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The Birkhoff-Neumann Embedding of Relatively Free Groups (*).

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To Professor Guido Zappa on his 90th birthday

ABSTRACT - Let G_r be the relatively free group of rank r in the variety generated by a finite group H. In this paper we determine the minimum positive integer $t(r)$ for which G_r is embedded in the direct product of $t(r)$ copies of H, when H is a minimal non-abelian group or a dihedral group $\boldsymbol{D}_n$ (n odd). Moreover the finite nilpotent groups of class c are characterized as those finite groups for which $t(r)$ is bounded from above by a polynomial of degree c.

1. Introduction.

Let H be a finite group, and let $\boldsymbol{V} = Var(H)$ be the variety generated by H. For $r \geq 1$, let $G_r = F_r(\boldsymbol{V})$ be the relatively free group in $\boldsymbol{V}$ with r free generators (see [12, p. 9]). By a result of G. Birkhoff [1, p. 441] and B.H. Neumann [11, p. 519] (see also [12, Theorem 15. 71]), we have an embedding

$$G_r \hookrightarrow H^{|H|^r}$$

of G_r as a subgroup of the direct product of $|H|^r$ copies of H.

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In general, much less direct factors of H will suffice to get such an embedding of G_r. For example, let $H = \mathbb{Z}_p$ be a group of prime order p. Then $\boldsymbol{V} = \boldsymbol{A}_p$, where $\boldsymbol{A}_e$ denotes the variety of all abelian groups of exponent dividing e. Moreover, $G_r = \mathbb{Z}_p^r$ and we have $G_r \cong H^r$.

In this paper, we shall consider the function $t(r)$ defined as follows. For every positive integer r, let $t(r)$ be the smallest positive integer t such that the relatively free group G_r of rank r in $\boldsymbol{V}$ can be embedded as a subgroup of H^t, the direct product of t copies of H. By the aforementioned result of Birkhoff-Neumann, $t(r)$ exists, and we have $t(r) \leq |H|^r$. If $H = \mathbb{Z}_p$ for a prime p, the above shows $t(r) = r$.

Note that the function t depends on the group H which generates $\boldsymbol{V}$, and it is not a function of $\boldsymbol{V}$ alone. Indeed, we can have $Var(H_1) = Var(H_2)$, but the corresponding functions t_1, t_2 differ (see the example below).

Let H be a finite abelian group of exponent e, and set $\boldsymbol{V} = Var(H)$. Then $\boldsymbol{V} = \boldsymbol{A}_e$ and $G_r = F_r(\boldsymbol{V})$ is homocyclic of exponent e and rank r. So $t(r) = \overline{e}\left(\dfrac{r}{\lambda(H)}\right)$ where $\overline{e}(x)$ denotes the minimum integer $\geq x$ and $\lambda(H)$ is the number of invariant factors of H whose order is equal to the exponent of H (see [3, p. 245]).

In this note we determine the structure of the relatively free groups in $Var(H)$ and the precise values of $t(r)$ for the minimal non-abelian finite groups H and the dihedral group $\boldsymbol{D}_n$ (n odd). In the case of the dihedral group $H = \boldsymbol{D}_p$, p prime and $r = 2$, this was already done by B. Fine [4].

The following theorem was suggested by an analysis of the material mentioned above.

THEOREM. *Let H be a finite group. Then H is nilpotent of class $\leq c$ if and only if $t(r)$ is bounded from above by a polynomial of degree $\leq c$ in r.*

The proof of this, as we will see in Section 5, can be obtained as a consequence of results of Higman [6].

2. The free groups in Var (H), with H a minimal non-abelian p-group.

In this chapter we study the variety $\boldsymbol{V}$ generated by a finite minimal non-abelian p-group H, that is a finite non-abelian p-group H such that every proper subgroup and every proper homomorphic image of H is abelian.

From [8, p. 309] we can deduce easily the following

PROPOSITION 1. *Let H be a minimal non-abelian p-group. Then one of the following holds:*

i) $H = Q_8$ *is the quaternion group of order* 8.
ii) $H = H_\alpha \cong \langle a, b | a^{p^\alpha} = b^p = 1, a^b = a^{1+p^{\alpha-1}} \rangle$ *for some* $\alpha \geq 2$.
iii) H *is of exponent* p *and order* p^3.

