Hölder Type Estimates for the $\overline{\partial}$-Equation in Strongly Pseudoconvex Domains.

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Abstract - In this paper we generalize the Hölder space with a majorant function, define its order, and prove the existence and regularity for the solutions of the Cauchy-Riemann equation in the generalized Hölder space over a bounded strongly pseudoconvex domain.

1. Introduction and regular majorant.

If $D$ is a bounded domain in $\mathbb{C}^n$, the Hölder space of order $\alpha$, $A_\alpha(D)$ ($0 < \alpha < 1$), is defined as the set of all functions $g$ on $D$ which satisfy for a constant $C = C_g > 0$ the condition

$$|g(z) - g(\zeta)| \leq C|z - \zeta|^{\alpha}, \quad z, \zeta \in D.$$

We first generalize this Hölder space following Dyakonov [Dya97] (also see Pavlović’s book [Pav04]). For this purpose we introduce the notion of a regular majorant. Let $\omega$ be a continuous increasing function on $[0, \infty)$. We assume $\omega(0) = 0$, and suppose that $\omega(t)/t$ is non-increasing

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and satisfies the inequality

\[(1.1) \quad \int_0^\delta \frac{\omega(t)}{t} \, dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} \, dt \leq C\omega(\delta), \quad \text{for any} \quad 0 < \delta < 1,\]

for a suitable constant \(C = C(\omega)\). Such a function \(\omega\) is called a regular majorant. Given a regular majorant \(\omega\), the Hölder type space, \(A(\omega, D)\) is defined as the family of all functions \(g\) on \(D\) such that

\[(1.2) \quad |g(z) - g(\zeta)| \leq C\omega(|z - \zeta|), \quad z, \zeta \in D.\]

The norm \(\|g\|_\omega\) of \(g \in A(\omega, D)\) is given by \(C_g + \|g\|_\infty\), where \(C_g \geq 0\) is the smallest constant satisfying (1.2) and \(\|g\|_\infty\) is the \(L^\infty\) norm in \(D\). Note that with this norm \(A(\omega, D)\) is a Banach space and \(A(\omega, D) \subset L^\infty(D)\) (see the chapter 10 of Pavlović’s book [Pav04]). We denote by \(A_q(\omega, D)\) the set of the differential forms of type \((0, q)\) whose coefficients are in \(A(\omega, D)\). We define the order of a regular majorant as follows:

**Definition 1.1.** We say that a regular majorant \(\omega\) has order \(z\) \((0 < z < 1)\) if there exist an \(z\) and a positive real number \(t_0\) such that

\[
z = \sup \left\{ \gamma : \frac{\omega(t)}{t^\gamma} \text{ is increasing} \forall t, \ 0 < t < t_0 \right\}
= \inf \left\{ \gamma : \frac{\omega(t)}{t^\gamma} \text{ is decreasing} \forall t, \ 0 < t < t_0 \right\}.
\]

If a regular majorant \(\omega\) has order \(z\), then we let \(\omega = \omega_z\) and call \(A(\omega_z, D)\) the Hölder type space of order \(z\). By definition of the order of a regular majorant, it is uniquely determined, if it exists. Now we state our main result of this paper.

**Theorem 1.2.** Let \(D \subset C^n (n \geq 2)\) be a strongly pseudoconvex domain with \(C^4\)-boundary and \(0 < z < 1/2\). If a regular majorant \(\omega_z\) has order \(z\) and \(f \in A_q(\omega_z, D)\) with \(\overline{D}f = 0 (1 \leq q \leq n)\), then there is a solution \(u \in A_{(q-1)}(t^{1/2}\omega_z, D)\) of \(\overline{D}u = f\) such that for some constant \(C = C(\omega_z)\)

\[(1.3) \quad \|u\|_{t^{1/2}\omega_z} \leq C\|f\|_{\omega_z}.
\]

The above inequality (1.3) generalizes the estimate by Henkin-Romanov [RH71] and Lieb-Range [LR80]

\[\|u\|_{t^{1/2}} \leq C\|f\|_{t^{1/2}}.\]
For the proof of the Hölder type estimate (1.3), we need a variant of the Hardy-Littlewood Lemma [HL84].

