Right Sided Ideals and Multilinear Polynomials with Derivations on Prime Rings

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Abstract - Let \( R \) be an associative prime ring of char \( R \neq 2 \) with center \( Z(R) \) and extended centroid \( C \), \( f(x_1, \ldots, x_n) \) a nonzero multilinear polynomial over \( C \) in \( n \) noncommuting variables, \( d \) a nonzero derivation of \( R \) and \( \rho \) a nonzero right ideal of \( R \). We prove that: (i) if \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0 \) for all \( x_1, \ldots, x_n \in \rho \) then \( \rho C = eRC \) for some idempotent element \( e \) in the socle of \( RC \) and \( f(x_1, \ldots, x_n) \) is central-valued in \( eRC \) unless \( d \) is an inner derivation induced by \( b \in Q \) such that \( b^2 = 0 \) and \( b \rho = 0 \); (ii) if \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R) \) for all \( x_1, \ldots, x_n \in \rho \) then \( \rho C = eRC \) for some idempotent element \( e \) in the socle of \( RC \) and either \( f(x_1, \ldots, x_n) \) is central in \( eRC \) or \( eRC \) satisfies the standard identity \( S_4(x_1, x_2, x_3, x_4) \) unless \( d \) is an inner derivation induced by \( b \in Q \) such that \( b^2 = 0 \) and \( b \rho = 0 \).

Throughout this paper, \( R \) always denotes a prime ring with extended centroid \( C \) and \( Q \) its two-sided Martindale ring of quotient. By \( d \) we mean a nonzero derivation of \( R \). For \( x, y \in R \), the commutator of \( x, y \) is denoted by \([x, y]\) and defined by \([x, y] = xy - yx\). We denote \([x, y]_2 = [[x, y], y] = [x, y]y - y[x, y]\).

A well known result proved by Posner [17] states that \( R \) must be commutative if \([d(x), x] \in Z(R) \) for all \( x \in R \). In [10] Lanski generalized the Posner’s result to a Lie ideal. More precisely Lanski proved that if \( L \) is a noncommutative Lie ideal of \( R \) such that \([d(x), x] \in Z(R) \) for all \( x \in L \),

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then \( \text{char } R = 2 \) and \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \), the standard identity. Note that a noncommutative Lie ideal of \( R \) contains all the commutators \([x_1, x_2]\) for \( x_1, x_2 \) in some nonzero ideal of \( R \) (see [10, Lemma 2 (i), (ii)]). So, it is natural to consider the situation when \([d(x), x] \in Z(R)\) for all commutators \( x = [x_1, x_2] \) or more general case \( x = f(x_1, \ldots, x_n) \) where \( f(x_1, \ldots, x_n) \) is a multilinear polynomial. In [11] Lee and Lee proved that if \([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)\) for all \( x_1, \ldots, x_n \) in some nonzero ideal of \( R \), then \( f(x_1, \ldots, x_n) \) is central-valued on \( R \), except when char \( R = 2 \) and \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \). Recently, De Filippis and Di Vincenzo (see [7]) consider the situation \( \delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0 \) for all \( x_1, \ldots, x_n \in R \), where \( d \) and \( \delta \) are two derivations of \( R \). The statement of De Filippis and Di Vincenzo’s theorem is the following:

**Theorem A ([7, Theorem 1]).** Let \( K \) be a noncommutative ring with unity, \( R \) a prime \( K \)-algebra of characteristic different from 2, \( d \) and \( \delta \) nonzero derivations of \( R \) and \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( K \). If \( \delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0 \) for all \( x_1, \ldots, x_n \in R \), then \( f(x_1, \ldots, x_n) \) is central-valued on \( R \).

In case \( \delta \) and \( d \) are two same derivations, the differential identity becomes \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0 \) for all \( x_1, \ldots, x_n \in R \). So, it is natural to ask, what happen in cases \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)\) for all \( x_1, \ldots, x_n \in R \) and \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)\) for all \( x_1, \ldots, x_n \in \rho \), where \( \rho \) is a non-zero right ideal of \( R \). In the present paper our object is to study these cases.

For the sake of completeness we recall some basic notations, definitions and some easy consequences of the result of Kharchenko [8] about the differential identities on a prime ring \( R \). First, we denote by \( \text{Der}(Q) \) the set of all derivations on \( Q \). By a derivation word \( A \) of \( R \) we mean \( A = d_1d_2d_3\ldots d_m \) for some derivations \( d_i \) of \( R \). For \( x \in R \), we denote by \( x^A \) the image of \( x \) under \( A \), that is \( x^A = (x^{d_1})^{d_2}\ldots x^{d_m} \). By a differential polynomial, we mean a generalized polynomial, with coefficients in \( Q \), of the form \( \Phi(x_i^{d_i}) \) involving noncommutative indeterminates \( x_i \) on which the derivations words \( d_i \) act as unary operations. \( \Phi(x_i^{d_i}) = 0 \) is said to be a differential identity on a subset \( T \) of \( Q \) if it vanishes for any assignment of values from \( T \) to its indeterminates \( x_i \).