PROPOSITION 2. *Let H be a minimal non-abelian p-group and let* $\boldsymbol{V} = Var(H)$. *If* $exp(H) = p^\alpha$, *then*

$$\boldsymbol{V} = Var(x^{p^\alpha}, [x^p, y], [x, y, z]).$$

PROOF. Let $\boldsymbol{W} = Var(x^{p^\alpha}, [x^p, y], [x, y, z])$. Clearly $\boldsymbol{V} \subseteq \boldsymbol{W}$. Note that $\boldsymbol{W}$ is nilpotent of finite exponent, so it is locally finite. If the above inclusion is proper, we can choose a finite group G in $\boldsymbol{W} \setminus \boldsymbol{V}$ of minimal possible order. By minimality of G we have that G is critical. Therefore G is a p-group generated by two elements with cyclic centre (see [12, Theorem 51.35]). By the laws of the variety $\boldsymbol{W}$ we have that G is of class two, G' is of exponent p and $G' \leq Z(G)$. Also $G^p \leq Z(G)$. So G' is of order p and $\Phi(G) \leq Z(G)$. Since G is generated by two elements, we have $\Phi(G) = Z(G)$ and the index of $Z(G)$ in G has to be p^2. It follows that $Z(G)$ has index p in every maximal subgroup of G. So every maximal subgroup of G is abelian. Moreover, since $Z(G)$ is cyclic and $|G'| = p$, also every proper homomorphic image of G is abelian. Thus G is minimal non-abelian of exponent p^γ with $\gamma \leq \alpha$.

If $\alpha = 1$, then G is of exponent p. Proposition 1 implies $G = H \in \boldsymbol{V}$, a contradiction.

If $H = H_\alpha$, one of the following holds:

a) $G = H_\gamma$, with $2 \leq \gamma \leq \alpha$.
b) G is of exponent p.
c) $G = Q_8$.

For a) we shall prove that all the groups $G = H_\gamma$, with $2 \leq \gamma \leq \alpha$, belong to the variety generated by H_α. For this it is sufficient to show that $H_{\alpha-1} \in Var(H_\alpha)$ for $\alpha \geq 3$. In fact $H_{\alpha-1}$ is isomorphic to a subgroup of $L = (H_\alpha \times \langle c \rangle)/\Delta$, where c is of order p^α and $\Delta = \langle a^{p^{\alpha-2}(p-1)} \cdot c^{-p^{\alpha-2}} \rangle$. For this note that the elements $x = \overline{a}\,\overline{c}$, $y = \overline{b}$ with $\overline{a} = a\Delta$, $\overline{b} = b\Delta$, $\overline{c} = c\Delta$ satisfy the following relations:

$$x^{p^{\alpha-1}} = (\overline{ac})^{p^{\alpha-1}} = (\overline{a}^{p^{\alpha-2}} \overline{c}^{p^{\alpha-2}})^p = (\overline{a}^{p^{\alpha-1}})^p = \overline{a}^{p^\alpha} = 1;$$

$$y^p = \overline{b}^p = 1;$$

$$[x, y] = [\overline{a}\,\overline{c}, \overline{b}] = [\overline{a}, \overline{b}] = \overline{a}^{p^{\alpha-1}} = \overline{a}^{p^{\alpha-2}} \overline{c}^{p^{\alpha-2}} = x^{p^{\alpha-2}}.$$

For b) we can repeat the above construction to show that G is isomorphic to a subgroup of
$$L = (H_2 \times \langle c \rangle)/\Delta,$$
where c is of order p^2 and $\Delta = \langle a^p c^p \rangle$, and x, y as above.

For c) an analogous argument shows that G is isomorphic to a subgroup of $L = (H_2 \times \langle c \rangle)/\Delta$, where c is of order 4, $\Delta = \langle a^2 c^2 \rangle$ and $x = \overline{a}$, $y = \overline{b}\,\overline{c}$. Thus $G \in Var(H_\alpha) = \boldsymbol{V}$, a contradiction.

Finally, let $H = Q_8$. If $G = Q_8$, then $G \in \boldsymbol{V}$, and we have a contradiction as above. Otherwise $G = H_2$ is the dihedral group of order 8. In this case we can repeat the above construction with $L = (Q_8 \times \langle c \rangle)/\Delta$, where c is of order 4, $\Delta = \langle a^2 c^2 \rangle$, $x = \overline{ac}$ and $y = \overline{bc}$. So we arrive at the same contradiction $G \in \boldsymbol{V}$. $\square$

PROPOSITION 3. *Let H be a minimal non-abelian p-group of exponent p^α, say, and $\boldsymbol{V} = Var(H)$. Let $G_r = F_r(\boldsymbol{V})$ be the free group of rank r in $\boldsymbol{V}$. Then we have*

a) G_r/G_r' is homocyclic of exponent p^α and rank r.

b) G_r' is elementary abelian of rank $\dfrac{1}{2}r(r-1)$.

c) $|G_r| = p^{\frac{1}{2}r(r+2\alpha-1)}$.