**Lemma 1.3.** Let \( D \subset \subset \mathbb{R}^n \) be a bounded domain with \( C^1 \) boundary. If \( g \) is a \( C^1(D) \)-function and \( \omega \) is a regular majorant of order \( \gamma \), \( 0 < \gamma < 1 \) such that for some constant \( c_g \) depending on \( g \),

\[
|dg(x)| \leq c_g \frac{\omega\gamma(|\rho(x)|)}{|\rho(x)|}, \quad x \in D,
\]

then we have

\[
|g(x) - g(y)| \leq c_g \omega\gamma(|x - y|).
\]

As convention we use the notation \( A \leq B \) or \( A \geq B \) if there are constants \( c_1, c_2 \), independent of the quantities under consideration, satisfying \( A \leq c_1B \) and \( A \geq c_2B \), respectively.

Before proving our theorem, we discuss some properties of a regular majorant and some examples.

**Example 1.4.** (i) The most typical example is a function \( \omega(t) = t^x \) \((0 < x < 1)\). Clearly, \( \omega \) is a regular majorant and has order \( x \).

(ii) A non-trivial example is the function, \( \omega(t) = t^x|\log t|^{\beta} \) on \([0, t_0]\) extended continuously for \( t > t_0 \) to be a regular majorant. Here \( 0 < x < 1 \), \(-\infty < \beta < \infty \) and \( t_0 \) must be chosen sufficiently small so that the function \( \omega \) should be a regular majorant (\( t_0 \) depends on \( x, \beta \)). Since \( \lim_{t \to 0^+} t^{\epsilon} |\log t|^{\beta} = 0 \) for any \( \epsilon > 0 \), it follows that \( \omega(t) = t^x|\log t|^{\beta} \) has order \( x \) for any choice of \( \beta \).

(iii) Define the function \( m(t) = 1/|\log t|^{\beta} \), \( \beta > 0 \) for \( 0 < t < t_0 \) and \( m(0) = 0 \). Then \( m(t) \) is continuous and increasing near 0, but it is not a regular majorant.

We end this section by describing useful properties of a regular majorant.

**Remark 1.5.** (i) If \( \omega, m \) are two regular majorants and have orders \( x, \beta \) respectively with \( 0 < x < \beta < 1 \), then letting \( \omega_x, m_\beta \), there exist \( t_0 > 0 \) and \( c \) such that \( m_\beta(t) \leq c\omega_x(t), \ 0 \leq t \leq t_0 \). Hence we have the inclusion \( A(m_\beta, D) \subset A(\omega_x, D) \). Note that if two regular majorants, \( \omega, m \) have the same order \( x \), then generally there is no inclusion relation between \( A(\omega, D) \) and \( A(m, D) \).

(ii) In our Theorem 1.2, for a general regular majorant of order 1/2, the
estimate \( \|u\|_{\mathcal{O}^2} \lesssim \|f\|_\infty \) does not hold. In fact, the celebrated Henkin’s theorem [RH71] holds only for the special regular majorant \( \omega_{1/2}(t) = |t|^{1/2} \) and this number 1/2 is the sharp bound [Ran86]. But there is a regular majorant \( m_{1/2}(t) = |t|^{1/2} \log t \) near the origin of order 1/2, which is strictly bigger than \( |t|^{1/2} \).

**Remark 1.6.** Let \( \omega \) be a regular majorant of order \( \alpha (0 < \alpha < 1/2) \), say \( \omega = \omega_x \). Then \( t^{1/2} \omega_x \) is also a regular majorant of order \( (\alpha + 1/2) \). In fact, \( t^{1/2} \omega_x \) is increasing and \( (t^{1/2} \omega_x) / t \) is non-increasing, since \( \omega_x / t^\gamma \), \( \gamma > \alpha \) is decreasing. Here we use the fact that \( \omega_x \) has order \( \alpha \). It remains to show that \( t^{1/2} \omega_x \) also satisfies (1.1). Since \( (t^{1/2} \omega_x) / t \) is non-increasing, we have for any \( \delta, (0 < \delta < 1) \),

\[
(1.4) \quad \int_0^{\delta} \frac{s^{1/2} \omega_x(s)}{s} ds \leq \delta^{1/2} \int_0^{\delta} \frac{\omega_x(s)}{s} ds \leq \delta^{1/2} \omega_x(\delta).
\]

