Moreover, the linear form $l_x$ is bounded independently of $x \in K$. We also have

$$0 \leq g(y) \leq g(x) + dg_x(y-x) + C\|y-x\|^2$$

for some $C > 0$, for all $x, y$ in $\mathbb{R}^n$. Taking the product, we get, for $x \in K$ and $y \in B^n$,

$$u(y)g(y) \leq u(x)g(x) + (g(x)l_x + u(x)dg_x) \cdot (y-x) + C\|y-x\|^2 + C\|y-x\|^3 + C\|y-x\|^4$$

where $C > 0$ is a constant independent of $x \in K$ and $y \in B^n$, which may change from line to line. As a consequence, setting $L_x = g(x)l_x + u(x)dg_x$, we obtain the inequality

$$(L) \quad (gu)(y) \leq (gu)(x) + L_x(y-x) + C\|y-x\|^2$$

for each $x \in K$ and $y \in B^n$. If we set $L_x = 0$ for $x \in \mathbb{R}^n - K$, the relation $(L)$ holds for each $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. For $x \in K$ and $y \in B^n$, we have already proved it. Since the linear forms $L_x, x \in K$ are uniformly bounded, we can assume that $L_x \cdot (y-x) + C\|y-x\|^2 \geq 0$ for all $x \in K$ and $y \in \mathbb{R}^n - B^n$ by taking $C$ large enough. Then, $(L)$ holds for all $x \in K$ and $y \in \mathbb{R}^n$. For $x \in \mathbb{R}^n - K$ and $y$ outside of the support of $g$, the relation $(L)$ holds in an obvious way, because $gu(x) = gu(y) = 0$, and $L_x = 0$. For $x \in \mathbb{R}^n - K$ and $y$ in the support of $g$, the relation holds provided that $C \geq \max (gu)/d^2$, where $d$ is the distance between the complement of $K$ and the support of $g$. This is a positive number since $K$ is a compact set containing the support of $g$ in its interior. We conclude that the function $(gu)$ is semi-concave on $\mathbb{R}^n$. □

For completeness, we also prove, following Fathi:

**Lemma 5.** Let $u : H \rightarrow \mathbb{R}$ be a locally bounded function which is both $k$-semi-concave and $k$-semi-convex. Then the function $u$ is $C^{1,1}$, and $6/k$ is a Lipschitz constant for the gradient of $u$.

**Proof.** It is well known that a locally bounded convex function is continuous. We conclude that $u$ is continuous. Let $u$ be a continuous function which is both $k$-semi-concave and $k$-semi-convex. Then, for each $x \in H,$
there exists a unique $l_x \in H$ such that

$$|u(x + y) - u(x) - l_x \cdot y| \leq \frac{\|y\|^2}{k}.$$  

We conclude that $l_x$ is the gradient of $u$ at $x$, and we have to prove that the map $x \mapsto l_x$ is Lipschitz. We have, for each $x$, $y$ and $z$ in $H$:

$$l_x \cdot (y + z) - \|y + z\|^2/k \leq u(x + y + z) - u(x) \leq l_x \cdot (y + z) + \|y + z\|^2/k$$

$$l_{(x+y)} \cdot (-y) - \|y\|^2/k \leq u(x) - u(x + y) \leq l_{(x+y)} \cdot (-y) + \|y\|^2/k$$

$$l_{(x+y)} \cdot (-z) - \|z\|^2/k \leq u(x + y) - u(x + y + z) \leq l_{(x+y)} \cdot (-z) + \|z\|^2/k.$$  

Taking the sum, we obtain

$$|l_{x+y} - l_x| \cdot (y + z) \leq \|y + z\|^2/k + \|y\|^2/k + \|z\|^2/k.$$  

By a change of variables, we get

$$|l_{x+y} - l_x| \cdot (z) \leq \|z\|^2/k + \|y\|^2/k + \|z + y\|^2/k.$$  

Taking $\|z\| = \|y\|$, we obtain

$$|l_{x+y} - l_x| \cdot (z) \leq 6\|z\| \|y\|^2/k$$  

for each $z$ such that $\|z\| = \|y\|$, we conclude that

$$\|l_{x+y} - l_x\| \leq 6\|y\|^2/k.$$  

\[ \square \]

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