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On Inert Subgroups of a Group.

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ABSTRACT - A subgroup H of a group G is called *inert* if $|H : H \cap H^g|$ is finite for all g in G. If every subgroup of G is inert, then G is said to be *inertial*. After giving an account of the basic properties of inert subgroups, we study the structure of inertial soluble groups. A classification is obtained for the groups which are finitely generated or have finite abelian total rank.

1. Introduction.

A subgroup H of a group G is said to be *inert* if $|H : H \cap H^g|$ is finite for all $g \in G$. This is equivalent to saying that H is commensurable with each of its conjugates. Obvious examples of inert subgroups are normal subgroups, finite subgroups and subgroups of finite index. Somewhat less obvious is the fact that permutable subgroups are inert (Lemma 3.4 below): here a subgroup H is said to be *permutable* in a group G if $HK = KH$ for all subgroups K of G. Recently inert subgroups have received attention in the literature, mainly in the context of locally finite groups—see [1], [2], [3], [6].

A group G will be called *inertial* if every subgroup of G is inert (such groups are termed totally inert in [3]). Clearly the class of inertial groups contains all finite groups, Dedekind groups and Tarski monsters, and so it is a highly complex class. On the other hand, Belyaev, Kuzucuoğlu and Seçkin [3] have shown that no infinite locally finite simple group can be inertial.

In the present work, after a discussion of the basic properties of inert subgroups, we investigate the structure of soluble inertial groups. While there are many groups of this type, we are able to give complete de-

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scriptions in the cases where the group is finitely generated or has finite abelian total rank.

2. Results.

We begin by noticing some uncomplicated types of inertial groups. Suppose that G is a group with an abelian normal subgroup A of finite index. In addition, assume that each $g \in G$ induces a power automorphism in A; thus $a^g = a^n$ for all $a \in A$ where $n = n(a) \in \mathbb{Z}$. Then *G is inertial.* For, if $H \leq G$, then $H \cap A \triangleleft G$ and $H/H \cap A$, being finite, is inert in $G/H \cap A$. Hence H is inert in G and G is inertial.

More generally, a group G is said to be inertial *of elementary type* if there are normal subgroups F and A, with $F \leq A$, such that $|F|$ and $|G : A|$ are finite, A/F is abelian and elements of G induce power automorphisms in A/F: clearly G is an inertial group.

Our first two results show that for wide classes of soluble-by-finite groups, the only inertial groups are those of elementary type.

THEOREM A. *Let G be a hyper-(abelian or finite) group such that $\tau(G) = 1$. Then G is inertial if and only if G is abelian or dihedral.*

Here $\tau(G)$ is the unique maximum normal torsion subgroup of G. Also a group G is called *dihedral* if there is an abelian group A such that $G \simeq \mathrm{Dih}\,(A)$, where

$$\mathrm{Dih}\,(A) = \big\langle t, A \mid t^2 = 1, a^t = a^{-1},\ a \in A \big\rangle,$$

the dihedral group on A.

THEOREM B. *Let G be a finitely generated hyper-(abelian-by-finite) group. Then G is inertial if and only if it has a torsion-free abelian normal subgroup of finite index in which elements of G induce power automorphisms.*

Thus, for the classes of groups in Theorems A and B, the only inertial groups are those of elementary type.

Inertial groups of non-elementary type

It is not difficult to find examples of inertial soluble groups which are not of elementary type. A simple example is the group $\langle t \rangle \ltimes A$ where A is of type p^∞, $\langle t \rangle$ is infinite cyclic and $a^t = a^{1+p}$, $(a \in A)$.

Recall that a soluble group with *finite abelian total rank* (FATR) is a group G with a series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ where each factor is abelian and the sum of the p-ranks

$$\sum_p r_p(G_{i+1}/G_i)$$

is finite for $i = 0, 1, \ldots, n-1$: in the summation here p is a prime or 0. Such groups have also been called $\mathfrak{S}_1$-*groups* and in the interests of conciseness we will use this terminology. (For background on $\mathfrak{S}_1$-groups see [5], Chapter 5.) Our interest centres on the class

$$\mathfrak{S}_1\mathfrak{F}$$

of $\mathfrak{S}_1$-by-finite groups: within this class there are, it turns out, inertial groups of three types which are not elementary. We will now explain how these non-elementary types may be constructed.