PROOF. Let $G = G_r$. Part a) follows from [12, Theorem 13.52]. For b), note that G is of class two, and hence, in the case $\exp(H) = p^\alpha$, we have $[x, y]^p = [x^p, y] = 1$ for all $x, y \in G$. So, G' is elementary abelian. As $d(G) = r$, we have that G' is of rank $\binom{r}{2} = \dfrac{1}{2}r(r-1)$. Part c) now follows from a) and b). $\square$

THEOREM 4. *Let H be a minimal non-abelian p-group, let $\boldsymbol{V} = Var(H)$, and let $G_r = F_r(\boldsymbol{V})$ the free group of rank $r \geq 2$ in $\boldsymbol{V}$.*

a) If $exp(H) = p^\alpha$, $\alpha \geq 1$, then $G_r = [N]L$ where
$$N = \langle n_1, n_2, \ldots, n_r \rangle \cong \mathbb{Z}_{p^\alpha} \times \underbrace{\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{r-1},$$
and L is isomorphic to the free group $F_{r-1}(\boldsymbol{V})$ of rank $r-1$ in $\boldsymbol{V}$.

b) $[N, L'] = 1$.

c) If $L = \langle x_1, \ldots, x_{r-1} \rangle$ with free generators $x_1, \ldots, x_{r-1}$, then the action of x_i on N is given by
$$\begin{aligned} n_1^{x_i} &= n_1 n_{i+1} \quad (i = 1, \ldots, r-1) \\ n_j^{x_i} &= n_j \qquad\quad (i = 1, \ldots, r-1;\ 2 \leq j \leq r). \end{aligned}$$

d) Let $x_r = n_1$. Then the elements $x_1, \dots, x_r$ are free generators of G_r.

PROOF. Let $G = [N]L$ with N and L as described in the statement of the theorem. We show that $G \in \boldsymbol{V}$ and $d(G) \leq r$. Then G is an epimorphic image of $F_r(\boldsymbol{V})$. On the other hand it follows from Proposition 3c) and from the definition of G that $|F_r(\boldsymbol{V})| = |G|$. From this we then can conclude $G \cong F_r(\boldsymbol{V})$.

We first show $d(G) \leq r$. Clearly $G = \langle n_1, \dots, n_r, x_1, \dots, x_{r-1} \rangle$. The relations of Part c) yield:

$$[n_1, x_1] = n_2,\ [n_1, x_2] = n_3, \dots, [n_1, x_{r-1}] = n_r.$$

So $n_2, \dots, n_r \in \langle n_1, x_1, x_2, \dots, x_{r-1} \rangle$, and we get $G = \langle n_1, x_1, x_2, \dots, x_{r-1} \rangle$.

Now we check that G satisfies the identities of $\boldsymbol{V}$. First, we show that G is of class two. Write automorphisms of N as $r \times r$-matrices in the obvious way. Note that L acts on N as the subgroup

$$A = \left\{ \begin{pmatrix} 1 & a_2 & \dots & a_r \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \middle|\ a_2, \dots, a_r \in \mathbb{Z}_p \right\}$$

of $Aut(N)$. Hence A is abelian, and so L' centralizes N. Since $G = [N]L$, we have $G' = [N, L]L'$. By the definition of the action of L on N, we have that $[N, L] \leq C_G(L)$. Since N is abelian and $[N, L] \leq N$, we get $[N, L] \leq Z(G)$. Therefore $G' \leq Z(G)$ and so G is nilpotent of class two.

We now check the law $x^{p^\alpha} = 1$. First, note that $[N, L] = \langle n_2, \dots, n_r \rangle$ is elementary abelian. As G is of class two, for all $n \in N$, $x \in L$ we have $(nx)^{p^\alpha} = n^{p^\alpha} x^{p^\alpha} [x, n]^{p^\alpha(p^\alpha - 1)/2} = 1$.

It remains to show that $[x^p, y] = 1$ is a law in G. Since G is nilpotent of class two, the law is equivalent to $[x, y]^p = 1$. Therefore it is sufficient to show that G' is elementary abelian. We have $G' = [N, L] \oplus L'$, where $[N, L]$ is elementary abelian by a previous observation and L' is elementary abelian by Proposition 3b).

For d) observe that, since G is the free group of rank r in $\boldsymbol{V}$ and G is finite, then the generators $x_1, \dots, x_{r-1}, x_r$ freely generate G. $\square$

LEMMA 5. *Let H and $\boldsymbol{V}$ be as in Theorem 4. Then we have*

a_1) If $exp(H) = p$, then $t(r) \geq \dfrac{1}{2} r(r-1)$.

a_2) If $exp(H) = p^\alpha$, with $\alpha \geq 2$, then $t(r) \geq \dfrac{1}{2} r(r+1)$.

