Some Simple Groups which are determined by the Set of their Character Degrees II

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To Guido Zappa on his 90th birthday

In part I of this paper we began to study the following conjecture.

**Conjecture.** Let $H$ be a simple nonabelian group. If $G$ is any group such that $G$ and $H$ have the same set of character degrees of irreducible characters over $\mathbb{C}$, then $G \cong H \times A$, where $A$ is abelian.

In part I we proved this conjecture if $H$ is a Suzuki group $Sz(q)$ or (using an unpublished result by F. Lübeck) if $H = SL(2, 2^f)$ for some $f$. In this paper we study the case $H = PSL(2, p^f)$, where $p$ is odd. As $PSL(2, 5) \cong PSL(2, 4) \cong A_5$, we can assume that $p^f > 5$.

The proof follows the same pattern as in part I. Also references to lemma 1 to lemma 6 refer to part I.

In section 4, we give a proof, which relies on the following lemma.

**Lemma 1.** Let $p$ be a prime. We say that a group $G$ has property $(p)$, if the following holds:

1. If $\chi \in \text{Irr } G$, then $\chi(1)$ is a power of $p$ or prime to $p$.
2. There exists some $\chi \in \text{Irr } G$ such that $\chi(1) = p^d > 1$.

If $G$ is a simple group with property $(p)$, then either $G \cong PSL(2, p^d)$, where $p^d > 3$, or $G \cong PSL(2, 2^f)$, where $p = 2^f - 1$ on $p = 2^f + 1$, or $p = 3$ and $2^f = 8$.

(As a simple group has some even degree, so for $p = 2$ condition (2) can be omitted).

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Proof. a) $\text{PSL}(2, q)$ has the degrees $q$, $q + 1$, $q - 1$ and if $2 \nmid q$ the odd one of $\frac{q + 1}{2}$. The cases, where $q + 1$ on $q - 1$ is a power of some prime, are the cases stated above. If the odd degree $\frac{q + 1}{2}$ is a prime-power $p^a$, then $p^a$ and $2p^a$ are degrees, so condition $(p)$ is not satisfied.

b) We have to show that all other simple groups do not have property $(p)$ for any prime $p$. All quasi-simple groups, which have some prime-power degree, have been determined in Malle-Zaleskii [10]. We have to go through the list of such groups, following the numbering in [10]. All statements about degrees of sporadic and other small simple groups come from the Atlas.

(I have to thank G. Malle for very detailed information).

(1) Let $G$ be a simple group of Lie type of characteristic $p$ and let $\chi$ be the Steinberg character of $G$ with $\chi(1) = |G|_p = q^d$.

The list in Curtis-Iwahori-Kilmoyer [2] on p. 111 gives in most cases a degree, which is divisible by $p$, but not a power of $p$. The exceptions are the following:

$\text{PSL}(2, q)$, as mentioned in a).

$B_l(2) = \text{Sp}(2l, 2)$ and $p = 2$. But by [2], p. 114 $B_l(2)$ has the degree

$$
\frac{2^{l(l-1)}(2^{l-1} + 1)(2^l + 1)}{2 \cdot 3},
$$

which is even for $l \geq 3$, but not a power of 2. Observe that $B_2(2) = \text{Sp}(4, 2) \cong S'_6$.

By [2], p. 114 $F_4(2)$ has the degree $2 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 17$.

$G_2(3)$ has the degrees $2^6$ and $3^6$, but also 56 and 21.

$B_2^1(q) = \text{Sz}(q)$ has the degree $(q - 1)2^n$, where $q = 2^{2n+1}$ (Suzuki [11]).

$G_2^1(q)$ has the degree $q(q^2 - q + 1)$, where $q = 3^{2n+1}$ (Ward [12], p. 87).

(Observe that as $A_8 \cong \text{PSL}(4, 2)$ the degree $2^6$ of $A_8$ has been handled.)

(2) The case $G = \text{PSL}(2, q)$ has been considered in section a).

