Compact Subgroups of $GL_n(\mathbb{C})$.

JEAN FRESNEL (*) - MARIUS VAN DER PUT (**) 

1. Introduction.

Let $G \subset GL_n(\mathbb{C})$ be a compact subgroup. Using the Haar measure on $G$ one obtains a positive definite Hermitian form on $\mathbb{C}^n$ which is invariant under $G$. In other words, $G$ is conjugated, with respect to $GL_n(\mathbb{C})$, to a subgroup of the standard unitary group $U_n(\mathbb{C})$. In particular, every $g \in G$ is semisimple and all its eigenvalues have absolute value 1.

The inverse problem was posed by K. Millet and I. Kaplansky (see [Ba]):

Suppose that the subgroup $G \subset GL_n(\mathbb{C})$ has the property that every $g \in G$ is semisimple and all its eigenvalues have absolute value 1. Is $G$ conjugated to a subgroup of $U_n(\mathbb{C})$?

For $n = 1, 2$ the answer is positive. A counterexample for $n \geq 3$ is given in ([Ba], Counterexample 1.10, p. 19). However, using the techniques of Burnside, it is shown in ([Ba], Corollary 1.8, p. 18), that $G$ is isomorphic to a subgroup of $U_n(\mathbb{C})$. The aim of this paper is to present a proof of the following positive result.

**Theorem 1.1.** Suppose that the subgroup $G \subset GL_n(\mathbb{C})$ satisfies:

(i) Every element of $G$ is semisimple and all its eigenvalues have absolute value 1.

(ii) $G$ is closed with respect to the ordinary topology of $GL_n(\mathbb{C})$.

Then $G$ is conjugated in $GL_n(\mathbb{C})$ to a subgroup of $U_n(\mathbb{C})$ and therefore compact.

The theorem has an almost immediate consequence.

(*) Indirizzo dell’A.: Laboratoire de Théorie des nombres et Algorithmique arithmétique, Université Bordeaux I, 351 cours de la Libération, 33405 Talence, France. E-mail: fresnel@math.u-bordeaux1.fr

(**) Indirizzo dell’A.: Department of Mathematics, University of Groningen, P.O.Box 800, 9700 AV Groningen, The Netherlands. E-mail: mvdput@math.rug.nl
COROLLARY 1.2. Let $E$ be an $n$-dimensional affine euclidean space and $G$ a closed subgroup of the group of all isometries of $E$. Suppose that each element of $G$ has a fixed point. Then the group $G$ is compact and has a fixed point.

PROOF. The action of $g \in G$ on $\mathbb{R}^n$ is given by $X \in \mathbb{R}^n \mapsto gX = UX + A$ with $U \in O_n(\mathbb{R})$, $A \in \mathbb{R}^n$. One associates to $g \in G$ the matrix $M(g) = \begin{pmatrix} U & A \\ 0 & 1 \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{R})$. All eigenvalues of $M(g)$ have absolute value 1. Since $U$ is semisimple, $M(g)$ is semisimple if and only if there exists a vector $X \in \mathbb{R}^n$ such that $M(g) \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} X \\ 1 \end{pmatrix}$. This property of $X$ is equivalent to $X$ is a fixed point for $g$. It follows that $M(g)$ is semisimple if and only if $g$ has a fixed point. The theorem implies that $\{M(g) | g \in G\}$ is compact. Then $G$ is compact and has a fixed point. \hfill \qed

2. A result on real Lie algebras.

PROPOSITION 2.1. $V$ is a complex vector space of dimension $n \geq 1$. Let $\mathfrak{g}$ be a real Lie subalgebra of $\text{End}_\mathbb{C}(V)$ satisfying:

(a) $i \cdot 1_V \not\subset \mathfrak{g}$

(b) If $V = V_1 \oplus V_2$ with $V_1, V_2$ complex vector spaces invariant under $\mathfrak{g}$, then $V_1 = 0$ or $V_2 = 0$.

(c) Every element of $\mathfrak{g}$ is semisimple and all its eigenvalues are in $i \cdot \mathbb{R}$.