PROOF. Let $G = G_r$ and $D = H^{t(r)}$. By Proposition 3b), we have that G' is elementary abelian of order $p^{\frac{1}{2}r(r-1)}$. As $D' \cong \mathbb{Z}_p^{t(r)}$ we have that

$$0 \leq rank \frac{G \cap D'}{G'} \leq t(r) - \frac{1}{2}r(r-1). \tag{1}$$

So in both cases

$$t(r) \geq \frac{1}{2}r(r-1).$$

Suppose now that $\exp(H) = p^\alpha$ with $\alpha \geq 2$. Note that

$$\Omega_1(G/G') \leq (G \cap D')/G'.$$

As $G/G' \cong \mathbb{Z}_{p^\alpha}^r$, this yields that $(G \cap D')/G'$ is of rank greater or equal to r. Using (1), this implies

$$t(r) - \frac{1}{2}r(r-1) \geq rank(G \cap D'/G') \geq r,$$

so that

$$t(r) \geq \frac{1}{2}r(r+1). \qquad \square$$

LEMMA 6. *Let H be a minimal non-abelian p-group.*

a_1) *If $exp(H) = p$, then $t(2) = 1$.*
a_2) *If $exp(H) = p^\alpha$, $\alpha \geq 2$, then $t(2) = 3$.*

PROOF. Let $G = F_2(\boldsymbol{V})$. If $\exp(H) = p$, we have $G = H$ by Proposition 3. Suppose $\exp(H) = p^\alpha$ with $\alpha \geq 2$. By Lemma 5 it suffices to show that $G \hookrightarrow H \times H \times H$. For this we observe that by Proposition 3a) we have $G/G' \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\alpha}$. So $G/G' \hookrightarrow H \times H$. Now we consider a normal subgroup K of G such that $G/K \cong H$ and $K \cap G' = 1$. Indeed, using the notation of Theorem 4 we choose $K = \langle n_1^2 n_2,\, x_1^2 n_2 \rangle$ if H is the quaternion group, and we set $K = \langle x_1^p,\, n_2^{-1} n_1^{p^{\alpha-1}} \rangle$ otherwise. Since $G' = \langle n_2 \rangle$ is of order p and $n_2 \notin K$, we have $K \cap G' = 1$. Hence $G = G/(K \cap G') \hookrightarrow H \times H \times H$. $\square$

THEOREM 7. *Let H and $\boldsymbol{V}$ be as in Theorem 4 and set $G_r = F_r(\boldsymbol{V})$.*

a_1) *If $exp(H) = p$, then $t(r) = \frac{1}{2}r(r-1)$.*
a_2) *If $exp(H) = p^\alpha$, $\alpha \geq 2$, then $t(r) = \frac{1}{2}r(r+1)$.*

PROOF. Let $G = G_r$ and $x_1, x_2, \ldots, x_r$ be free generators of G_r, as in Theorem 4d). Set $y_{i,j} = [x_i, x_j]$ for $i > j$. Observe that, with the notation of

Theorem 4, we have

$$n_1 = x_r$$

and

$$n_i = y_{r,i-1} \quad (i = 2, \ldots, r).$$

Then the elements x_i $(i = 1, \ldots, r)$ and the $y_{i,j}$ satisfy the following conditions:

$$x_i^p = y_{i,j}^p = [x_i, y_{l,k}] = 1$$

where $2 \leq l \leq r,\ 1 \leq k \leq l-1$, in the case of $\exp(H) = p$; and

$$x_i^{p^\alpha} = y_{i,j}^p = [x_i, y_{l,k}] = 1$$

where $2 \leq l \leq r,\ 1 \leq k \leq l-1$, in the case of $\exp\,(H) = p^\alpha,\ \alpha \geq 2$.

By Theorem 4, we have that G is a split extension $G = [N]L$ where $L = G_{r-1}$ is the free group of rank $r-1$ in the free generators $x_1, x_2, \ldots, x_{r-1}$ and

$$N = \langle x_r \rangle \times \langle y_{r,1} \rangle \times .. \times \langle y_{r,r-1} \rangle.$$

Moreover we have that L' centralizes N.

By Lemma 6, the claim is true for $r = 2$. So let $r \geq 3$. First consider the case in which H is of exponent p.