Now let \( D_{int} \) be the \( C \)-subspace of \( \text{Der}(Q) \) consisting of all inner derivations on \( Q \). By Kharchenko’s theorem [8, Theorem 2], we have the following result:
Let $R$ be a prime ring of characteristic different from 2. If two nonzero derivations $d$ and $\delta$ are $C$-linearly independent modulo $D_{int}$ and $\Phi(x_i^{A_i})$ is a differential identity on $R$, where $A_i$ are derivations words of the following form $\delta, d, \delta^2, \delta d, d^2$, then $\Phi(y_{ji})$ is a generalized polynomial identity on $R$, where $y_{ji}$ are distinct indeterminates.

As a particular case, we have:

(i) either $d \in D_{int}$

or

(ii) $R$ satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)$

Denote by $Q \ast C \{X_1, \ldots, X_n\}$ the free product of the $C$-algebra $Q$ and $C\{X_1, \ldots, X_n\}$, the free $C$-algebra in noncommuting indeterminates $X_1, \ldots, X_n$.

Since $f(x_1, \ldots, x_n)$ is a multilinear polynomial, we can write

$$f(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n + \sum_{1 \neq \sigma \in S_n} x_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where $S_n$ is the permutation group over $n$ elements and any $x_{\sigma} \in C$. We denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $x_{\sigma}$ with $d(x_{\sigma}.1)$. In this way we have

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)$$

and

$$d^2(f(x_1, \ldots, x_n)) = d(f^d(x_1, \ldots, x_n)) + d\left( \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n) \right)$$

$$= f^{d^2}(x_1, \ldots, x_n) + \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n)$$

$$+ \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n)$$

$$+ \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n)$$

$$= f^{d^2}(x_1, \ldots, x_n) + 2 \sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n)$$

$$+ 2 \sum_{i < j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n) + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n).$$
1. The case for $\rho = R$.

**Lemma 1.1.** Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over a field $F$ of characteristic $\neq 2$, $b \in R$ and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $F$. If $k \geq 2$ and $[[b, [b, f(x_1, \ldots, x_n)]], f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in R$ or if $k \geq 3$ and $[[b, [b, f(x_1, \ldots, x_n)]], f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in R$, then either $b \in F \cdot I_k$ or $f(x_1, \ldots, x_n)$ is central-valued on $R$.

**Proof.** Let $b = (b_{ij})_{k \times k}$. Let $e_{ij}$ be the usual matrix unit with 1 in $(i, j)$ entry and zero else where. Now we proceed to show that $b \in Z(R)$ if $\in f(x_1, \ldots, x_n)$ is non central valued on $R$.

For simplicity, we write $f(x_1, \ldots, x_n) = f(x)$, where $x = (x_1, \ldots, x_n)$ $R^n = R \times \cdots \times R$ ($n$ times). Then by assumption,

$$[[b, [b, f(x)]]], f(x)] = [b^2f(x) - 2bf(x)b + f(x)b^2, f(x)] \in Z(R)$$

for all $x \in R^n$. Since $f(x_1, \ldots, x_n)$ is assumed to be noncentral on $R$, by [15, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r = (r_1, \ldots, r_n)$ in $R$ such that $f(r) - f(r_1, \ldots, r_n) = xe_{ij} \neq 0$ where $0 \neq x \in F$ and $i \neq j$. Thus

$$[b^2xe_{ij} - 2bxe_{ij}b + xe_{ij}b^2, xe_{ij}] \in Z(R).$$

Since the rank of $[b^2xe_{ij} - 2bxe_{ij}b + xe_{ij}b^2, xe_{ij}]$ is $\leq 2$, $[b^2xe_{ij} - 2bxe_{ij}b + xe_{ij}b^2, xe_{ij}] = 0$. Left multiplying by $e_{ij}$, we get $0 = e_{ij}(-2bxe_{ij}bxe_{ij}) = -2x^2b_{ij}^2e_{ij}$. Since char $F \neq 2$, $b_{ij} = 0$. For $s \neq t$, let $\sigma$ be a permutation in the symmetric group $S_m$ such that $\sigma(i) = s$ and $\sigma(j) = t$. Let $\psi$ be the automorphism of $R$ defined by $x^{\psi} = \left(\sum_{p,q} \xi_{pq} e_{pq}\right)^{\psi} = \sum_{p,q} \xi_{pq} e_{\sigma(p), \sigma(q)}$. Then

$$f(r^{\psi}) = f(r_1^{\psi}, \ldots, r_n^{\psi}) = f(r)^{\psi} = xe_{st} \neq 0$$

and we have as above $b_{ts} = 0$ for $s \neq t$. Thus $b$ is a diagonal matrix. For any $F$-automorphism $\theta$ of $R$, $b^\theta$ enjoys the same property as $b$ does, namely, $[[b^\theta, [b^\theta, f(x)]]], f(x)] \in Z(R)$ for all $x \in R^n$. Hence, $b^\theta$ must be diagonal. Write $b = \sum_{i=1}^k a_{ii} e_{ii}$; then for each $j \neq 1$, we have

$$(1 + e_{ij})b(1 - e_{ij}) = \sum_{i=1}^k a_{ij} e_{ii} + (b_{ij} - b_{11})e_{ij}$$

diagonal. Therefore, $b_{ij} = b_{11}$ and so $b$ is a scalar matrix.
**Lemma 1.2.** Let $R$ be a prime ring of characteristic different from 2 and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$. If for any $i = 1, \ldots, n$,

$$[f(x_1, \ldots, z_i, \ldots, x_n), f(x_1, \ldots, x_n)] = 0$$

for all $x_1, \ldots, x_n, z_i \in R$, then the polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$.