Construction

Let D be a divisible abelian p-group with finite positive rank and let F be a torsion-free abelian group, also of finite positive rank. By the *spectrum* of a group G

$$\mathrm{sp}\,(G),$$

we mean the set of primes p for which G has a Prüfer p^∞-group as an image.

I Split extension type

Let Q be an extension of a finite group by either F or $\mathrm{Dih}\,(F)$, and let Q act on D as a group of power automorphisms. Assume that

$$p \notin \mathrm{sp}\,(C_Q(D))$$

and define

$$G = Q \ltimes D.$$

Then G is an $\mathfrak{S}_1\mathfrak{F}$-group and in fact it is inertial, as is proved in Lemma 10.1 below.

II Central extension types

Let $Q = F$ or $\mathrm{Dih}\,(F)$ and regard D as a trivial Q-module. Since D is divisible, $\mathrm{Ext}\,(Q_{ab}, D) = 0$ and the Universal Coefficients Theorem shows that

$$H^2(Q, D) \simeq \mathrm{Hom}\,(M(Q), D),$$

where $M(Q)$ is the Schur multiplier of Q.

Let $\delta \in \mathrm{Hom}\,(M(Q), D)$ have infinite order (or equivalently, infinite image). If $D_0 < D$ and $F_0 \leq F$ we denote by δ_{D_0} and δ_{F_0} the natural induced homomorphisms from $M(Q)$ to D/D_0 and $M(F_0)$ to D respectively.

Assume that δ satisfies the following conditions.

(a) If $Q = F$, (so that $M(Q) = F \wedge F$), then $(F_0 \wedge f)^{\delta_{D_0}}$ is finite for all $f \in F$, whenever $\delta_{D_0, F_0} = 0$.

(b) If $Q = \mathrm{Dih}\,(F)$, then $p \notin \mathrm{sp}\,(F_0)$ whenever δ_{F_0} is not surjective.

The existence of such homomorphisms δ is a question that will be addressed in §11. Next choose a central extension with cohomology class δ,

$$\delta : \quad D \rightarrowtail G \twoheadrightarrow Q.$$

Then G will be called a group of *central extension type IIa* or *IIb*, according to whether (a) or (b) applies. Clearly G is an $\mathfrak{S}_1$-group and it is shown in Lemma 10.2 that G is inertial.

Primary $\mathfrak{S}_1\mathfrak{F}$-groups

Let G be any $\mathfrak{S}_1\mathfrak{F}$-group. Then G has a maximum divisible abelian torsion subgroup D and $\tau(G/D) = \tau(G)/D$ is finite. It follows from Theorem A that if G is inertial, then G/D is inertial of elementary type, so it may be assumed that $D \neq 1$. It is shown in Lemma 6.2 that G is inertial if and only if $G/D_{p'}$ is inertial for each $p \in \pi(D)$. Furthermore, $D/D_{p'}$ is the maximum divisible abelian torsion subgroup of $G/D_{p'}$ and this subgroup is a p-group with finite rank.

An $\mathfrak{S}_1\mathfrak{F}$-group whose maximum divisible abelian torsion subgroup is a p-group will be termed *p-primary*. Thus in order to characterize the inertial $\mathfrak{S}_1\mathfrak{F}$-groups of non-elementary type, it suffices to describe those which are p-primary. This is accomplished in the following theorem, which is the main result of the paper.

THEOREM C. *Let G be a p-primary $\mathfrak{S}_1\mathfrak{F}$-group. Then G is inertial if and only if either G is of elementary type or else it is an extension of a finite group by a group of one of the types I, IIa, IIb.*

The proof of Theorem C is accomplished in §§6–10. Basic results on inert subgroups are presented in §3, while initial structural information about inertial groups appears in §4. The proofs of Theorems A and B are given in §5.