For $k = 1, \ldots, r-1$ set

$$S_k = \langle L',\, X_k,\, Y_k \rangle,$$

where

$$X_k = \{x_1, \ldots, x_{r-1}\} \setminus \{x_k\}; \quad Y_k = \{y_{r,j} \mid 1 \leq j < r\} \setminus \{y_{r,k}\}.$$

Observe that S_k is normal in G. In fact

$$N \leq C_G(L');\ \ X_k^{x_i} \subseteq L'X_k\ (i = 1, \ldots, r-1);\ \ X_k^{x_r} \subseteq X_k Y_k.$$

Moreover G/S_k is a non-abelian group of class 2 and exponent p on two generators. So $G/S_k \cong H$.

Now

$$\Big(\bigcap_{k=1}^{r-1} S_k\Big) \cap N = \bigcap_{k=1}^{r-1} (S_k \cap N)$$

Since $\langle L', X_k \rangle \subseteq L$ and L normalizes $\langle Y_k \rangle$ we have

$$\bigcap_{k=1}^{r-1} (S_k \cap N) = \bigcap_{k=1}^{r-1} \langle Y_k \rangle = 1.$$

By induction, $G/N \cong L$ is embeddable in the direct product of $\frac{(r-1)(r-2)}{2}$ copies of H. Hence G is embeddable in the direct product of $\frac{(r-1)(r-2)}{2} + (r-1) = \frac{r(r-1)}{2}$ copies of H.

For the other cases we can proceed in a similar way. If H is a minimal non-abelian p-group of order $p^{\alpha+1}$ and exponent p^α other than quaternion group, for $k = 1, \dots, r-1$ we choose

$$X_k = \{x_1, \dots, x_{k-1}, {x_k}^p, x_{k+1}, \dots, x_{r-1}\};$$
$$Y_k = \{y_{r,1}, \dots, y_{r,k-1}, y_{r,k}^{-1}\, x_r^{p^{\alpha-1}}, y_{r,k+1}, \dots, y_{r,r-1}\}$$

Moreover we set

$$X_r = \{x_2, \dots, x_{r-1}, x_r^p\}; \quad Y_r = \{y_{r,1}^{-1}\, x_1^{p^{\alpha-1}}, y_{r,2}, \dots, y_{r,r-1}\}.$$

If H is the quaternion group, for $k = 1, \dots, r-1$ we put

$$X_k = \{x_1, \dots, x_{k-1}, {x_k}^2 x_r^2, x_{k+1}, \dots, x_{r-1}\};$$
$$Y_k = \{y_{r,1}, \dots, y_{r,k-1}, y_{r,k}\, x_r^2, y_{r,k+1}, \dots, y_{r,r-1}\}$$

and we set

$$X_r = \{x_1, \dots, x_{r-1}, x_1^2\, x_r^2\}; \quad Y_r = \{y_{r,1}\, x_1^2, y_{r,2}, \dots, y_{r,r-1}\}.$$

In both cases we obtain as above

$$\Big(\bigcap_{k=1}^{r} S_k\Big) \cap N = \bigcap_{k=1}^{r-1} (S_k \cap N) \cap \langle Y_r \rangle = \bigcap_{k=1}^{r} \langle Y_k \rangle = 1.$$

Hence G is embeddable in the direct product of $\frac{r(r-1)}{2} + r = \frac{r(r+1)}{2}$ copies of H. $\square$

3. The free groups in the variety generated by a minimal non-nilpotent group.

In this section we will use the following well known results:

LEMMA 8. *Let H be a minimal non-nilpotent group (that is a finite group H such that every proper subgroup and every proper homomorphic image of H is nilpotent). Then $H = [N]Q$ is a Frobenius group in which Q*

is cyclic of prime order q and N is an irreducible $\mathbb{Z}_pQ$-module, for some prime $p \neq q$, on which Q acts faithfully.

LEMMA 9. *Let H be as in Lemma 8. We have $\boldsymbol{V} = Var(H) = \boldsymbol{A}_p\boldsymbol{A}_q$. If $G_r = F_r(\boldsymbol{V})$ then $|G_r| = p^{1+(r-1)q^r} q^r$.*

PROOF. See [7, p. 175 f.] and [12, 21.13].