(3) Let $G = \text{PSL}(n, q)$, where $q > 2$, $n$ is an odd prime such that $(n, q - 1) = 1$ and $\chi(1) = \frac{q^n - 1}{q - 1}$ is a power of a prime (which happens sometimes). By Carter [1], 13.8 for $n \geq 4$ the group $\text{PSL}(n, q)$ has the
Some simple groups which are determined etc. 3

degree
\[
q^2 \frac{(q^n - 1)}{q - 1} \frac{(q^{n-3} - 1)}{q^2 - 1}.
\]

This is a multiple of the prime-power \(\frac{q^n - 1}{q - 1}\), but also a multiple of \(q\).

For \(\text{PSL}(3, q)\) we have the degrees \(q^2 + q + 1\) and \(q(q^2 + q + 1)\).

(4) Let \(G = \text{PSU}(n, q)\), where \(n\) is an odd prime such that \((n, q + 1) = 1\) and \(\chi(1) = \frac{q^n + 1}{q + 1}\) is the power of a prime. By the Lusztig- Srinivasan correspondence between degrees of \(\text{PSL}(n, q)\) and \(\text{PSU}(n, q)\), replacing \(q\) by \(-q\), as \(n\) is odd, we obtain from (3) for \(n \geq 4\) the degree
\[
q^2 \frac{(q^n + 1)}{q + 1} \frac{(q^{n-3} - 1)}{q^2 - 1}
\]
for \(\text{PSU}(n, q)\).

\(\text{PSU}(3, q)\) has the degrees \(q^2 - q + 1\) and \(q(q^2 - q + 1)\) (Klemm [9]).

(5) Let \(G = \text{Sp}(2n, q)\), \(n > 1\) a power of 2 and \(\chi(1) = \frac{q^n + 1}{2}\) a power of a prime. By [1], p. 111 \(\text{Sp}(2n, q)\) has the degree
\[
m = q \frac{(q^{n-1} + 1)}{q + 1} \frac{(q^n + 1)}{2}.
\]

As \(n\) is a power of 2, so \(n - 1\) is odd and hence
\[
\frac{q^{n-1} + 1}{q + 1} = 1 - q + q^2 - \ldots + q^{n-2}
\]
is integral. Hence \(m\) is divisible by \(q\) and \(\frac{q^n + 1}{2}\).

(6) Let \(G = \text{Sp}(2n, 3)\), \(n > 1\) a prime, \(\chi(1) = \frac{3^n - 1}{2}\) a power of a prime. By [2], p. 111 \(\text{Sp}(2n, 3)\) has the degree
\[
3 \frac{(3^n - 1)}{2} \frac{(3^{n-1} + 1)}{2}.
\]

(7) Let \(G = A_n\), where \(n = p^d + 1 \geq 10\) and \(\chi(1) = p^d\). Then \(A_n\) has also the degree.
\[
(n - 1) \frac{n - 2}{2} = p^d \frac{(p^d - 1)}{2}.
\]

Observe that \(A_8 \cong \text{PSL}(4, 2)\) has the degrees 7 and \(2^6\), but also 14. The cases
$A_5 \cong \text{PSL}(2, 5)$ and $A_6 \cong \text{PSL}(2, 9)$ have been considered in a).

(11) Let $G = \text{Sp}(6, 2)$ and $\chi(1) = 7$. Then $G$ has the degree 21. (The degrees $2^6$ and $3^3$ of $\text{Sp}(6, 2)$ are treated in (1) and (17).)

(12) $A_6 \cong \text{PSL}(2, 9)$ has already been mentioned.

(14) $M_{11}$ and $M_{12}$ have the degree 11, but also 44 resp. 66.

(15) $M_{11}$ and $M_{12}$ have the degree $2^4$, but also 44 resp. 66.

PSL(3, 3) has the degree $2^4$, but also 12.

(16) $M_{24}$, Co$_2$ and Co$_3$ have the degree 23, but also 11 \cdot 23 resp. 99 \cdot 23 resp. 8019 \cdot 23.