**Proof.** Let $a$ be a noncentral element of $R$. Then replacing $z_i$ with $[a, x_i]$ we have that for any $i = 1, \ldots, n$

$$[f(x_1, \ldots, [a, x_i], \ldots, x_n), f(x_1, \ldots, x_n)] = 0$$

and so

$$\left[ \sum_{i=0}^{n} f(x_1, \ldots, [a, x_i], \ldots, x_n), f(x_1, \ldots, x_n) \right] = 0$$

which implies $[a, f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in R$. By [11, Theorem], $f(x_1, \ldots, x_n)$ is central-valued on $R$.

**Theorem 1.3.** Let $R$ be a prime ring of characteristic different from 2, $d$ a nonzero derivation of $R$, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$. If

$$[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R) \quad \text{for all } x_1, \ldots, x_n \in R,$$

then either $f(x_1, \ldots, x_n)$ is central-valued on $R$ or $R$ satisfies the standard identity $S_4(x_1, x_2, x_3, x_4)$.

**Proof.** Let $I$ be any nonzero two-sided ideal of $R$. If for every $r_1, \ldots, r_n \in I, \ [d^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$, then by [14], this generalized differential identity is also satisfied by $Q$ and hence by $R$ as well. By Theorem A, $f(r_1, \ldots, r_n)$ is then central-valued on $R$ and we are done. Now we assume that for some $r_1, \ldots, r_n \in I, \ 0 \neq [d^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in I \cap Z(R)$. Thus $I \cap Z(R) \neq 0$. Let $K$ be a nonzero two-sided ideal of $R_Z$, the ring of central quotients of $R$. Since $K \cap R$ is a nonzero two-sided ideal of $R$, $(K \cap R) \cap Z(R) \neq 0$. Therefore, $K$ contains an invertible element in $R_Z$ and so $R_Z$ is a simple ring with identity 1.
By assumption, $R$ satisfies the differential identity
\[
g(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n), d^2(x_1), \ldots, d^2(x_n))
= \left[ f^{d^2}(x_1, \ldots, x_n) + 2 \sum_{i} f^{d}(x_1, \ldots, d(x_i), \ldots, x_n) \right.
+ 2 \sum_{i<j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n)
+ \left. \sum_{i} f(x_1, \ldots, d^2(x_i), \ldots, x_n), f(x_1, \ldots, x_n) \right], x_{n+1}.\]

If $d$ is not $Q$-inner, then by Kharchenko’s theorem [8],
\[
\left[ f^{d^2}(x_1, \ldots, x_n) + 2 \sum_{i} f^{d}(x_1, \ldots, y_i, \ldots, x_n) \right.
+ 2 \sum_{i<j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n)
+ \left. \sum_{i} f(x_1, \ldots, z_i, \ldots, x_n), f(x_1, \ldots, x_n) \right], x_{n+1} = 0
\tag{1}
\]

for all $x_i, y_i, z_i, x_{n+1} \in R$ for $i = 1, 2, \ldots, n$. In particular, for any $i$, assuming $y_1 = \cdots = y_{i-1} = y_{i+1} = \cdots = y_n = 0, z_1 = \cdots = z_n = 0$, we have
\[
\left[ f^{d^2}(x_1, \ldots, x_n) + 2f^d(x_1, \ldots, y_i, \ldots, x_n), f(x_1, \ldots, x_n) \right], x_{n+1} = 0
\]
and so
\[
\left[ f^{d^2}(x_1, \ldots, x_n) + 2 \sum_{i} f^{d}(x_1, \ldots, y_i, \ldots, x_n), f(x_1, \ldots, x_n) \right], x_{n+1} = 0
\]
for all $x_i, y_i, x_{n+1} \in R, i = 1, 2, \ldots, n$. Thus from (1), we obtain
\[
\left[ 2 \sum_{i<j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n) \right.
+ \left. \sum_{i} f(x_1, \ldots, z_i, \ldots, x_n), f(x_1, \ldots, x_n) \right], x_{n+1} = 0 \tag{2}
\]
for all $x_i, y_i, z_i, x_{n+1} \in R$ for $i = 1, 2, \ldots, n$.

By localizing $R$ at $Z(R)$, we obtain that (2) is also an identity of $R_Z$. Since $R$ and $R_Z$ satisfy the same polynomial identities, in order to prove that $R$ satisfies $S_4$, we may assume that $R$ is a simple ring with 1. Thus $R$ satisfies the identity (2). Now putting $y_i = [b, x_i] = \delta(x_i)$ and $z_i = [b, [b, x_i]] =
= δ²(x_i), i = 1, 2, ..., n for some b ∉ Z(R), where δ is an inner derivation induced by some b ∈ R, we obtain that R satisfies

\[[δ²(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}] = 0.\]

Thus by Martindale's theorem [16], R is a primitive ring with a minimal right ideal, whose commuting ring D is a division ring which is finite dimensional over Z(R). However, since R is simple with 1, R must be Artinian. Hence R = D_k', the ring of k' × k' matrices over D, for some k' ≥ 1. Again, by [9, Lemma 2], it follows that there exists a field F such that R ⊆ M_k(F), the ring of all k × k matrices over the field F, and M_k(F) satisfies

\[[δ²(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}] = 0.\]

If k ≥ 3, then by Lemma 1.1, we have b ∈ Z(R), a contradiction. Thus k = 2, that is, R satisfies S_4(x_1, x_2, x_3, x_4).