THEOREM 10. *Let $H = [N]Q$ be a Frobenius group as in the beginning of this section, and assume that N is elementary abelian of order p^n. Let G_r be the free group of rank r in Var(H). Then $t(r) = \dfrac{(r-1)(q^r-1)}{n} + \bar{e}\left(\dfrac{r}{n}\right)$, where $\bar{e}\left(\dfrac{r}{n}\right)$ denotes the smallest integer $\geq \dfrac{r}{n}$.*

PROOF. Let $t = t(r)$. Clearly $|G_r|$ must divide $|H|^t = p^{nt}q^t$. As $|G_r| = p^{1+(r-1)q^r} q^r$, comparing exponents, the latter condition implies

$$1 + (r-1)q^r \leq nt \quad \text{and} \quad r \leq t. \tag{2}$$

Now

$$r \leq 1 + q + \cdots + q^{r-1} = \frac{q^r - 1}{q-1} \leq \frac{q^r}{q-1};$$

but, by Fermat's Theorem, n divides $q - 1$, so

$$\frac{q^r}{q-1} \leq \frac{q^r}{n}.$$

It turns out that, for $r \geq 2$ the following holds:

$$r \leq \frac{q^r}{n} \leq \frac{1 + (r-1)q^r}{n}.$$

Therefore the two conditions (2) for $r \geq 2$ are equivalent to:

$$t \geq \frac{1 + (r-1)q^r}{n} = \frac{(r-1)(q^r-1)}{n} + \frac{r}{n}.$$

Now, since n divides $q - 1$, n divides also $(r-1)(q^r-1)$. Thus t and $\dfrac{(r-1)(q^r-1)}{n}$ are integers, so

$$t \geq \frac{(r-1)(q^r-1)}{n} + \bar{e}\left(\frac{r}{n}\right).$$

By [2, p. 129], we have $G_r \cong \mathbb{Z}_p \times \left(\mathbb{Z}_p^{r-1} wr \mathbb{Z}_q^r\right)$. Splitting off the center of the wreath product, we arrive at a decomposition $G_r = E \times W_0$, where $E \cong \mathbb{Z}_p^r$, and W_0 by [13, Theorem 6.1] is directly indecomposable. Obviously, E is isomorphic to a subgroup of $H^{\bar{e}(\frac{r}{n})}$. So it suffices to show that W_0 is embeddable in H^l, where $l = \dfrac{(r-1)(q^r-1)}{n}$. Indeed, in this case, G_r is isomorphic to a subgroup of $H^{l+\bar{e}(\frac{r}{n})}$. That is

$$t \leq \frac{(r-1)(q^r-1)}{n} + \bar{e}\Big(\frac{r}{n}\Big),$$

and, comparing with the above relation for t, we obtain the theorem.

Now $W_0 = [M]Q^r$, where M is elementary abelian of order $p^{(r-1)(q^r-1)}$. Thus M is decomposable into $l = \dfrac{(r-1)(q^r-1)}{n}$ irreducible Q^r-modules M_i $(1 \leq i \leq l)$.

Let K_i be the kernel of the action of Q^r on M_i, then $|K_i| = q^{r-1}$. Moreover the subgroup

$$R_i = \Big(\bigoplus_{j\neq i} M_j\Big)K_j$$

is normal in W_0, and we have $W_0/R_i \cong H$ for all indices i. We claim that

$$D = \bigcap_{i=1}^{l} R_i = 1;$$

consequently W_0 is isomorphic to a subgroup of H^l. For this we first show that D is a q-group. Let $x \in D$ be a p-element. Then $x \in R_i$ for all i, and so $x \in \bigoplus_{j\neq i} M_j$. But the intersection of these groups is trivial, hence $x = 1$. So D is a normal q-subgroup of W_0 and, since W_0 is indecomposable, we arrive at $D = 1$ as claimed. □

4. The free groups in Var ($\boldsymbol{D}_n$), n odd.

Let $G_r = F_r(Var(\boldsymbol{D}_n))$ be the relatively free group of rank $r \geq 1$ in the variety generated by the dihedral group $\boldsymbol{D}_n$ where n is odd. Then $Var(\boldsymbol{D}_n) = \boldsymbol{A}_n\boldsymbol{A}_2$ (see [9]). We have $G_r = [N]Q$ where, by the theorem of Nielsen-Schreier, a minimal set of generators of N consists of $r' = (r-1)2^r + 1$ elements and Q is an elementary abelian group of rank r. By [12, 21.13] we have that $|G_r| = 2^r n^{(r-1)2^r+1}$ and in particular N is homocyclic.