On the other hand, for a given \( 0 < \alpha < 1/2 \), we can choose a sufficiently small \( \epsilon \) such that \( \alpha < 1/2 - \epsilon \). It follows from the order of \( \omega_x \) that \( \omega_x / t^{(1/2 - \epsilon)} \) is decreasing. Hence we obtain

\[
(1.5) \quad \delta \int_0^{\infty} \frac{s^{1/2} \omega_x(s)}{s^2} ds = \delta \int_0^{\infty} \frac{\omega_x(s)}{s^{1/2 - \epsilon}} \frac{1}{s^{1+\epsilon}} ds \approx \delta^{1/2} \omega_x(\delta) \cdot \delta^{1/2 - \epsilon} \int_0^{\infty} \frac{1}{s^{1+\epsilon}} ds \lesssim \delta^{1/2} \omega_x(\delta).
\]

By (1.4) and (1.5), \( t^{1/2} \omega_x \) is a regular majorant of order \( (\alpha + 1/2) \).

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2. Henkin’s solution operator of the \( \bar{\partial} \)-equation.

In this section, we introduce the Henkin’s solution operator [HL84] of the \( \bar{\partial} \)-equation and prove the integral estimates for the solution operator in a strongly pseudoconvex domain in \( \mathbb{C}^n \). Let \( D \) be defined by a function \( \rho \), i.e., \( D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \), where \( \rho \in C^4 \) and \( \nabla \rho \neq 0 \) on \( \partial D \).

To construct the integral formula for solutions of the \( \bar{\partial} \)-equation in a strongly pseudoconvex domain, we need a support function (see [HL84]). For
the global support function, we follow Fornaess construction [For76]. He showed that there exist a neighborhood $U$ of $\overline{D}$ and a function $\hat{\phi}(\cdot, \cdot) \in C^3(U \times U)$ such that for all $\zeta \in U$, $\hat{\phi}(\zeta, \cdot)$ is holomorphic in $U$ and $\hat{\phi}(\zeta, z) = \langle \Phi, \zeta - z \rangle$, where we define $\Phi = \Phi(\zeta, z) = (\phi_1(\zeta, z), \ldots, \phi_n(\zeta, z))$ and $\langle \Phi, \zeta - z \rangle = \sum_{j=1}^{n} \phi_j(\zeta, z)(\zeta_j - z_j)$. In [For76], Fornaess also showed that $\phi_j \in C^3(U \times U)$ is holomorphic in $z$ and there is a constant $c$ such that for all $z \in \overline{D}$ and $\zeta \in D$ we have

$$2\text{Re} \hat{\phi}(\zeta, z) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2$$

and $d_z \hat{\phi}(\zeta, z)|_{z=\zeta} = \partial \rho(\zeta)$. Suppose that $f \in A_q(\omega_z, D) (1 \leq q \leq n)$ and $\overline{\partial} f = 0$. Then $f$ is uniformly continuous in $D$. Using the above global support function $\hat{\phi}$, we define Henkin kernel $H(\zeta, z)$ as follows:

$$H(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\langle \zeta - \tilde{z}, d\zeta \rangle}{|\zeta - z|^2} \wedge \frac{\langle \Phi, d\zeta \rangle}{\langle \Phi, \zeta - z \rangle} \wedge \sum_{k+\ell=n-2}^{\infty} \left( \frac{\langle \zeta - \tilde{z}, d\zeta \rangle}{|\zeta - z|^2} \right)^k \left( \frac{\langle \partial_{\zeta, \overline{\zeta}} \Phi, d\zeta \rangle}{|\zeta - z|^2} \right)^\ell,$$

where $\partial_{\zeta, \overline{\zeta}} \Phi = \partial_{\zeta} \Phi$ and $d\zeta = (d\zeta_1, \ldots, d\zeta_n)$. Note that $\Phi$ is holomorphic in $z$. We also define the Bochner-Martinelli kernel:

$$K(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\langle \zeta - \tilde{z}, d\zeta \rangle}{|\zeta - z|^2} \wedge \left( \frac{\langle d\zeta - d\tilde{z}, d\zeta \rangle}{|\zeta - z|^2} \right)^{n-1}.$$