Similarly, the same conclusion can be drawn in case d is an Q-inner derivation induced by some b ∈ Q.

2. The case for one-sided ideal.

We begin with the following lemmas

**Lemma 2.1.** Let ρ be a nonzero right ideal of R and d a derivation of R. Then the following conditions are equivalent:

(i) d is an inner derivation induced by some b ∈ Q such that bp = 0;
(ii) d(ρ)ρ = 0.

For its proof, we refer to [2, Lemma].

**Lemma 2.2.** Let R be a prime ring, ρ a nonzero right ideal of R, f(x_1, \ldots, x_l) a multilinear polynomial over C, a ∈ R and n a fixed positive integer. If f(x_1, \ldots, x_l)^n a = 0 for all x_1, \ldots, x_l ∈ ρ, then either a = 0 or f(ρ)ρ = 0.

For its proof, we refer to [3, Lemma 2 (II)].

**Lemma 2.3.** Let R be a prime ring. If \[d²(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \subseteq Z(R) for all x_1, \ldots, x_n ∈ ρ, then R satisfies nontrivial generalized polynomial identity unless d is an inner derivation induced by b ∈ Q such that b² = 0 and bp = 0.
PROOF. Suppose on the contrary that \( R \) does not satisfy any nontrivial generalized polynomial identity (GPI). Thus we may assume that \( R \) is noncommutative, otherwise \( R \) satisfies trivially a nontrivial GPI. Now we consider the following two cases:

**Case I.** Suppose that \( d \) is a \( Q \)-inner derivation induced by an element \( b \in Q \) such that \( b^2 \neq 0 \). Then for any \( x_0 \in \rho \)

\[
[[b, [b, f(x_0X_1, \ldots, x_0X_n)], f(x_0X_1, \ldots, x_0X_n)] \in Z(R)
\]

that is

\[
[[b^2f(x_0X_1, \ldots, x_0X_n) - 2bf(x_0X_1, \ldots, x_0X_n)b + f(x_0X_1, \ldots, x_0X_n)b^2, f(x_0X_1, \ldots, x_0X_n)], x_0X_{n+1}] = 0
\]  

is a GPI for \( R \), so it is the zero element in \( Q \ast C \{X_1, \ldots, X_{n+1}\} \). Denote \( l_R(\rho) \) the left annihilator of \( \rho \) in \( R \). Suppose first that \( \{1, b, b^2\} \) are linearly \( C \)-independent modulo \( l_R(\rho) \), that is \( (zb^2 + \beta b + \gamma)\rho = 0 \) if and only if \( z = \beta = \gamma = 0 \). Since \( R \) is not a GPI-ring, a fortiori it can not be a PI-ring. Thus, by [13, Lemma 3] there exists \( x_0 \in \rho \) such that \( \{b^2x_0, bx_0, x_0\} \) are linearly \( C \)-independent. Then we have that

\[
[[b^2f(x_0X_1, \ldots, x_0X_n) - 2bf(x_0X_1, \ldots, x_0X_n)b + f(x_0X_1, \ldots, x_0X_n)b^2, f(x_0X_1, \ldots, x_0X_n)], x_0X_{n+1}] = 0
\]

is a nontrivial GPI for \( R \), a contradiction.

Therefore, \( \{1, b, b^2\} \) are linearly \( C \)-independent modulo \( l_R(\rho) \), that is there exist \( z, \beta, \gamma \in C \), not all zero, such that \( (zb^2 + \beta b + \gamma)\rho = 0 \). Suppose that \( z = 0 \). Then \( \beta \neq 0 \), otherwise \( \gamma = 0 \). Thus by \( (\beta b + \gamma)\rho = 0 \), we have that \( (b + \beta^{-1}\gamma)\rho = 0 \). Since \( b \) and \( b + \beta^{-1}\gamma \) induce the same inner derivation, we may replace \( b \) by \( b + \beta^{-1}\gamma \) in the basic hypothesis. Therefore, in any case we may suppose \( b\rho = 0 \) and then from (3), \( R \) satisfies \( x_0X_{n+1}f^2(x_0X_1, \ldots, x_0X_n)b^2 = 0 \). Since \( R \) does not satisfy any nontrivial GPI, \( b^2 = 0 \), a contradiction.