LEMMA 11. *Let $G_r = F_r(Var(\boldsymbol{D}_n))$. Then N is the direct product of r Q-modules each of which is homocyclic, $r-1$ are regular modules (i.e. Q acts regularly on a minimal set of generators) and one is a trivial Q-module.*

PROOF. By the preliminaries we have that N is homocyclic of order $n^{r'}$ (using the above notation). Let $\{p_1, \ldots, p_s\}$ be the set of all distinct prime divisors of n with $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ and $P_i \in Syl_{p_i}(N)$ where $1 \le i \le s$. Then Q acts on each P_i. let $\overline{P_i} = P_i/\Phi(P_i)$. By [5] $\overline{P_i}$ is the direct sum of $r-1$ regular $\boldsymbol{F}_{p_i}Q$-modules and a trivial $\boldsymbol{F}_{p_i}Q$-module. Let $\overline{P_{i,j}}$ a regular direct factor of $\overline{P_i}$ with $1 \le j \le r-1$ and $\overline{P_{i,r}}$ the trivial direct factor. Denote by $\{\overline{v}_1^{i,j}, \ldots, \overline{v}_{2^r}^{i,j}\}$ a basis of $\overline{P_{i,j}}$ on which Q acts regularly, such that $\overline{v}_r^{i,j} = \overline{v}_1^{i,j} a_r$ with $a_r \in Q$, $1 \le r \le 2^r$ and $a_1 = 1$. Let $v_1^{i,j} \in P_i$ with $v_1^{i,j}\Phi(P_i) = \overline{v_1}^{i,j}$ and $v_r^{i,j} = v_1^{i,j} a_r$ with $1 \le i \le 2^r$. Moreover let $P_{i,j} = \langle v_r^{i,j} | 1 \le r \le 2^r \rangle$. Then $P_{i,j}/\Phi(P_{i,j}) = \overline{P_{i,j}}$ and $P_{i,j} = \langle v_1^{i,j} \rangle \times \ldots \times \langle v_{2^r}^{i,j} \rangle$. For, $P_{i,j}\Phi(P_i)/\Phi(P_i) \cong P_{i,j}/(P_{i,j} \cap \Phi(P_i))$ and $\Phi(P_{i,j}) \le P_{i,j} \cap \Phi(P_i)$. Then $\overline{P_{i,j}} \cong \dfrac{P_{i,j}/\Phi(P_{i,j})}{(P_{i,j} \cap \Phi(P_i))/\Phi(P_{i,j})}$ and $P_{i,j}/\Phi(P_{i,j})$ cannot have dimension less than 2^r which is the dimension of $\overline{P_{i,j}}$, It follows that $\overline{P_{i,j}} = \dfrac{P_{i,j}}{\Phi(P_{i,j})}$. On the other hand, since $\{\overline{v}_r^{i,j} | 1 \le r \le 2^r\}$ is a basis for $\overline{P_{i,j}}$, we have that $\{v_r^{i,j} | 1 \le r \le 2^r\}$ is a minimal set of generators of $P_{i,j}$. Moreover $P_i = P_{i,1}P_{i,2}\ldots P_{i,r}$. Then $|P_i| = |P_{i,1}P_{i,2}\ldots P_{i,r}| = (p_i^{\alpha_i})^{2^r(r-1)+1}$. Since $|P_{i,j}| \le (p_i^{\alpha_i})^{2^r}$ we must have $P_{i,l} \cap P_{i,m} = 1$ for all indices $1 \le l < m \le r$ and $|P_{ij}| = (p_i^{\alpha_i})^{2^r}$ for $1 \le j \le r-1$. In particular $P_{i,j}$ is homocyclic for all j with $1 \le j \le r$ and $P_i = P_{i,1} \times \ldots \times P_{i,(r-1)} \times P_{i,r}$. Let c_i be an element of P_i such that $\overline{c_i} = c_i\Phi(P_i)/\Phi(P_i)$ is a generator of $P_{i,r}$ which is centralized by Q. Then $P_{i,r}$ is fixed by Q. The elements

$$\{v_1^{1,j} \cdots (v_r^{2,j}) \cdots v_r^{s,j} \,|\, 1 \le j \le r-1 \ \text{ and } \ 1 \le r \le 2^r\}$$

and the element $c_1 \cdots c_s$ are generators of N, and Q acts regularly on the set of elements

$$v_1^{1,j} \cdots v_1^{s,j}, \ldots, v_{2^r}^{1,j} \cdots v_{2^r}^{s,j},\ 1 \le j \le r-1. \qquad \square$$

LEMMA 12. *Let $G_r = [N]Q$ the relatively free group of rank r in the variety $Var(\boldsymbol{D}_n)$ with n odd. Then there is a base $\mathcal{B} = \{v_1, v_2, \ldots, v_{r'}\}$ of N such that every element of Q fixes $\dfrac{r'+1}{2}$ elements of $\mathcal{B}$ and inverts the other $\dfrac{r'-1}{2}$ elements.*

