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Mutually Permutable Products of two Nilpotent Groups.

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Dedicated to Professor G. Zappa on the occasion of his ninetieth birthday

1. Introduction.

Throughout the paper only finite groups are considered. Two subgroups A and B of a group G are called *mutually permutable* if and only if $AY = YA$ and $XB = BX$ is true for all $Y \subseteq B$ and $X \subseteq A$; they are called *totally permutable* if $XY = YX$ for all $Y \subseteq B$ and $X \leq A$. These two concepts are more general than those of normal products and central products, for instance the dihedral group of order 6 is a product of two nilpotent totally permutable subgroups. In both cases, under some restrictions on the structure of the factors, the product is a supersoluble group (see Carocca [7] or [6, Corollary]), a class of groups considered much earlier by Zappa (see [14] and [15]). Asaad and Shaalan [1] study sufficient conditions for totally and mutually permutable products of two supersoluble subgroups to be supersoluble. More precisely they prove:

Let G be the mutually permutable product of the supersoluble subgroups A and B. If either the product is totally permutable or A or B is nilpotent, then G is supersoluble.

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The above results have been studied within the framework of formation theory. This was the beginning of an intensive study of this kind of products, even in the infinite case (see [3,4,5,8]). Moreover Ballester-Bolinches, Cossey and Esteban-Romero have shown that totally permutable products of nilpotent groups are extensions of an abelian by a nilpotent group (see [2, Theorem 1]). We are interested here to find more information about the structure of the product of two nilpotent mutually permutable subgroups. In particular, if G is the mutually permutable product of two nilpotent subgroups of odd order, then we obtain that G is abelian by nilpotent.

2. Results.

First of all recall that if $\mathcal{N}$ denotes the class of all nilpotent groups, then the $\mathcal{N}$-residual of a group G, denoted by $G^{\mathcal{N}}$, is the intersection of all normal subgroups N of G such that $G/N \in \mathcal{N}$. It is clear that $G/G^{\mathcal{N}} \in \mathcal{N}$. In the following G' will denote the derived subgroup of G and $G^2 = \; = < g^2 : g \in G >$.

LEMMA. *Let $G = AB$ be the mutually permutable product of the nilpotent subgroups A and B. Assume that $G^{\mathcal{N}}$ is a p-group and that $G^{\mathcal{N}} \leq A$. Then G satisfies one of the following statements: (a) G^2 is nilpotent; (b) $G^{\mathcal{N}}$ is abelian.*

PROOF. Assume the result is not true and G is a minimal counterexample, so that G is not nilpotent, $G^{\mathcal{N}} \neq 1$ and p is a prime divisor of G; also, by supersolubility [1], $p > 2$. We begin with two statements and collect the consequences afterwards.

(i) $\boldsymbol{O}_{p'}(G) = 1$.

The conditions of the Lemma carry over to quotient groups. Assume that $\boldsymbol{O}_{p'}(G) = M \neq 1$. Then $M \cap G^{\mathcal{N}} = 1$ and M belongs to the hypercenter of G since M and $MG^{\mathcal{N}}/G^{\mathcal{N}}$ are operator isomorphic. Now the nilpotence of $(G/M)^2$ leads to the nilpotence of G^2M/M and G^2; on the other hand, if $(G/M)^{\mathcal{N}} = (G^{\mathcal{N}} \times M)/M$ is abelian, so is $G^{\mathcal{N}}$. Thus the statement is true by minimality of G.

In particular, the Fitting subgroup $\boldsymbol{F}(G)$ of G is a p-group and p is the biggest prime dividing the order of G.

(ii) $\boldsymbol{O}^p(G) = G$.

Let T be any normal subgroup of G; by [6], Lemma 1 (i),(ii) we have that $(T \cap A)(T \cap B)$ is a normal subgroup of G and that it is the product of the two mutually permutable subgroups $T \cap A$ and $T \cap B$. Furthermore $|G : (T \cap A)(T \cap B)|$ is a divisor of $|G : T|^2$. Applying this to the smallest normal subgroup of p-power index $\boldsymbol{O}^p(G) = K$, we have $K = (K \cap A)(K \cap B)$, and for $K \neq G$ we find that the Lemma is true for K. If K^2 is nilpotent, so is $G^2 = K^2\boldsymbol{F}(G)$; likewise $K^{\mathcal{N}} = G^{\mathcal{N}}$ and if one is abelian, the other is, too. Again minimality of G yields the result.

Now we collect the consequences: $\boldsymbol{F}(G)$ is the Sylow p - subgroup P of G by (i), furthermore $\boldsymbol{F}(G) = G^{\mathcal{N}}$ by (ii) and therefore $\boldsymbol{F}(G) = G^{\mathcal{N}} = G'$. Since $C_G(\boldsymbol{F}(G)) \subseteq \boldsymbol{F}(G)$ and $G^{\mathcal{N}} \subseteq A$, we obtain that A does not possess elements of order prime to p, so $A = G^{\mathcal{N}} = P$ and $G = AH$ where H is the Hall p' - subgroup of B. Moreover $G/A \cong H$ is abelian of exponent dividing $p-1$.