Next suppose that \( z \neq 0 \). In this case there exist \( \lambda, \mu \in C \) such that \( b^2x_0 = \lambda bx_0 + \mu x_0 \) for all \( x_0 \in \rho \). If \( bx_0 \) and \( x_0 \) are linearly \( C \)-dependent for all \( x_0 \in \rho \), then again we obtain \( b\rho = 0 \) and so \( b^2 = 0 \). Therefore choose \( x_0 \in \rho \) such that \( bx_0 \) and \( x_0 \) are linearly \( C \)-independent. Then replacing \( b^2x_0 \) with \( \lambda bx_0 + \mu x_0 \), we obtain from (3)
that $R$ satisfies
\[
\{ (\lambda b + \mu)^2(x_0X_1, \ldots, x_0X_n) - 2bf(x_0X_1, \ldots, x_0X_n)bf(x_0X_1, \ldots, x_0X_n) \\
+ f(x_0X_1, \ldots, x_0X_n)(\lambda b + \mu)f(x_0X_1, \ldots, x_0X_n) \\
- \{ f(x_0X_1, \ldots, x_0X_n)(\lambda b + \mu)f(x_0X_1, \ldots, x_0X_n) \\
- 2f(x_0X_1, \ldots, x_0X_n)bf(x_0X_1, \ldots, x_0X_n)b + f^2(x_0X_1, \ldots, x_0X_n)b^2 \}, x_0X_{n+1} \}
\]

This is a nontrivial GPI for $R$, because the term
\[(\lambda b f^2(x_0X_1, \ldots, x_0X_n) - 2bf(x_0X_1, \ldots, x_0X_n)bf(x_0X_1, \ldots, x_0X_n))x_0X_{n+1}\]
appears nontrivially, a contradiction.

**Case II.** Suppose that $d$ is an inner derivation induced by an element $b \in Q$ such that $b^2 = 0$. Thus we have that $[- 2bf(X_1, \ldots, X_n)b, f(X_1, \ldots, X_n)] \in Z(R)$ is satisfied by $\rho$. In case there exists $x_0 \in \rho$ such that $\{ bx_0, x_0 \}$ are linearly $C$-independent, we have that $[- 2bf(x_0X_1, \ldots, x_0X_n)b, f(x_0X_1, \ldots, x_0X_n), x_0X_{n+1}]$ is a non trivial GPI for $R$, a contradiction. Hence $\{ bx_0, x_0 \}$ are linearly $C$-dependent for all $x_0 \in \rho$, that is there exists $x \in C$ such that $(b - x)\rho = 0$. Thus we have that $[zf^2(X_1, \ldots, X_n)(x - b), x_0X_{n+1}]$ is satisfied by $\rho$, in particular $R$ satisfies:
\[[zf^2(x_0X_1, \ldots, x_0X_n)(x - b), f(x_0X_1, \ldots, x_0X_n)] = zf^3(X_1, \ldots, X_n)(x - b)\]
for any $x_0 \in \rho$. Since $R$ is not GPI, it follows that either $b = x \in C$, which is a contradiction, or $x = 0$ which means $b\rho = 0$, as required.

**Case III.** Suppose that $d$ is an inner derivation induced by an element $b \in Q$ such that $b\rho = 0$. Thus we have that $[- f^2(X_1, \ldots, X_n)b^2, X_{n+1}]$ is satisfied by $\rho$, in particular $R$ satisfies:
\[[ - f^2(x_0X_1, \ldots, x_0X_n)b^2, f(x_0X_1, \ldots, x_0X_n)] = f^3(x_0X_1, \ldots, x_0X_n)b^2\]
for any $x_0 \in \rho$. Again since $R$ is not GPI we conclude that $b^2 = 0$.

**Case IV.** Next suppose that $d$ is not $Q$-inner derivation. By our assumption we have that $R$ satisfies
\[0 = \left[ [ f^d(xX_1, \ldots, xX_n) + 2 \sum_i f^d(xX_1, \ldots, d(x)X_i + xd(X_i), \ldots, xX_n) \\
+ 2 \sum_{i<j} f(xX_1, \ldots, d(x)X_i + xd(X_i), \ldots, d(x)X_j + xd(X_j), \ldots, xX_n) \\
+ \sum_i f(xX_1, \ldots, d^2(x)X_i + 2d(x)d(X_i) + xd^2(X_i), \ldots, xX_n), f(xX_1, \ldots, xX_n)] \right] \cdot X_{n+1} \left[.\right] .\]
By Kharchenko’s theorem [8],
\[
\left[ f^{d^2}(xX_1, \ldots, xX_n) + 2 \sum_i f^d(xX_1, \ldots, d(x)X_i + xr_i, \ldots, xX_n) \\
+ 2 \sum_{i < j} f(xX_1, \ldots, d(x)X_i + xr_i, \ldots, d(x)X_j + xr_j, \ldots, xX_n) \\
+ \sum_i f(xX_1, \ldots, d^2(x)X_i + 2d(x)r_i + xs_i, \ldots, xX_n), f(xX_1, \ldots, xX_n), X_{n+1} \right] = 0
\]
for all \(X_1, \ldots, X_n, r_1, \ldots, r_n, s_1, \ldots, s_n \in R\). In particular, for \(r_1 = \ldots = r_n = 0\), we have
\[
\left[ f^{d^2}(xX_1, \ldots, xX_n) + 2 \sum_i f^d(xX_1, \ldots, d(x)X_i, \ldots, xX_n) \\
2 \sum_{i < j} f(xX_1, \ldots, d(x)X_i, \ldots, d(x)X_j, \ldots, xX_n) + \sum_i f(xX_1, \ldots, d^2(x)X_i, \ldots, xX_n) + \\
+ \sum_i f(xX_1, \ldots, xs_i, \ldots, xX_n), f(xX_1, \ldots, xX_n), X_{n+1} \right] = 0.
\]

Hence \(R\) satisfies the blended component
\[
[[f(xs_1, \ldots, xX_n), f(xX_1, \ldots, xX_n)], X_{n+1}] = 0
\]
which is a nontrivial GPI for \(R\), a contradiction.