For the construction of the above kernels, see [Ran86] or [CS01]. We have the Henkin’s solution operator $Sf = Kf - Hf$ of the $\overline{\partial}$-equation, where

$$Hf(z) = \int_{\zeta \in bD} f(\zeta) \wedge H(\zeta, z), \quad Kf(z) = \int_{\zeta \in D} f(\zeta) \wedge K(\zeta, z).$$

We remark that the fact that the support function $\hat{\phi}(\zeta, z)$ is holomorphic in $z$ is very crucial in the construction of the solution operator $Sf$ of the $\overline{\partial}$-equation.

To prove the Hölder type estimate (1.3) of the main Theorem 1.2, we use Lemma 1.3. Hence, we have to estimate the differential of the Henkin solution operator, $d_z Sf$. Using the fact that $|\zeta - z|^2 \lesssim |\hat{\phi}(\zeta, z)|$ for $(\zeta, z) \in bD \times \overline{D}$, straightforward computations give the kernel estimate (for the details, see [Ran86])

$$|d_z H(\zeta, z)| \lesssim \frac{1}{|\hat{\phi}(\zeta, z)|^2 |\zeta - z|^{2n-3}}, \quad (\zeta, z) \in bD \times \overline{D}.$$
Remark 2.1. Without loss of generality, we assume that the differential of Henkin kernel, \( d_z H(\zeta, z) \) has the following form:

\[
(2.7) \quad d_z H(\zeta, z) = \frac{A(\zeta, z)}{(\Phi(\zeta, z))^2 |\zeta - z|^{2n-2}},
\]

where \( A(\cdot, z) \) belongs to \( C^1(\overline{D}) \) and satisfies \( |A(\zeta, z)| \lesssim |\zeta - z| \). Actually, \( d_z H(\zeta, z) \) contains more terms whose singularity order is lower than that of (2.7) and so we can ignore other terms (refer to §3. of chapter 4 in [Ran86]).

Generally, the Bochner-Martinelli integral, \( Kf \), has a good regularity, so \( K \) is a bounded operator from \( L^\infty \)-forms to \( A_\alpha \)-forms for any \( 0 < \alpha < 1 \). This kind of regularity still holds for a regular majorant of order \( \alpha (0 < \alpha < 1) \). Hence, we only prove the estimate for the differential of Henkin kernel, \( d_z \mathcal{H}f \), which is the main part of this paper.

Proposition 2.2. For any \( \alpha \) with \( 0 < \alpha < 1/2 \), there exists a constant \( C_\alpha > 0 \) such that

\[
(2.8) \quad |d_z \mathcal{H}f(z)| \leq C_\alpha \| f \|_{A_\alpha} \frac{\omega_2(|\rho(z)|)}{|\rho(z)|^{1/2}} \quad \text{for} \quad z \in D.
\]

Proof. Since the singularities of the Henkin kernel are located in the diagonal \( bD \times bD \), to show the inequality (2.8), it suffices to estimate the integral of (2.8) near boundary points. Fix a point \( z \in D \) which is sufficiently close to the boundary of \( D \) and choose a ball \( B(z, r) \) with \( B(z, r) \cap bD \neq \emptyset \), in which we have a \( C^1 \) coordinates system \( (t_1, \ldots, t_{2n}) = t = t(\zeta, z) \) such that \( t_1 = -\rho(\zeta), t_2 = \text{Im} \Phi(\zeta, z), t(z, z) = (\rho(z), \ldots, 0) \), and \( |t(\zeta, z)| < 1 \) for \( \zeta \in B(z, r) \). (For the detail, see [HL84].) Moreover, this coordinate system \( t \) satisfies

\[
|t| \lesssim |\zeta - z| \lesssim |t|, \quad \zeta \in B(z, r) \cap bD.
\]