Then the following holds:

(1) $\mathfrak{g}$ is a real semisimple Lie algebra, $\mathfrak{g} := \mathbb{C} \otimes \mathbb{R} \mathfrak{g}$ is a complex semisimple Lie algebra and the canonical map $\mathfrak{g} \rightarrow \text{End}_\mathbb{C}(V)$ is injective.

(2) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\tilde{\mathfrak{g}} := \mathbb{C} \otimes \mathfrak{h}$ is a Cartan subalgebra for the complex Lie algebra of $\mathfrak{g}$. Let $R$ be the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$. Then

(2a) $a(h) \in i \cdot \mathbb{R}$ for every $h \in \mathfrak{h}$ and $a \in R$,

(2b) for every $a \in R$, the real Lie subalgebra of $\mathfrak{g}$, generated by $\mathfrak{g} \cap (\mathfrak{g}_a \oplus \mathfrak{g}_{-a})$ is isomorphic to $\mathfrak{sl}_2$.

(3) There exists a positive definite Hermitian form $F$ such that for all $x, y \in V$ and $g \in \mathfrak{g}$ one has $F(gx, y) + F(x, gy) = 0$. 

Proof. (1). Suppose that $\mathfrak{g}$ is not semisimple. Then $\mathfrak{g}$ has a non zero solvable ideal. Let $\alpha \neq 0$ be a minimal solvable ideal, then $[\alpha, \alpha] = 0$. Since the elements of $\alpha$ are semisimple and commute there is a decomposition $V := \mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$ and there are distinct $\mathbb{R}$-linear maps $\lambda_j : \alpha \to \mathbb{i} \cdot \mathbb{R}$ such that the action of $\alpha$ on $V$ is given by

$$a \left( \sum_{j=1}^r v_j \right) = \sum \lambda_j(a)v_j$$

for $a \in \alpha$ and $v_j \in V_j$ for all $j$.

Choose an element $a \in \alpha$ such that, say, $\lambda_1(a) = \mathbb{i}$ and the $\lambda_j(a)$ are distinct. For $g \in \mathfrak{g}$ one writes $b := [g, a] = ga - ag \in \alpha$. Consider for a given $u \in V_j$ the expression $g(u) = \sum v_k$ with all $v_k \in V_k$. Now $\lambda_j(b)u = b(u) = (ga - ag)(u) = \lambda_j(a)\sum v_k - \sum \lambda_k(a)v_k$. This implies $v_k = 0$ for $k \neq j$ and $\lambda_j(b)u = 0$. Thus the spaces $V_j$ are invariant under $\mathfrak{g}$. Condition (b) implies $r = 1$. Then $a = \mathbb{i} \cdot 1_V$, which contradicts condition (a). One concludes that $\mathfrak{g}$ is semisimple.

According to [F-H], $\mathfrak{g} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ is semisimple, too. An element of $\mathfrak{g}$ can uniquely be written as $1 \otimes a + \mathbb{i} \otimes b$ with $a, b \in \mathfrak{g}$. If the image of this element is 0 in End$_{\mathbb{C}}(V)$, then $a = -ib$. This implies $a = b = 0$ since $a$ and $b$ have their eigenvalues in $\mathbb{i} \cdot \mathbb{R}$ and are semisimple.

(2). The first statement of (2) is immediate. We recall (see [F-H]) that the Cartan decomposition (or root decomposition) $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha} \mathfrak{g}_\alpha)$ has the following properties: For any non zero linear map $a : \mathfrak{h} \to \mathbb{C}$ one has

$$\mathfrak{g}_\alpha := \{ g \in \mathfrak{g} \mid [h, g] = a(h)g \text{ for all } h \in \mathfrak{h} \}.$$

If $\mathfrak{g}_\alpha \neq 0$ then $a$ is called a root and in that case dim$_{\mathbb{C}} \mathfrak{g}_\alpha = 1$. If $a$ is a root, then $ca$ with $c \in \mathbb{C}^*$ is a root if and only if $c = \pm 1$.