**Theorem 2.4.** Let \(R\) be an associative prime ring of char \(R \neq 2\) with center \(Z(R)\) and extended centroid \(C\), \(f(x_1, \ldots, x_n)\) a nonzero multilinear polynomial over \(C\) in \(n\) noncommuting variables, \(d\) a nonzero derivation of \(R\) and \(\rho\) a nonzero right ideal of \(R\). If \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0\)
for all \(x_1, \ldots, x_n \in \rho\) then \(\rho C = eRC\) for some idempotent \(e\) in the socle of \(RC\) and \(f(x_1, \ldots, x_n)\) is central-valued on \(eRC\) unless \(d\) is an inner derivation induced by \(b \in Q\) such that \(b^2 = 0\) and \(b\rho = 0\).

**Proof.** Suppose \(d\) is not a \(Q\)-inner derivation induced by an element \(b \in Q\) such that \(b^2 = 0\) and \(b\rho = 0\).

Now assume first that \(f(\rho)\rho = 0\), that is \(f(x_1, \ldots, x_n)x_{n+1} = 0\) for all \(x_1, x_2, \ldots, x_{n+1} \in \rho\). Then by [12, Proposition], \(\rho C = eRC\) for some idempotent \(e \in \text{soc}(RC)\). Since \(f(\rho)\rho = 0\), we have \(f(\rho R)\rho R = 0\) and hence \(f(\rho Q)\rho Q = 0\) by [4, Theorem 2]. In particular, \(f(\rho C)\rho C = 0\), or equivalently, \(f(eRC)e = 0\). Then \(f(eRC)e = 0\), that is, \(f(x_1, \ldots, x_n)\) is a PI for \(eRC\) and, a fortiori, central valued on \(eRC\).

Next assume that \(f(\rho)\rho \neq 0\), that is \(f(x_1, \ldots, x_n)x_{n+1} \) is not an identity for \(\rho\) and then we derive a contradiction. By Lemma 2.3, \(R\) is a GPI-ring
and so is also $Q$ (see [1] and [4]). By [16], $Q$ is a primitive ring with $H = soc(Q) \neq 0$. Moreover, we may assume $f(\rho H) \rho H \neq 0$, otherwise by [1] and [4], $f(\rho Q) \rho Q = 0$, which is a contradiction. Choose $a_0, a_1, \ldots, a_n \in \rho H$ such that $f(a_1, \ldots, a_n) \neq 0$. Let $a \in \rho H$. Since $H$ is a regular ring, there exists $e^2 = e \in H$ such that

$$eH = aH + a_0H + a_1H + \cdots + a_nH.$$  

Then $e \in \rho H$ and $a = ea, a_i = ea_i$ for $i = 0, 1, \ldots, n$. Thus we have $f(eHe) = f(eH)e \neq 0$. By our assumption and by [14, Theorem 2], we also assume that

$$[d^2(f(x_1, \ldots, x_n), f(x_1, \ldots, x_n))]$$

is an identity for $\rho Q$. In particular $[d^2(f(x_1, \ldots, x_n), f(x_1, \ldots, x_n))]$ is an identity for $\rho H$ and so for $eH$. It follows that, for all $r_1, \ldots, r_n \in H$,

$$0 = [d^2(f(er_1, \ldots, er_n)), f(er_1, \ldots, er_n)].$$

We may write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})x_n + h(x_1, \ldots, x_n)$, where $x_n$ never appears as last variable in any monomials of $h$. Let $r \in H$. Then replacing $r_n$ with $r(1 - e)$, we have

(4) $$0 = [d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)].$$

Now, we know the fact that $d(x(1 - e))e = -x(1 - e)d(e)$ and $(1 - e)d(ex) = (1 - e)d(e)ex$ and so

$$(1 - e)d^2(ex(1 - e))e = (1 - e)d\{d(e)ex(1 - e) + ed(ex(1 - e))\}e$$

$$= (1 - e)d(e)d(ex(1 - e))e + (1 - e)d(e)d(ex(1 - e))e$$

$$= -2(1 - e)d(e)ex(1 - e)d(e).$$

Thus using this facts, we have from (4),

$$0 = (1 - e)[d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)]$$

$$= (1 - e)d^2(t(er_1, \ldots, er_{n-1})er(1 - e))t(er_1, \ldots, er_{n-1})er(1 - e)$$

$$= -2(1 - e)d(e)t(er_1, \ldots, er_{n-1})er(1 - e)d(e)t(er_1, \ldots, er_{n-1})er(1 - e)$$

$$= -2((1 - e)d(e)t(er_1, \ldots, er_{n-1})er)^2(1 - e).$$

This implies

$$0 = -2\{(1 - e)d(e)t(er_1, \ldots, er_{n-1})er\}^3$$

that is

$$0 = -2\{(1 - e)d(e)t(er_1, \ldots, er_{n-1})eH\}^3.$$
By [6], \((1 - e)d(e)t(\chi_1, \ldots, \chi_{n-1})eH = 0\) which implies
\((1 - e)d(e)t(\chi_1e, \ldots, \chi_{n-1}e) = 0\).