(17) $A_9$, Sp(6, 2) and $^2F_4(2)'$ have the degree $3^3$, but also $2 \cdot 3 \cdot 7$ resp. 21 resp. $2 \cdot 3 \cdot 13$.

(18) PSU(3, 3) has the degree $2^5$, but also 28.

(19) $G_2(3)$ has the degree $2^6$, but also $2 \cdot 3 \cdot 13$.

Hence all cases have been considered.

**4. The group PSL(2, $p^f$), where $p \neq 2$.**

For $p^f > 5$ we have

$$\text{cd} \ PSL(2, p^f) = \{1, p^f - 1, p^f, p^f + 1, r\},$$

where

$$r = \frac{p^f - 1}{2} \quad \text{if} \ p^f \equiv -1 \pmod{4},$$

$$r = \frac{p^f + 1}{2} \quad \text{if} \ p^f \equiv 1 \pmod{4}.$$

Observe that $r$ is odd.

If $p^f \neq 9$, then SL(2, $p^f$) is the Schur covering group of PSL(2, $p^f$). The degrees of the properly projective irreducible representations of PSL(2, $p^f$) are $p^f - 1, p^f + 1$ and $s$, where

$$s = \frac{p^f + 1}{2} \quad \text{if} \ p^f \equiv -1 \pmod{4},$$

$$s = \frac{p^f - 1}{2} \quad \text{if} \ p^f \equiv 1 \pmod{4}.$$
(see Dornhoff [3], p. 228). But observe that \( \text{PSL}(2, 9) \cong A_6 \) has a Schur multiplier of order 6 and has some more degrees of projective representations (Atlas, p. 5).

**Theorem 3.** Suppose that \( p > 2, p^f > 5 \) and
\[
\text{cd } G = \text{cd } \text{PSL}(2, p^f) = \{1, p^f - 1, p^f, p^f + 1, r\},
\]
where
\[
r = \begin{cases} 
  \frac{p^f - 1}{2} & \text{if } p^f \equiv -1 \pmod{4} \\
  \frac{p^f + 1}{2} & \text{if } p^f \equiv 1 \pmod{4}.
\end{cases}
\]

Then
\[
G \cong \text{PSL}(2, p^f) \times A,
\]
where \( A \) is abelian.

**Proof.** Step 1. We have \( G' = G'' \):

Otherwise let \( G/N \) be a solvable, minimal nonabelian factor group of \( G \).

Suppose at first that \( G/N \) is a \( q \)-group for some prime \( q \). If \( q = p \) and \( \chi \in \text{Irr}(G) \) such that \( \chi(1) = p^f - 1 \), then \( \chi_N \in \text{Irr}(N) \) and by lemma 2 of part I therefore \( \chi \tau \in \text{Irr}(G) \) for all \( \tau \in \text{Irr}(G/N) \). As now \( p^f \in \text{cd } G/N \), we obtain the forbidden degree
\[
\chi(1) \tau(1) = (p^f - 1)p^f.
\]
If \( q \neq p \), we apply the same argument to \( \chi \in \text{Irr}(G) \) with \( \chi(1) = p^f \).

Hence by lemma 4 of part I we can assume that \( G/N \) is a Frobenius group with elementary abelian Frobenius kernel \( F/N \). Then \( |G/F| \in \text{cd } G \) and \( |F/N| = q^a \) for some prime \( q \), where \( |G/F| \) divides \( q^a - 1 \).

If
\[
|G/F| \in \{p^f - 1, p^f, p^f + 1\},
\]
then as \( p^f > 3 \) no proper multiple of \( |G/F| \) is in \( \text{cd } G \). Hence by lemma 4 of part I, if \( \chi \in \text{Irr } G \) and \( q \nmid \chi(1) \), then \( \chi(1) \) divides \( |G/F| \).