Fix an element $h \in \text{End}(V)$. The eigenvalues of the linear map $\text{End}(V) \to \text{End}(V)$, defined by $g \mapsto \text{ad}(h)(g) := [h, g]$, are the differences of the eigenvalues of $h$. In particular for $h \in \mathfrak{h}$ and $a \in \mathbb{R}$ one has $a(h) \in \mathbb{i} \cdot \mathbb{R}$. This proves (2a).

One writes $a_1, -a_1, \ldots, a_r, -a_r$ for the roots. Any $g \in \mathfrak{g}$ has a unique decomposition $g = g_0 + \sum_{j=1}^r (g_{a_j} + g_{-a_j})$ with $g_0 \in \mathfrak{h}$, $g_{\pm a_j} \in \mathfrak{g}_{\pm a_j}$.

Choose a generic element $h_0 \in \mathfrak{h}$, i.e., the $2r$ elements $\pm a_j(h_0) \in \mathbb{i} \cdot \mathbb{R}$ are distinct. For $m \geq 1$ one has

$$\text{ad}(h_0)^m(g) = \sum_j a_j(h_0)^mg_{a_j} + (-a_j(h_0))^mg_{-a_j} \in \mathfrak{g}.$$
Using this relation for \( m = 2n, \ n = 1, \ldots, r \) and observing that the
\( a_j(h_0)^2 \in \mathbb{R}^*, \ j = 1, \ldots, r \) are distinct, one finds that all \( g_{a_j} + g_{-a_j} \) are in \( \mathfrak{g} \). Then also \( g_0 \in \mathfrak{g} \). Similarly, one finds that each \( ig_{a_j} - ig_{-a_j} \in \mathfrak{g} \).

Now we study the real vector space \( T_j := \mathfrak{g} \cap (\mathfrak{g}_{a_j} + \mathfrak{g}_{-a_j}) \). As shown above, any element of \( \mathfrak{g}_{\pm a_j} \) is nilpotent. Since the elements of \( \mathfrak{g} \) are semisimple one has \( \mathfrak{g} \cap (\mathfrak{g}_{\pm a_j} = 0 \). In particular the two projections \( T_j \rightarrow \mathfrak{g}_{\pm a_j} \) are injective. We conclude from this that \( T_j \) has a real basis of the form \( X_{a_j} + X_{-a_j}, iX_{a_j} - iX_{-a_j} \), where \( X_{\pm a_j} \) are non zero elements of \( \mathfrak{g}_{\pm a_j} \).

The complex Lie algebra generated by \( X_{\pm a_j} \) is easily seen to be the complex Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \). One easily verifies that the real Lie algebra generated by \( X_{a_j} + X_{-a_j}, iX_{a_j} - iX_{-a_j} \) is isomorphic to \( \mathfrak{sl}_2 \). This proves (2b).

(3). One applies [F-H], Proposition 26.4. The condition (i) of that proposition is (2a) and (2b). The equivalent condition (iii) states that the real Lie algebra associated to \( \mathfrak{g} \) is compact. This implies the existence of a positive definite Hermitian form \( F \) on \( V \) such that \( F(gx, y) + F(x, gy) = 0 \) holds for all \( x, y \in V \) and \( g \in \mathfrak{g} \).

3. Proof of the theorem.

The case \( G \) connected.

Put \( \mathfrak{g} := \{ A \in \text{Mat}_n(\mathbb{C}) | \exp(tA) \in G \text{ for all } t \in \mathbb{R} \} \). According to ([M-T], Proposition 3.4.2 and 3.4.2.1.), \( \mathfrak{g} \) is a real Lie subalgebra of \( \text{Mat}_n(\mathbb{C}) \) and moreover \( G \) is generated by \( \{ \exp(g) | g \in \mathfrak{g} \} \). The elements \( g \in \mathfrak{g} \) are clearly semisimple and all their eigenvalues are in \( \mathfrak{i} \cdot \mathbb{R} \).

Let \( V := \mathbb{C}^n = V_1 \oplus \cdots \oplus V_r \) denote a maximal decomposition into (non trivial) complex subspaces invariant under \( \mathfrak{g} \). This decomposition is also invariant under the action of \( G \). It suffices to prove the theorem for the restriction of \( G \) to each \( V_j \). In other words we may suppose that \( r = 1 \). Thus \( \mathfrak{g} \) satisfies the conditions (b) and (c) of Proposition 2.1.