Also, note that the new coordinate system satisfies \( t(\zeta, z) = (0, t') \) for \( \zeta \in B(z, r) \cap bD \), where \( t' = (t_2, \ldots, t_{2n}) \). By Remark 2.1, we have to show that

\[
I(z) = \left| \int_{bD \cap B(z, r)} \frac{f(\zeta)\chi(\zeta)A(\zeta, z)}{(\Phi(\zeta, z))^2 |\zeta - z|^{2n-2}} dV(\zeta) \right| \lesssim \| f \|_{A_\alpha} \frac{\omega_2(\delta(z))}{\delta(z)^{1/2}},
\]

where \( \chi \) is a compactly supported cut-off function in \( B(z, r) \). For this kind of estimate of Hölder type, we choose \( \zeta' \in B(z, r) \cap bD \) satisfying \( t(\zeta', z) = (0, 0, t_3, \ldots, t_{2n}) \). This gives the obvious estimate, \( I(z) \leq I_1(z) + \)
+ I_2(z), where

\[ I_1(z) = \left| \int_{bD \cap B(z, r)} \frac{(f(\zeta) - f(\zeta')) \chi(\zeta) A(\zeta, z)}{(\hat{\phi}(\zeta, z))^2 |\zeta - z|^{2n-2}} \, dV(\zeta) \right|, \]

\[ I_2(z) = \left| \int_{bD \cap B(z, r)} \frac{f(\zeta') \chi(\zeta) A(\zeta, z)}{(\hat{\phi}(\zeta, z))^2 |\zeta - z|^{2n-2}} \, dV(\zeta) \right|. \]

It follows from the definition of \( \| \cdot \|_{\omega_2} \) and the inequality \( |A(\zeta, z)| \lesssim |\zeta - z| \), that

\[ I_1(z) \leq \| f \|_{\omega_2} \int_{bD \cap B(z, r)} \frac{\omega_2(|\zeta - \zeta'|)}{|\hat{\phi}(\zeta, z)|^2 |\zeta - z|^{2n-3}} \, dV(\zeta). \tag{2.9} \]

To estimate the integral of the right hand side of (2.9), we use the coordinate system \( t \), the inequality (2.6), and introduce polar coordinates in \( t'' = (t_3, \ldots, t_{2n}) \in \mathbb{R}^{2n-2} \), and also set \( r = |t''| \). Then we have

\[ I_1(z) \lesssim \| f \|_{\omega_2} \int_{|t'| < 1} \frac{\omega_2(|t_2|)}{|t_2| + |t'|^2 + |\rho(z)|^2 |t'|^{2n-3}} \, dV(t') \]

\[ \lesssim \| f \|_{\omega_2} \int_{|t'| < 1} \omega_2(|t_2|) \left[ \int_0^1 \frac{r^{2n-3} \, dr}{(|t_2| + r^2 + |\rho(z)|^2)^{2n-3}} \right] \, dt_2 \]

\[ \lesssim \| f \|_{\omega_2} \int_0^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2. \]

We may assume that \( 0 < |\rho(z)| < 1 \), since \( z \in D \) is close to the boundary. We decompose the integral as follows:

\[ \int_0^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 = \int_0^{|\rho(z)|} \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 + \int_{|\rho(z)|}^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2. \]

Since \( \omega_2 \) is a regular majorant, by the first term of the left hand side of (1.1), we have

\[ \int_0^{|\rho(z)|} \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 \lesssim \frac{1}{|\rho(z)|^{1/2}} \int_0^{|\rho(z)|} \frac{\omega_2(t_2)}{t_2} \, dt_2 \lesssim \omega_2(|\rho(z)|) |\rho(z)|^{1/2}. \]

Similarly, since \( s^{1/2} \omega_2(s) \) is also a regular majorant, by the second term of
the left hand side of (1.1), it follows that

$$
\int_{|\rho(z)|}^{1} \frac{\omega_z(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 = \int_{|\rho(z)|}^{1} \frac{\frac{1}{2} t_2^{1/2} \omega_z(t_2)}{t_2^2} \left( \frac{t_2}{t_2 + |\rho(z)|} \right)^{3/2} \, dt_2 \\
\leq \int_{|\rho(z)|}^{1} \frac{\frac{1}{2} t_2^{1/2} \omega_z(t_2)}{t_2^2} \, dt_2 \\
\leq \frac{|\rho(z)|^{1/2} \omega_z(|\rho(z)|)}{|\rho(z)|} = \omega_z(|\rho(z)|).$$

These inequalities imply $I_1(z) \lesssim \|f\|_\infty \omega_z(|\rho(z)|)/|\rho(z)|^{1/2}.$

For $I_2(z),$ we need a somewhat different method. The integration by parts allows one to lower the singularity order of the Henkin kernel. This kind of method was used in [Ran92].