(iii) $[A, H] = A$.

This follows from $A = [G, A] = [H, A]A' \subseteq [H, A]\Phi(A)$.

(iv) $A \cap B \subseteq A'$.

By [10, Proposition I(1.7), p. 206] applied to A/A' and H we obtain $A/A' = ([A, H]A'/A')C_{A/A'}(H)$ with trivial intersection of the factors since the p-group is abelian. Now (iii) yields $(B \cap A)A'/A' \subseteq C_{A/A'}(H) = 1$ and this is (iv).

Let $T = [A', A]$. We have the following strengthening of (iv):

(v) $(A \cap B)T = A'$.

First, $(A \cap B)T$ is normalized by H and by A. Every subgroup U with $(A \cap B)T \subseteq U \subseteq A$ is normalized by B since $U = UB \cap A$. So H induces power automorphisms on $A/(A \cap B)T$. By a result of Cooper (see [9, Corollary 2.2.2, p. 339]) we have $[A, [A, H]] \subseteq (A \cap B)T$. Now (v) follows from (iii) and (iv).

(vi) If A is nonabelian, then G^2 is nilpotent.

Choose an element $x \neq 1$ of H, it induces a non-identity automorphism on A/A'. Since all subgroups D with $A' \subseteq D \subseteq A$ are normalized by B and therefore by x, there is a number s such that $x^{-1}uxA' = u^sA'$ for all $u \in A$; also $x^{-1}[u_1, u_2]xT = [u_1^s, u_2^s]T = [u_1, u_2]^{s^2}T$ for all $u_1, u_2 \in A$. But x centralizes A'/T, so $s^2 \equiv 1$ mod. $\exp(A'/T)$ and $x^{-1}uxA' = u^{-1}A'$ for all $u \in A$. Now $[x^2, A] \subseteq A'$; by a famous result of P. Hall [12, Theorem 7] we have: If the normal subgroup M of the group T is nilpotent and T/M' is nilpotent, then T is nilpotent. Applying this to A and $\langle x^2, A\rangle$ yields that $\langle x^2, A\rangle$ is nilpotent and in particular the p' - element x^2 centralizes A. So $x^2 = 1$ and $H^2 = 1$ since $x \in H$ was arbitrary. Therefore $G^2 = AH^2 = A$ is nilpotent.

We have shown that there is no minimal counterexample. Now the proof is complete.

Notice that under the conditions of the Lemma the following is always true:

(+) $G^{\mathcal{N}} \cap B = 1$ if and only if $G^{\mathcal{N}}$ is abelian.

We are now in the position to prove our

THEOREM. *Let $G = AB$ be the mutually permutable product of A and B with A and B nilpotent. Then $(G^2)^{\mathcal{N}}$ is abelian. In particular, if G is of odd order we obtain $(G^{\mathcal{N}})$ is abelian.*

PROOF. Assume the result is not true and that G is a minimal counterexample. Again the result of Asaad and Shaalan [1] yields that G is supersoluble. So we have that $G^{\mathcal{N}} \subseteq G' \subseteq \boldsymbol{F}(G)$ and $P \subseteq \boldsymbol{F}(G)$ if P is the Sylow p-subgroup of G with p the largest prime divisor of the order of G. Clearly, every quotient group G/N of G inherits the hypothesis. We deduce at once

(i) G possesses only one minimal normal subgroup.

Assume that N_1 and N_2 are two different minimal normal subgroups, then $((G^2)^{\mathcal{N}})' \subseteq N_i$ for $i = 1, 2$ and $((G^2)^{\mathcal{N}})' \subseteq N_1 \cap N_2 = 1$.

By (i), we have now that $P = \boldsymbol{F}(G)$ and therefore $G^{\mathcal{N}} \subseteq G' \subseteq \subseteq P \subset G^2 \subseteq G$.

It is a celebrated result of Wielandt (see [13, Satz 4.6, p. 676]) that in the product of two nilpotent groups X, Y the corresponding Hall π-subgroups permute with each other and that their product is a Hall π-subgroup of XY, for every set of primes π. In our case we have $P = (P \cap A)(P \cap B)$ and the product of the two p-complements of A and B respectively gives rise to a p-complement Q of G, with $(Q \cap A)(Q \cap B) = Q$, and $G = PQ$. The quotient group $G/P \cong Q$ is abelian by (i) and supersolubility of G.

We study now the subgroup BP. Our intention is to prove that BP satisfies the hypothesis of the Lemma. We consider the intersections of A and B with P and Q and obtain

(ii) $B \cap Q$ normalizes $A \cap P$.

The subgroup $A \cap P$ is the Sylow p-subgroup of $(B \cap Q)A = = (B \cap Q)(A \cap Q)(A \cap P) = Q(A \cap P)$, and (ii) follows.

(iii) $BP = (B \cap Q)P = (A \cap P)B$ satisfies the conditions of the Lemma, taking $(A \cap P)$ and B instead of A and B.