Suppose at first \( |G/F| = p^f \), hence \( q \neq p \). If \( q \nmid p^f - 1 \), we obtain the contradiction that \( p^f - 1 \) divides \( p^f \). If \( q \nmid p^f + 1 \), we get the contradiction
that $p^f + 1$ divides $p^f$. Hence $q$ divides $p^f - 1$ and $p^f + 1$, hence $q = 2$. But as $2
ot| r$, hence lemma 4 of part I provides the contradiction that $r = \frac{p^f + 1}{2}$ divides $p^f$.

Suppose next that $|G/F| = p^f \pm 1$. If $q \neq p$, we take $\chi \in \text{Irr} G$ such that $\chi(1) = p^f$ to obtain the contradiction that $p^f$ divides $|G/F|$. If $q = p$, we obtain either the contradiction that $p^f + 1$ divides $p^f - 1$ or $p^f - 1$ divides $p^f + 1$. (Observe that $p^f > 3$).

There remains only the case that

$$|G/F| = r = \frac{p^f + 1}{2}.$$ 

By lemma 4 of part I, if $\psi \in \text{Irr} F$, then either

$$|G/F|\psi(1) \in \text{cd} G,$$

so $\psi(1) \leq 2$, or $q$ divides $\psi(1)$.

Let $\chi \in \text{Irr} G$ such that

$$\chi(1) = \begin{cases} 
  p^f - 1 & \text{if } |G/F| = \frac{p^f + 1}{2} \\
  p^f + 1 & \text{if } |G/F| = \frac{p^f - 1}{2}
\end{cases}$$

As $2
ot| |G/F|$, so

$$\left(|G/F|, \chi(1)\right) = 1.$$ 

Therefore $\chi_F \in \text{Irr} F$ and $\chi(1) > 2$. Hence $q$ divides $\chi(1)$, so $q \neq p$. If $\tau \in \text{Irr} G$ and $\tau(1) = p^f$, then also $\tau_F \in \text{Irr} F$, but $q
ot| \tau(1)$.

This contradiction shows $G' = G''$.

**Step 2.** If $G'/M$ is a chief factor of $G$ then

$$G'/M \cong \text{PSL}(2, p^f);$$

As $G' = G''$, we have

$$G'/M = S_1 \times \ldots \times S_k,$$

where $S_i \cong S$ is simple and nonabelian. The degrees of $S$ divide degrees of $G$, hence are prime to $p$ or powers of $p$.

Suppose at first that $p
ot| |S|$. Let $\chi \in \text{Irr} G$ and $\chi(1) = p^f$. As $G' = G''$, so $\chi_{G'}$ cannot split in characters of $G'$ of degrees 1. Hence

$$\chi_{G'} = \sum_i \varphi_i, \quad \varphi_i \in \text{Irr} G'$$
and \( \varphi_1(1) = p^r > 1 \). As by our assumption \( \varphi_1(1) \) is prime to \( |G'/M| \), so \((\varphi_1)_M \in \text{Irr} M \) and \( \varphi_1 \tau \in \text{Irr} G' \) for all \( \tau \in \text{Irr} (G'/M) \). As \( G'/M \) is a non-abelian \( p' \)-group, there exists \( \tau \in \text{Irr} (G'/M) \) such that \( \tau(1) > 1 \) and \( p \not| \tau(1) \). But then \( \varphi_1 \tau \) has a “mixed” degree, a contradiction.

Hence \( p|S| \). By the theorem of Ito and Michler there exists \( \sigma \in \text{Irr} S \) such that \( p|\sigma(1) \). Hence \( \sigma(1) \) is a power of \( p \) larger than 1. Therefore we can apply lemma 1 of this paper.

**Case 1.** Suppose \( S \cong \text{PSL}(2, p^m) \) for some \( m \). Obviously \( k = 1 \), hence

\[
G'/M \cong \text{PSL}(2, p^m).
\]

If \( \psi \in \text{Irr} (G'/M) \) and \( \psi(1) = p^m \), then \( \psi(1) \) divides \( \chi(1) \) for some \( \chi \in \text{Irr} G \). Hence \( \chi(1) = p^f \), so \( m \leq f \).