We see that

$$\frac{1}{\varphi^2} = - \left( \frac{\partial \varphi}{\partial t_2} \right)^{-1} \frac{\partial}{\partial t_2} \left( \frac{1}{\varphi} \right).$$

Therefore, by integration by parts, we have

\begin{equation}
I_2(z) \leq \int_{|t'| \leq 1} \left| \int_{|t'| \leq 1} f(0, 0, t', t'') \frac{1}{\varphi} \frac{\partial}{\partial t_2} \left( \frac{\partial \varphi}{\partial t_2} \right)^{-2} \frac{\chi(t') A(t', z)}{|t'|^{2n-2}} \, dt' \right| \, dt''
\end{equation}

$$= \int_{|t'| \leq 1} \left| \int_{|t'| \leq 1} f(0, 0, t') \frac{1}{\varphi} \frac{\partial}{\partial t_2} \left( \frac{\partial \varphi}{\partial t_2} \right)^{-2} \frac{\chi(t') A(t', z)}{|t'|^{2n-2}} \, dt' \right| \, dt''
$$

$$= \int_{|t'| \leq 1} f(0, 0, t') \frac{1}{\varphi} \left( \frac{\partial \varphi}{\partial t_2} \right)^{-2} B(t', z) \, dt' ,$$

where

$$B(t', z) = - \frac{\partial^2 \varphi}{\partial t_2^2} \frac{\chi(t') A(t', z)}{|t'|^{2n-2}} + \frac{\partial \varphi}{\partial t_2} \frac{\partial}{\partial t_2} \left( \frac{\chi(t') A(t', z)}{|t'|^{2n-2}} \right).$$

In the second equality of (2.10), we use the fact that $f(0, 0, t'')$ does not depend on $t_2.$ Since $t_2 = \operatorname{Im} \varphi,$ we have $|\partial \varphi / \partial t_2| \geq 1.$ Therefore, we have

$$I_2(z) \lesssim \|f\|_\infty \int_{|t'| \leq 1} \frac{dt'}{|\varphi|^{2n-2}}.$$
For the moment, we assume that for any \( \varepsilon > 0 \),

\[
J(z) = \int_{|t'| \leq 1} \frac{dt'}{|\phi||t'|^{2n-2}} \leq |\rho(z)|^{-\varepsilon},
\]

which will be proved later as an independent lemma. Since (2.11) holds for arbitrary \( \varepsilon > 0 \), one can choose \( \varepsilon > 0 \) so that \( 0 < 1/2 - \varepsilon < \alpha \). Moreover, \( \omega_2 \), \( 0 < \alpha < 1/2 \), is a regular majorant and so \( \omega_2(t)/t^{1/2-\varepsilon} \) is increasing, or equivalently, \( |\rho(z)|^{-\varepsilon} \leq \omega_2(|\rho(z)|)/|\rho(z)|^{1/2} \). It follows that

\[
I_2(z) \leq \|f\|_{\infty} J(z) \leq \|f\|_{\alpha_\omega} \frac{\omega_2(|\rho(z)|)}{|\rho(z)|^{1/2}}.
\]

These two estimates for \( I_1(z) \) and \( I_2(z) \) complete the proof. \( \square \)

We end this section with the proof of (2.11).

**Lemma 2.3.** For any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
J(z) \leq C_\varepsilon |\rho(z)|^{-\varepsilon}.
\]

**Proof.** We have

\[
J(z) \leq \int_{|t'| \leq 1} \frac{dt'}{(|t_2| + |\rho(z)| + |t'|^2)|t'|^{2n-2}}
\]

\[
\leq \int_{|(t_2, t_3, t_4)| \leq 1} \frac{dt_2 dt_3 dt_4}{(|t_2| + |\rho(z)|)(t_2^2 + t_3^2 + t_4^2)}.
\]