PROOF. By the previous lemma, we can restrict attention to a module N

on which Q acts regularly. We have $|Q| = 2^r$. We proceed by induction on r. First of all we see that the result is true for $r = 1$. For, let $\mathcal{B} = \{v_1, v_2\}$ a base of N on which $Q = \langle a \rangle$ ($|Q| = 2$) acts regularly. So $v_1 a = v_2$ and $v_2 a = v_1$. Consider the elements $v_1 v_2$ and $v_1 v_2^{-1}$. Then $\langle v_1 v_2,\, v_1 v_2^{-1} \rangle = N$. We have $(v_1 v_2)a = v_2 v_1$ and $(v_1 v_2^{-1})a = v_2 v_1^{-1} = (v_1 v_2^{-1})^{-1}$. Now let Q be an elementary abelian group of order 2^r ($r > 1$) which acts regularly on a base $\mathcal{B}$ of a homocyclic group N. Let $\mathcal{B} = \{v_1, \dots, v_{2^r}\}$. We can assume $v_i = v_1 a_i$ with $a_i \in Q$ and $a_1 = 1$. Moreover, we can order the elements of Q so that the set of elements $\{a_1, \dots, a_{2^{r-1}}\}$ is a subgroup Q_0 of Q. Then $[N]Q \cong N_0 wr_{Q_0} Q$, is the twisted wreath product of the group N_0 generated by the elements $v_1, \dots, v_{2^{r-1}}$ by the group Q. By induction, we have that N_0 possesses a base $\mathcal{B}_0 = w_1, \dots, w_{2^{r-1}}$ such that every element of Q_0 fixes half of the elements of $\mathcal{B}_0$ and inverts all the others. Now let $\{1, b\}$ be a transversal of Q_0 in Q. We can identify a base of N with the set of ordered pair (w_i, d) where d can be either 1 or b. Consider the set $\mathcal{C}$ of elements $\{(w_i, 1)(w_i, b), (w_i, 1)(w_i, b)^{-1}\}$ with $1 \leq i \leq 2^{r-1}$. Then $\mathcal{C}$ is a base of N. If $a \in Q_0$ then $(w_i, 1)(w_i, b)^{\varepsilon} a = (w_i a, 1)(w_i a, b)^{\varepsilon}$ where ε is either 1 or -1. If $a \in Q \backslash Q_0$ then $a = hb$ with $h \in Q_0$. Then $(w_i, 1)(w_i, b)^{\varepsilon} hb = (w_i h, b)(w_i h, 1)^{\varepsilon}$. So any element of Q fixes half of the elements of $\mathcal{B}$ and inverts the others. □

THEOREM 13. *Let $H = \boldsymbol{D}_n$ with n odd and let G_r be the free group of rank r in Var (H). Then $t(r) = (r-1)2^r + 1$.*

PROOF. Let $t = t(r)$ and $r' = (r-1)2^r + 1$. Clearly $|G_r|$ must divide $|H|^t = 2^t n^t$. As $|G_r| = n^{r'} 2^r$ and $r < r'$ we have, comparing exponents $r' \leq t$.

By Lemma 12 we have $G_r = [N]Q \leq [N]\hat{Q}$ where

$$\hat{Q} \cong \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{r'}$$

acts on N as the group of all diagonal matrices with entries ± 1. It is easy to see that $[N]\hat{Q} \cong (H)^{r'}$. So $t \leq r'$. Hence $t = r' = (r-1)2^r + 1$. □

5. On the growth of the function $t(r)$.

We end this note with the proof of the theorem stated in the introduction.

PROOF OF THE THEOREM (1) Let G_r be the free group in the variety

(1) We thank an unknown referee for greatly improving upon our original result.

$Var(H)$, and let $|G_r| = p_1^{e_1} \cdots p_s^{e_s}$ be the prime decomposition of its order. Assume that we have an embedding $\eta : G \hookrightarrow H^{t(r)}$. Let $\pi_i : H^{t(r)} \to H$ be the i-th projection. Set $\phi_i = \pi_i \eta$, and let $K_i := Ker(\phi_i)$. As η is an embedding, the intersection of all the K_i is the trivial group. By Lagrange's Theorem, already the intersection of at most $e_1 + \ldots + e_s$ of these kernels is trivial. We have

$$f(r) := \log_2 |G_r| = \log_2 p_1^{e_1} \cdots p_s^{e_s} \geq e_1 + \ldots + e_s \,,$$

and hence we get $t(r) \leq f(r)$.

On the other hand $|G_r| \leq |H|^{t(r)}$, so $f(r) \leq t(r) \log_2 |H|$. It follows that if one of $f(r)$ and $t(r)$ is bounded by a polynomial of degree c in r, then so is the other. By Higman's result [6, p. 154] we have that H is nilpotent of class $\leq c$ if and only if $f(r)$ is a polynomial of degree $\leq c$. $\square$

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