Suppose \( m < f \). We consider \( \overline{G} = G/C_G(G'/M) \). Then \( \overline{G}' \cong \text{PSL}(2, p^m) \) and \( |\overline{G} : \overline{G}'| \) divides \( |\text{Out} \text{PSL}(2, p^m)| = 2m \).

Take at first \( \psi_1 \in \text{Irr} \overline{G} \) such that \( \psi_1(1) = p^m \). If we choose \( \chi_1 \in \text{Irr} \overline{G} \) such that \( (\psi_1, (\chi_1)_{\overline{G}})_{\overline{G}} > 0 \), then \( \chi_1(1) = p^f \) and

\[
(\chi_1)_{\overline{G}} = \sum_{i=1}^{k} \psi_1^{q_i},
\]

where \( k \) divides \( |\overline{G} : \overline{G}'| \). Hence \( \chi(1) = p^f = kp^m \) divides \( 2mp^m \). As \( p \) is odd, so \( p^f \) divides \( mp^m \).

Take \( \psi_2 \in \text{Irr} \overline{G} \) such that \( \psi_2(1) = p^m - 1 \). As \( \psi_2(1) \) divides some degree of \( G \), so \( p^m - 1 \) divides \( p^{2f} - 1 \). Therefore \( m \) divides \( 2f \). As \( m < f \), so \( m \leq \frac{2}{3} f \). Hence

\[
p^f \leq mp^m \leq \frac{2}{3} fp^{2f/3}.
\]

This produces the contradiction

\[
3^{f/3} \leq p^{f/3} \leq \frac{2}{3} f.
\]

Hence \( m = f \) and \( G'/M \cong \text{PSL}(2, p^f) \).

**Case 2.** Now we have to exclude the possibility that \( S \cong \text{PSL}(2, 2^n) \), where \( p = 2^n - 1 \) or \( p = 2^n + 1 \) or \( p = 3 \) and \( S \cong \text{PSL}(2, 8) \). Obviously \( k = 1 \), hence

\[
G'/M \cong \text{PSL}(2, 2^n).
\]
We put $\overline{G} = G/C_G(G'/M)$. Then $|\overline{G} : \overline{G}'|$ divides $|\text{Out PSL}(2, 2^k)| = s$. Let $
abla \in \text{Irr } \overline{G}'$ and $\psi(1) = p$ or $\psi(1) = 3^2$. If $\chi \in \text{Irr } \overline{G}$ and $\chi$ is above $\psi$, then $\chi(1) = e\psi(1)$, where $e$ divides $s$. As $p$ divides $\chi(1)$, so $\chi(1) = p^f$ divides $s\psi(1)$.

If $\psi(1) = p$, we obtain $s \geq p^{f-1}$ and then

$$p = 2^s \pm 1 \geq 2^{p^{f-1}} \pm 1,$$

which implies $f = 1$.

Suppose at first that $p = 2^s + 1$. If $\tau \in \text{Irr } \overline{G}'$ and $\tau(1) = 2^s - 1$, then $\tau(1)$ divides $p - 1 = 2^s$ or $p + 1 = 2^s + 2$. This implies

$$2^s + 2 = 2(2^s - 1),$$

a contradiction as $p^f = p = 2^s + 1 > 5$.

Next suppose that $p = 2^s - 1$. If $\tau \in \text{Irr } \overline{G}'$ and $\tau(1) = 2^s + 1$, we again obtain that $2^s + 1$ divides $p + 1 = 2^s$ or $p - 1 = 2^s - 2$, in both cases a contradiction. Hence there remains only the possibility that $p = 3$ and $S \cong \text{PSL}(2, 8)$. We take $\psi \in \text{Irr } \overline{G}'$ such that $\psi(1) = 3^2$ and $\chi \in \text{Irr } \overline{G}$ such that $\chi$ is above $\psi$. Then $\psi(1) = 3^2$ divides $\chi(1) = 3^f$ and $\chi(1)$ divides $3\psi(1) = 3^3$. As $7 \in \text{cd PSL}(2, 8)$, so $7$ divides some degree of $\overline{G}$. As

$$\text{cd } \overline{G} \subseteq \text{cd } G = \{1, 3^f - 1, 3^f + 1, r\},$$

where $r = \frac{1}{2}(3^f \pm 1)$, so $f = 3$ and

$$\text{cd } G = \{1, 13, 26, 27, 28\}.$$

But there exists $\rho \in \text{Irr } \overline{G}'$ such that $\rho(1) = 8$, and $8$ does not divide any degree of $G$.