Again, using polar coordinates in \( (t_3, t_4) \), say \( x = |(t_3, t_4)| \), one obtains

\[
J(z) \leq \int_{|t_2| \leq 1} \frac{1}{|t_2| + |\rho(z)|} \left( \int_0^1 \frac{x \, dx}{t_2^2 + x^2} \right) dt_2
\]

\[
\leq \int_0^1 \frac{|\log t_2|}{(t_2 + |\rho(z)|)} \, dt_2
\]

\[
\leq C_\varepsilon \int_0^1 \frac{t_2^{-\varepsilon}}{(t_2 + |\rho(z)|)} \, dt_2.
\]
By the change of variable \( s = t_2/|\rho(z)| \), we have

\[
J(z) \leq \frac{1}{|t_2| + |\rho(z)|} \int_0^1 dt_2
\]

\[
\leq |\rho(z)|^{-\varepsilon} \int_0^\infty ds \frac{ds}{(1 + s)s^\varepsilon} \leq C_\varepsilon |\rho(z)|^{-\varepsilon}.
\]

3. Proof of Theorem 1.2.

In this section, we complete the proof of our main Theorem 1.2 using Proposition 2.2 and Lemma 1.3.

The inequality (2.8) in Proposition 2.2 implies that

\[
|dH f(z)| \leq c_z \|f\|_\infty \frac{|\rho(z)|^{1/2} \omega_\varepsilon(|\rho(z)|)}{|\rho(z)|}.
\]

Therefore by Lemma 1.3 and regularities of the operator \( \mathcal{H} f \) in the Hölder type spaces we can prove the inequality (1.3) of Theorem 1.2.

Finally, we include a brief sketch of the proof of Lemma 1.3.

**Proof.** Because \( \overline{D} \) is compact, by the local coordinate change argument, it suffices to show the following in the special domain

\[
D(k) = \{ (x_1, x') \in \mathbb{R}^n : 0 < x_1 < k, \ |x'| < k \}:
\]

if

\[
|dg(x)| \leq c_g \frac{\omega_\varepsilon(x_1)}{x_1}
\]

for \( x, y \in D(k/2) \) with \( |x - y| \leq k/2 \), then we have

\[
|g(x) - g(y)| \leq c \cdot c_g \omega_\varepsilon(|x - y|).
\]

To show this, fix two points \( x, y \in D(k/2) \) with \( |x - y| \leq k/2 \) and let \( d = |x - y| \). Here we may assume that \( k \leq 1/2 \) and by symmetry we may also suppose \( x_1 \leq y_1 \).

First it follows from (3.12) that

\[
|g(x_1, x') - g(x_1 + d, x')| \leq \int_{x_1}^{x_1 + d} \left| \frac{\partial g}{\partial x_1} (t, x') \right| dt
\]

\[
\leq c_g \int_{x_1}^{x_1 + d} \frac{\omega_\varepsilon(t)}{t} dt \leq c \cdot c_g \omega_\varepsilon(d).
\]
In fact, if $0 < d \leq x_1$, then
\[
\int_{x_1}^{x_1 + d} \frac{\omega_j(t)}{t} \, dt \leq d \frac{\omega_j(x_1)}{x_1} \leq \omega_j(d),
\]
since $\omega_j(t)/t$ is decreasing. If $0 < x_1 \leq d$, then
\[
\int_{x_1}^{x_1 + d} \frac{\omega_j(t)}{t} \, dt \leq \int_{0}^{d} \frac{\omega_j(t)}{t} \, dt \leq \omega_j(d). \tag{3.15}
\]
Since $\omega_j(t)/t$ is decreasing, the first inequality of (3.15) holds and by (1.1) the second inequality of (3.15) is also true.

Next, by the Mean Value Theorem and (3.12), since $\omega_j(t)/t$ is decreasing, we have
\[
|g(x_1 + d, x') - g(y_1 + d, y')| \leq c_g d \frac{\omega_j(a_1)}{a_1} \leq c_g \omega_j(d) \tag{3.16}
\]
for some $a_1$ in the line segment between $x_1 + d$ and $y_1 + d$. Since
\[
|g(x) - g(y)| \leq |g(x_1, x') - g(x_1 + d, x')| + |g(x_1 + d, x') - g(y_1 + d, y')| + |g(y_1 + d, y') - g(y_1, y')|,
\]
(3.13) follows from the estimates (3.14) and (3.16). \hfill \Box

REFERENCES


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