Since \(eH e\) is a simple Artinian ring and \(t(eH e) \neq 0\) is invariant under the action of all inner automorphisms of \(eH e\), by [5, Lemma 2], \((1 - e)d(e) = 0\) and so \(d(e) = ed(e) \in eH\). Thus \(d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq \rho H\) and \(d(a) = d(ea) \in d(eH) \subseteq \rho H\). Therefore, \(d(\rho H) \subseteq \rho H\). Denote the left annihilator of \(\rho H\) in \(H\) by \(l_H(\rho H)\). Then \(\rho H = \frac{\rho H}{\rho H \cap l_H(\rho H)}\), a prime \(C\)-algebra with the derivation \(\overline{d}\) such that \(\overline{d}(\overline{x}) = \overline{d(x)}\), for all \(x \in \rho H\). By assumption, we have that
\[\overline{d}^2(f(\overline{x_1}, \ldots, \overline{x_n}), f(\overline{x_1}, \ldots, \overline{x_n})) = 0\]
for all \(\overline{x_1}, \ldots, \overline{x_n} \in \rho H\). By Theorem A, either \(\overline{d} = 0\) or \(f(\overline{x_1}, \ldots, \overline{x_n})\) is central-valued on \(\rho H\).

If \(\overline{d} = 0\), then \(d(\rho H)\rho H = 0\) and so \(d(\rho)\rho = 0\). By Lemma 2.1, \(d\) is an inner derivation induced by an element \(b \in Q\) such that \(b\rho = 0\). Then for all \(x_1, \ldots, x_n \in \rho\), we have by assumption that
\[0 = [[b, [b, f(x_1, \ldots, x_n)]]], f(x_1, \ldots, x_n)] = -f^2(x_1, \ldots, x_n)b^2\]
By [3, Lemma 4], either \(b^2 = 0\) or \(f(\rho)\rho = 0\). In both cases we have contradiction.

If \(f(\overline{x_1}, \ldots, \overline{x_n})\) is central-valued on \(\rho H\), then \(\rho H\), as well as \(\rho\), satisfies \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} = 0\). Then \(\rho C = eRC\) for some idempotent element \(e \in \text{soc}(RC)\) by [12, Proposition] and \(f(x_1, \ldots, x_n)\) is central-valued on \(eRC\) and we are done.

**Theorem 2.5.** Let \(R\) be an associative prime ring of char \(R \neq 2\) with center \(Z(R)\) and extended centroid \(C\), \(f(x_1, \ldots, x_n)\) a nonzero multilinear polynomial over \(C\) in \(n\) noncommuting variables, \(d\) a nonzero derivation of \(R\) and \(\rho\) a nonzero right ideal of \(R\). If \([d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)\) for all \(x_1, \ldots, x_n \in \rho\) then \(\rho C = eRC\) for some idempotent \(e\) in the socle of \(RC\) and either \(f(x_1, \ldots, x_n)\) is central-valued on \(eRC\) or \(eRC\) satisfies \(S_4(x_1, x_2, x_3, x_4)\) unless \(d\) is an inner derivation induced by \(b \in Q\) such that \(b^2 = 0\) and \(b\rho = 0\).

**Proof.** Suppose \(d\) is not a \(Q\)-inner derivation induced by an element \(b \in Q\) such that \(b^2 = 0\) and \(b\rho = 0\).

If \([f(\rho), \rho] \neq 0\), that is \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} = 0\) for all
$x_1, x_2, \ldots, x_{n+2} \in \rho$, then by [12, Proposition], $\rho C = eRC$ for some idempotent $e \in \text{soc}(RC)$ and $f(x_1, \ldots, x_n)$ is central-valued on $eRCe$.

So, assume that $[f(\rho), \rho] \neq 0$, that is $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for $\rho$ and then we derive that $eRCe$ satisfies $S_4$. By Lemma 2.3, $R$ is a GPI-ring and so is also $Q$ (see [1] and [4]). By [16], $Q$ is a primitive ring with $H = \text{soc}(Q) \neq 0$. Moreover, we may assume $[f(\rho H), \rho H] \neq 0$, otherwise by [1] and [4], $[f(\rho Q), \rho Q] = 0$, which is a contradiction. Choose $a_1, \ldots, a_{n+2}, b_1, \ldots, b_5 \in \rho H$ such that $[f(a_1, \ldots, a_n), a_{n+1}]a_{n+2} \neq 0$ and $S_4(b_1, b_2, b_3, b_4)b_5 \neq 0$. Let $a \in \rho H$. Since $H$ is a regular ring, there exists $e^2 = e \in H$ such that

$$eH = aH + a_1H + \cdots + a_{n+2}H + b_1H + \cdots + b_5H.$$

Then $e \in \rho H$ and $a = ea, a_i = ea_i$ for $i = 1, \ldots, n + 2, b_i = eb_i$ for $i = 1, \ldots, 5$. Thus we have $f(eHe) = f(eH)e \neq 0$. Moreover, by [14, Theorem 2], we may also assume that

$$[[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n), x_{n+1}], x_{n+1}]$$

is an identity for $\rho Q$. In particular, $[[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}]$ is an identity for $\rho H$ and so for $eH$. It follows that, for all $r_1, \ldots, r_{n+1} \in H$,

$$0 = [[d^2(f(er_1, \ldots, er_n)), f(er_1, \ldots, er_n)], er_{n+1}].$$

We may write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})x_n + h(x_1, \ldots, x_n)$, where $x_n$ never appears as last variable in any monomials of $h$. Let $r \in H$. Then replacing $r_n$ with $r(1 - e)$ and $r_{n+1}$ with $r_{n+1}(1 - e)$, we have

$$0 = [[d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)], er_{n+1}(1 - e)].$$