Hence this case is impossible.

**Step 3.** If $\vartheta \in \text{Irr } M$, then $I_{G'}(\vartheta) = G'$ and therefore $M' = [M, G']$:

Suppose $I = I_{G'}(\vartheta) < G'$ for some $\vartheta \in \text{Irr } M$ and

$$\vartheta^I = \sum_i \varphi_i, \quad \varphi_i \in \text{Irr } I.$$

Then $\varphi^G_i \in \text{Irr } G'$, hence

$$|G' : I| \cdot \varphi_i(1) \in \text{cd } G'.$$

The proper subgroups of $G'/M \cong \text{PSL}(2, p^f)$ are of indices at least $p^f + 1$, with the exceptions (observe $p^f > 5$) that

$$p^f = |G' : I| = 7 \quad \text{and} \quad I/M \cong S_4,$$

$$p^f = |G' : I| = 11 \quad \text{and} \quad I/M \cong A_5,$$

$$p^f = 9, \quad |G' : I| = 6 \quad \text{and} \quad I/M \cong A_5.$$
Some simple groups which are determined etc.

(Huppert [6], p. 214.) The last possibility is excluded as 6 does not divide any degree of \( \text{PSL}(2, 9) \). So in all cases \( \varphi_i(1) = 1 \), hence \( \varphi_i \) is an extension of \( \vartheta \) to \( I \). Therefore

\[
(\varphi_i \tau)^G \in \text{Irr} \ G'
\]

for all \( \tau \in \text{Irr} \ I/M \). So \( |G' : I| \cdot \tau(1) \) divides some degree of \( G \).

If \( |G' : I| = p^f + 1 \), then \( I/M \) is metabelian of order \( p^f(p^f - 1)/2 \), so there exists \( \tau \in \text{Irr} \ I/M \) such that \( \tau(1) = \frac{1}{2}(p^f - 1) \). But this provides a contradiction as \( \frac{1}{2}(p^f + 1)(p^f - 1) \) does not divide any degree of \( G \).

In the remaining exceptional cases we can choose

\[
\tau(1) = 3 \quad \text{if} \quad p^f = 7 \quad \text{and} \quad I/M \cong S_4,
\]

\[
\tau(1) = 4 \quad \text{if} \quad p^f = 11 \quad \text{and} \quad I/M \cong A_5.
\]

This produces the forbidden degrees \( 3 \cdot 7 \) resp. \( 4 \cdot 11 \) of \( G' \).

Hence \( I_{G'}(\vartheta) = G' \) for all \( \vartheta \in \text{Irr} \ M \). Therefore by lemma 6 of part I we obtain \( M' = [M, G'] \).

**Step 4.** We have \( |M/M'| \leq 2 \):

By lemma 6 of part I, \( |M/M'| \) is bounded by the order of the Schur multiplier of \( \text{PSL}(2, p^f) \). Therefore \( |M/M'| \leq 2 \) if \( p^f \neq 9 \) (Huppert [6], p. 646). But the Schur multiplier of \( \text{PSL}(2, 9) \cong A_6 \) has order 6.

Suppose \( p^f = 9 \) and \( |M/M'| > 2 \). From the Atlas, p. 5 we obtain the degrees of the irreducible characters of \( G'/M \), which do not have \( M/M' \) in their kernel, namely

\[
3, 6, 9, 15 \quad \text{if} \quad |M/M'| = 3,
\]

\[
6, 12 \quad \text{if} \quad |M/M'| = 6.
\]

In this case we have

\[
\text{cd} \ G = \text{cd} \ \text{PSL}(2, 9) = \{1, 5, 8, 9, 10\}.
\]

As 6 does not divide any degree of \( G \), so \( |M/M'| > 2 \) is not possible in the case \( p^f = 9 \).