Notice that $[B \cap Q, P]$ is a normal subgroup of $(B \cap Q)P = BP$. Now $[B \cap Q, P] = [B \cap Q, (A \cap P)(B \cap P)] = [B \cap Q, A \cap P] \subseteq A \cap P$ by (ii). But also $(BP)^{\mathcal{N}} \subseteq [B \cap Q, P] \subseteq A \cap P$ (notice that $B \cap Q$ acts on P and then $P = [B \cap Q, P]C_P(B \cap Q)$, which shows that $BP/[B \cap Q, P]$ is nilpotent). By symmetry we have also

(iv) AP satisfies the conditions of the Lemma, taking there $(B \cap P)$ and A instead of A and B.

(v) AP and BP are normal subgroups of G.
This follows since G/P is abelian.

Now we will exclude all possibilities for the counterexample G.

(vi) $(AP)^{\mathcal{N}}$ can not be abelian.
Assume first that the normal subgroup AP of G is nilpotent. This means $AP \subseteq \boldsymbol{F}(G) = P$ and $P = A(B \cap P)$. Let H be the Hall p'-subgroup of B. Then $G^{\mathcal{N}} = [G^{\mathcal{N}}, H] \subseteq [P, H] = [A, H] \subseteq A$. By our Lemma, $G^{\mathcal{N}}$ is abelian or G^2 is nilpotent, so also $(G^2)^{\mathcal{N}}$ is abelian.

Now consider the case that $(AP)^{\mathcal{N}} = W$ is nontrivial and abelian. If $(BP)^2$ is nilpotent, then $1 \subseteq W \subseteq (AP)(BP)^2 \subseteq G$ is a sequence of normal subgroups of G with W abelian, $(AP)(BP)^2/W$ nilpotent, and $G/(AP)(BP)^2$ of exponent 2. So $(G^2)^{\mathcal{N}}$ is abelian. If also $(BP)^{\mathcal{N}}$ is nontrivial and abelian, we have on one hand by (+) and (iv) $W \cap A = 1$ and $W \subseteq B \cap P$, on the other hand by (+) and (iii), $(BP)^{\mathcal{N}} \cap B = 1$ and $(BP)^{\mathcal{N}} \subseteq A \cap P$. Now $(BP)^{\mathcal{N}} \cap W = 1$ contrary to the existence of only one minimal normal subgroup.

We come to the conclusion

(vii) There is no counterexample to the Theorem.
By (vi) we know that the only possibility left is that $(AP)^2$ and $(BP)^2$ are nilpotent. But then $(AP)^2(BP)^2 = G^2$ is nilpotent. This finishes the proof of the Theorem.

The nilpotent residual of G need not be abelian in the situation considered. In the following we give two examples:

Example 1. Consider the group $B = \langle a, b \mid a^3 = [b, a, a], b^3 = [b, a, a]^2, [b, a]^3 = [b, a, a]^3 = [b, a, b] = 1\rangle$. B has order 3^4 and nilpotency class 3.

1. All elements of $B \setminus B'$ have order 9. Notice that $(ab^k)^3 = a^3 b^{3k}[b,a,a]^k = [b,a,a]^m$ with $m = 1 + 2k + k \equiv 1 \bmod o([b,a,a])$. Thus $B^3 = \langle [b,a,a] \rangle$.

2. $B' = \langle [a,b], [b,a,a] \rangle$.

3. Let α be the automorphism of B given by $a^\alpha = a^2$, $b^\alpha = b^2[b,a,a]^2 = b^5$. Note that α defines an automorphism of the group B by realizing that $[a,b]^\alpha = [a^2, b^5] = [a^{-1}, b^{-1}]$ is conjugate to $[a,b]$.

4. Let $d = [b,a][b^{-1}, a^{-1}]$. It is clear by the above remark that $d^\alpha = d$. Therefore we can consider $A = \langle d, \alpha \rangle = \langle d \rangle \times \langle \alpha \rangle$.

Let $G = [B]\langle \alpha \rangle$ be the corresponding semidirect product. The group G is a mutually permutable product of A and B. Moreover, $Core_G(A \cap B) = 1$ and $G^{\mathcal{N}} = B$ is not abelian.

EXAMPLE 2. Let p be an odd prime; denote by P an extraspecial group of order p^{2t+1} with $t \geq 1$ and of exponent p. It is clear that the subgroup $\Phi(P) = P' = Z(P)$ has order p. Moreover P has nilpotency class 2 and the quotient $P/Z(P)$ is elementary abelian. Now the application of [10, 20.8 (b), p. 81] yields that there exists an automorphism α of order 2 of P such that α fixes the elements of $Z(P)$ and induces inversion on $P/Z(P)$.

Consider now $G = [P]\langle \alpha \rangle$ the corresponding semidirect product. Denote $B = P$ and $A = \langle \alpha, Z(P) \rangle$. Then G is the mutually permutable product of A and B both nilpotent. Moreover $A \cap B = Z(P)$ and $G^{\mathcal{N}} = P$ is not abelian.

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