If \( \mathfrak{i} \cdot 1_V \in \mathfrak{g} \), then one replaces \( \mathfrak{g} \) by \( \mathfrak{g}^* := \{ g \in \mathfrak{g} | Tr(g) = 0 \} \). The latter is again a real Lie algebra, satisfies (a)\textendash)(c) and moreover \( \mathfrak{g} = \mathfrak{g}^* \oplus \mathfrak{R} \cdot 1 \).

The positive definite Hermitian form of part (3) of Proposition 2.1 has clearly the property \( F(gx, gy) = F(x, y) \) for all \( g \in G \) and \( x, y \in V \).

The general case.

Now \( G \) is a closed subgroup of \( \text{GL}_n(\mathbb{C}) \) (for the ordinary topology) such that every element of \( G \) is semisimple and such that all its eigenvalues have
absolute value 1. Let $G^o$ denote the component of the identity of $G$. According to the previous case, the group $G^o$ is compact.

**Lemma 3.1.** $G/G^o$ is a torsion group, i.e., all its elements have finite order.

**Proof.** Let $g$ be an element of $G$. Choose a basis $e_1, \ldots, e_n$ of eigenvectors of $g$. The group $T$, consisting of all elements $t \in \text{GL}_n(\mathbb{C})$ such that $te_j = c_j e_j$, $|c_j| = 1$ for all $j$, is compact. The topological closure $H \subset \text{GL}_n(\mathbb{C})$ of the group generated by $g$ is a closed subgroup of $T$ and therefore compact. The component of the identity $H^o$ of $H$ has finite index in $H$, since $H$ is compact. Moreover, $H^o \subset G^o$. It follows that the image of $g$ in $G/G^o$ has finite order. \qed

The group $G^o$ is conjugated to a subgroup of $U_n(\mathbb{C})$ and hence compact. One considers the real vector space $\text{Herm}$ consisting of the Hermitian forms $F$ on $V$. The group $G$ acts linearly on $\text{Herm}$ by $(gF)(x,y) := F(gx, gy)$. The real linear subspace $\text{Herm}_{G^o}$ consisting of the $G^o$-invariant Hermitian forms is not 0 and contains in fact a positive definite Hermitian form. The space $\text{Herm}_{G^o}$ is invariant under $G$, since $G^o$ is a normal subgroup of $G$. The action of $G$ on $\text{Herm}_{G^o}$ induces a homomorphism $G \rightarrow \text{GL}(\text{Herm}_{G^o})$ with kernel $G^+$ and image $I$. Since $G^+ \supset G^o$ the group $I$ is a torsion group. $G^+$ leaves a positive definite Hermitian form invariant and is closed. Therefore $G^+$ is compact.

We will need the following classical result and refer to ([Fr], p. 209, or [C-R] p. 252, or [S]) for a proof.

**Lemma 3.2** (Schur’s theorem). Let $H$ be a torsion subgroup of $\text{GL}_n(F)$, for some field $F$. Then:

Any finitely generated subgroup $J$ of $H$ is finite. As a consequence, $H$ is the filtered union of its finite subgroups.

We apply the lemma to $I$. Let $J \subset I$ be a finite subgroup. Its preimage $J^* \subset G$ is compact and the subspace $\text{Herm}_{J^*}$ of the $J^*$-invariant elements of $\text{Herm}$ is not 0 and contains a positive definite Hermitian form. For finite subgroups $J_1 \subset J_2$ of $I$ one has $\text{Herm}_{J_1} \supset \text{Herm}_{J_2}$. Since the spaces $\text{Herm}_{J^*}$ have finite dimension and $I$ is the filtered union of its finite subgroups, there exists a finite subgroup $J_0$ of $I$ such that $\text{Herm}_{J_0} = \text{Herm}_K$, for every finite subgroup $K \subset I$, containing $J_0$. This implies the existence of a positive definite Hermitian form invariant under $G$. 
REFERENCES


Manoscritto pervenuto in redazione il 4 luglio 2005.