Therefore \( |M/M'| \leq 2 \) in all cases.

**Step 5.** We claim that \( \text{cd} \ M \subseteq \{1, 2\} \) and hence \( M'' = E' \):

Suppose \( \vartheta \in \text{Irr} \ M \) and \( \vartheta(1) > 1 \). If \( \vartheta \) allows an extension \( \vartheta_0 \) to \( G' \), then \( \vartheta_0 \tau \in \text{Irr} \ G' \) for all \( \tau \in \text{Irr} \ (G'/M) \). Taking \( \tau(1) = p^f + 1 \), we obtain

\[
(p^f + 1)\vartheta(1) \in \text{cd} \ G'.
\]
But as \( \vartheta(1) > 1 \), so \((p^f + 1)\vartheta(1)\) does not divide any degree of \( G \). Hence \( \vartheta \) does not allow any extension to \( G' \).

If \( \varphi \in \text{Irr} \, G' \) and \((\varphi_M, \vartheta)_M > 0\), then, as \( I_G(\vartheta) = G' \) by lemma 3c) of part I, \( \varphi = \vartheta_0 \tau_0 \), where \( \vartheta_0 \) and \( \tau_0 \) are characters of irreducible projective representations of \( G' \), \( \vartheta_0(1) = \vartheta(1) \) and \( \tau_0 \) is the character of an irreducible, projective, non-ordinary representation of \( G'/M \cong \text{PSL}(2, p^f) \). The degrees of these representations are \( p^f - 1, p^f + 1 \) and \( s \), where

\[
\begin{align*}
s &= \frac{p^f + 1}{2} \quad \text{if } p^f \equiv -1 \pmod{4}, \\
s &= \frac{p^f - 1}{2} \quad \text{if } p^f \equiv 1 \pmod{4}.
\end{align*}
\]

(See Dornhoff [3], p. 228.) As \( \vartheta(1) > 1 \), so

\((p^f + 1)\vartheta(1) \not\in \text{cd} \, G'\).

There remains only the possibility that \( s\vartheta(1) \in \text{cd} \, G' \), which implies \( \vartheta(1) \leq 2 \). Hence \( \text{cd} \, M \subseteq \{1, 2\} \), and therefore \( M'' = E \) (Isaacs [8], p. 202).

**Step 6.** We have \( \text{cd} \, G/M = \text{cd} \, G\):

As

\[
\text{cd} \, G'/M = \{1, p^f - 1, p^f, p^f + 1, r\}
\]

and \( \text{cd} \, G/M \subseteq \text{cd} \, G \), we see immediately that

\[
p^f - 1, p^f, p^f + 1 \in \text{cd} \, G/M.
\]

Take \( \chi \in \text{Irr} \, G \) such that

\[
\chi(1) = r = \frac{p^f + 1}{2}.
\]

As \( r \) is odd and \( \text{cd} \, M \subseteq \{1, 2\} \) by step 5, so

\[
\chi_M = \sum_i \lambda_i, \quad \lambda_i(1) = 1.
\]

Hence \( M' \leq \ker \chi \). If \( M = M' \), then \( \chi \in \text{Irr} \, G/M \) and

\[
\chi(1) = r \in \text{cd} \, G/M.
\]

Suppose \(|M/M'| = 2\). As \( G' = G'' \), so
\[ \chi_{G'} = \sum_i \rho_j, \quad \rho_j \in \text{Irr } G', \quad \rho_j(1) > 1. \]

If \( \rho_j \) is faithful on \( M/M' \), then \( \rho_j(1) \) is one of the values \( p^f \pm 1 \) or \( s \). But as \( \rho_j(1) \) divides \( \chi(1) = r \), this is impossible. Hence \( M \leq \ker \rho_j \), so \( \chi \in \text{Irr } G/M \) also in this case.