Now, we know the fact that $d(x(1 - e)) = -x(1 - e)d(e), (1 - e)d(ex) = (1 - e)d(e)ex$ and $(1 - e)d^2(ex(1 - e)) = -2(1 - e)d(e)ex(1 - e)d(e)$. Thus using these facts, we have from (5),

$$0 = [[d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)], er_{n+1}(1 - e)]$$

$$= [d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)]er_{n+1}(1 - e)$$

$$- er_{n+1}(1 - e)[d^2(t(er_1, \ldots, er_{n-1})er(1 - e)), t(er_1, \ldots, er_{n-1})er(1 - e)]$$

$$= - t(er_1, \ldots, er_{n-1})er(1 - e)d^2(t(er_1, \ldots, er_{n-1})er(1 - e))er_{n+1}(1 - e)$$

$$- er_{n+1}(1 - e)d^2(t(er_1, \ldots, er_{n-1})er(1 - e), t(er_1, \ldots, er_{n-1})er(1 - e)$$

$$= t(er_1, \ldots, er_{n-1})er(1 - e)d(e)t(er_1, \ldots, er_{n-1})er(1 - e)d(e)r_{n+1}(1 - e)$$

$$+ er_{n+1}(1 - e)d(e)t(er_1, \ldots, er_{n-1})er(1 - e)d(e)t(er_1, \ldots, er_{n-1})er(1 - e).$$
Replacing $r_{n+1}$ with $t(er_1, \ldots, er_{n-1})er$ in the above relation, we get

$$2t(er_1, \ldots, er_{n-1})er(1 - e)d(e)t(er_1, \ldots, er_{n-1})er)^2(1 - e) = 0.$$ 

This implies

$$2(1 - e)d(e)t(er_1, \ldots, er_{n-1})er)^4 = 0$$

that is

$$2\{(1 - e)d(e)t(er_1, \ldots, er_{n-1})eH\}^4 = 0.$$ 

By [6], $(1 - e)d(e)t(er_1, \ldots, er_{n-1})eH = 0$ which implies

$$(1 - e)d(e)t(er_1e, \ldots, er_{n-1}e) = 0.$$ 

Since $eHe$ is a simple Artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of $eHe$, by [5, Lemma 2], $(1 - e)d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq \rho H$ and $d(a) = d(ea) \in d(eH) \subseteq \rho H$. Therefore, $d(\rho H) \subseteq \rho H$. Denote the left annihilator of $\rho H$ in $H$ by $l_H(\rho H)$. Then $\overline{\rho H} = \frac{\rho H}{\rho H \cap l_H(\rho H)}$, a prime $C$-algebra

with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in \rho H$. By assumption, we have that

$$[[\overline{d^2f(x_1, \ldots, x_n)}, f(x_1, \ldots, x_n)], x_{n+1}] = 0$$

for all $x_1, \ldots, x_n \in \overline{\rho H}$. By Theorem 1.3, either $\overline{d} = 0$ or $f(x_1, \ldots, x_n)$ is central-valued on $\overline{\rho H}$ or $\overline{\rho H}$ satisfies the standard identity $S_4(x_1, \ldots, x_4)$.

If $\overline{d} = 0$, then as in the proof of Theorem 2.4, we have $d(\rho)\rho = 0$ and hence by Lemma 2.1, $d$ is an inner derivation induced by an element $b \in Q$ such that $b\rho = 0$. Thus for all $r_1, \ldots, r_n \in \rho H$,

$$[\overline{d^2f(r_1, \ldots, r_n)}, f(r_1, \ldots, r_n)] = -f(r_1, \ldots, r_n)^2b^2 \in C.$$ 

Commuting both sides with $f(r_1, \ldots, r_n)$, we obtain $f(r_1, \ldots, r_n)^3b^2 = 0$. In this case by Lemma 2.2, since $b^2 \neq 0$, $f(\rho H)\rho H = 0$. If $f(\rho H)\rho H = 0$, then $[f(\rho H), \rho H]\rho H = 0$, a contradiction.

If $f(x_1, \ldots, x_n)$ is central-valued on $\overline{\rho H}$, then we obtain that

$$[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$$

is an identity for $\rho$, a contradiction.

Finally, if $S_4(x_1, \ldots, x_4)$ is an identity for $\overline{\rho H}$, $S_4(x_1, \ldots, x_4)x_5$ is an identity for $\rho H$ and so for $\rho C$ and this contradicts the choices of the elements $b_1, \ldots, b_5 \in \rho H$. Therefore, we conclude that in any case $\rho C$ satisfies a polynomial identity, hence by [12, Proposition], there exists an idempotent $e \in Soc(\rho C)$ such that $\rho C = e\rho C$, as desired.
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