**Step 7.** We now claim \( G/M = G'/M \times C_{G/M}(G'/M) \):

The characters of \( G'/M \) are all invariant under \( G \), for otherwise by fusion of characters of \( G'/M \) we obtain a forbidden degree which is a proper multiple of \( p^f \) or \( p^f \pm 1 \), or by fusion of the two characters of degree \( r = \frac{1}{2}(p^f \pm 1) \) we lose the degree \( r \), which by step 6 is a degree of \( G/M \). As the characters of \( G'/M \) separate the conjugacy classes, so \( G \) preserves the conjugacy classes of \( G'/M \). Hence by a theorem of Feit and Seitz [4], \( G \) induces only inner automorphisms on \( G'/M \). (As \( G'/M \cong \text{PSL}(2, p^f) \), this can be seen directly.) Therefore

\[ G/M = G'/M \times C_{G/M}(G'/M). \]

**Step 8.** We have \( M = E \), and hence

\[ G \cong \text{PSL}(2, p^f) \times A, \]

where \( A \) obviously is abelian.

By step 4 we know \(|M/M'| \leq 2. \) Suppose \(|M/M'| = 2. \) If \( p^f \neq 9 \), then \( G'/M' \) is the uniquely determined Schur covering group of \( G'/M \cong \text{PSL}(2, p^f) \), so

\[ G'/M' \cong \text{SL}(2, p^f). \]

This also is true for \( p^f = 9 \) and \(|M/M'| = 2. \)

We put \( C/M = C_{G/M}(G'/M) \). If \( x \in G'/M' \) and \( \text{ord } x = p, y \in C/M' \), then \( x^y \in xM/M' \). As \( \text{ord } x^y = p > 2 \), this implies \( x^y = x \). Therefore \( C/M' \) centralizes \( G'/M' \). Hence by step 7

\[ G/M' = G'/M' \cdot C/M' \]

is a central product with amalgamated subgroup \( M/M' = \langle zM' \rangle \). Take \( \psi \in \text{Irr } C/M' \) such that \( \psi(z) = -\psi(1). \) If \( \chi \in \text{Irr } G'/M' \) and \( \chi(z) = -\chi(1) \), then

\[ \chi \psi \in \text{Irr } (G'/M' \times C/M'). \]
and

$$\chi(z) \psi(z^{-1}) = \chi(1) \psi(1).$$

Hence \((z, z^{-1}) \in \ker \chi \psi\). Therefore \(\chi \psi\) is a character of

$$(G'(M' \times C/M')/((zM', z^{-1}M')) \cong G'/M' \cdot C/M' = G/M'. \quad \Box$$

Hence \(\chi(1) \psi(1) \in \text{cd } G\). The characters \(\chi\) of \(G'/M'\) with \(M/M'\) not in their kernel have the degrees \(p^f - 1\), \(p^f + 1\) and \(s\). Hence \((p^f - 1)\psi(1), (p^f + 1)\psi(1)\) and \(s\psi(1)\) are in \(\text{cd } G\). This implies \(\psi(1) = 1\) and then the contradiction \(s \in \text{cd } G\). Hence \(|M/M'| = 1\), so by step 5 \(M = M' = M'' = E\). 

Added during proof. Recently I was informed of the following results: Let \(G\) be nonsolvable. 

a) If the degree graph \(\Delta(G)\) has 3 components, then \(G \cong \text{PSL}(2, 2^f) \times \times A\), where \(A\) is abelian.

b) If \(\Delta(G)\) has 2 components, then \(\text{PSL}(2, p^f)\) is the only nonsolvable composition factor of \(G\).

(M. L. Lewis, D. L. White, J. Algebra 266 (2003), 51-76 and 283 (2005), 80-92.)

As \(\Delta(G)\) for solvable \(G\) has at most 2 components, so a) proves theorem 2 of [7].

REFERENCES

Some simple groups which are determined etc.


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