Notation.

$\tau(G)$: the maximum torsion normal subgroup of G.
$\mathrm{Fit}(G)$: the Fitting subgroup of G.
$M(G)$: the Schur multiplier of G.
$\mathrm{Dih}\,(A)$: the dihedral group on A.
$\pi(G)$: the set of primes dividing orders of elements of G.
$\mathrm{sp}\,(G)$: the *spectrum* of G, i.e., the set of primes p for which G has a p^∞-image.
$r_p(A)$: the *p-rank* of A.

3. Basic Results on Inert Subgroups.

We begin with an easy result that is used constantly in verifying that a subgroup is inert.

LEMMA 3.1. *Let H and K be subgroups of a group G such that $H \leq K$ and $|K : H|$ is finite. Then H is inert in G if and only if K is.*

PROOF. Let $g \in G$. Suppose first that $|K : K \cap K^g|$ is finite; then so is

$$|H(K \cap K^g) : K \cap K^g| = |H : H \cap K^g|.$$

Also $|H \cap K^g : H \cap H^g| \leq |K : H|$, so that $|H : H \cap H^g|$ is finite.

Conversely, assume that $|H : H \cap H^g|$ is finite. Then so is $|K : H \cap H^g|$ and hence $|K : K \cap K^g|$ is finite.

COROLLARY 3.2 ([3], Lemma 3). *A subgroup which is commensurable with an inert subgroup of a group is itself inert.*

PROOF. Let $H, K \leq G$ with H inert in G and K commensurable with H. Then $H \cap K$ is inert in G by Lemma 3.1 since $|H : H \cap K|$ is finite. By the same result K is inert in G.

LEMMA 3.3. *If H and K are inert subgroups of a group G, then $H \cap K$ is inert in G.*

PROOF. Let $g \in G$. Then $|H : H \cap H^g|$ is finite and thus so is $|H \cap K : H \cap K \cap H^g|$. Also $|K : K \cap K^g|$ is finite, whence

$$|H \cap K \cap \ H^g : H \cap K \cap H^g \cap K^g|$$

is finite. Therefore $|H \cap K : (H \cap K) \cap (H \cap K)^g|$ is finite.

On the other hand, *the intersection of infinitely many inert subgroups need not be inert*. For example, in a polycyclic group every subgroup is the intersection of subgroups of finite index, but not every subgroup of a polycyclic group need be inert: this is shown by the free nilpotent group of class and rank 2.

We note that, in contrast to Lemma 3.3, *the join of a pair of inert subgroups need not be inert.*

EXAMPLE. Let $G = \langle x, y, a, b \rangle$ where $x^2 = 1$, $y^x = y^{-1}$, $a^x = a^{-1}$, $b^x = b$, $a^y = ab$, $b^y = b$ and $[a, b] = 1$. Then $T = \langle x, y \rangle \simeq \mathrm{Dih}\,(\mathbb{Z})$ and $A = \langle a, b \rangle$ is free abelian of rank 2. Furthermore $G = T \ltimes A$. Notice that T is generated by the finite subgroups $\langle x \rangle$ and $\langle xy \rangle$. However T is not inert in G: for by an easy calculation $T \cap T^a = C_T(a) = 1$.

An interesting source of inert subgroups is the permutable subgroups of a group.

LEMMA 3.4. *Permutable subgroups are inert.*

PROOF. Let H be a permutable subgroup of a group G and let $g \in G$. Then

$$H^{\langle g \rangle} = H^{\langle g \rangle} \cap (H\langle g \rangle) = H(H^{\langle g \rangle} \cap \langle g \rangle)$$

and

$$|H^{\langle g \rangle} : H| = |H^{\langle g \rangle} \cap \langle g \rangle : H \cap \langle g \rangle|,$$

which is finite unless g has infinite order and $H \cap \langle g \rangle = 1$. But under these circumstances $H^g = H$ by a result of Stonehewer [7]. Thus in any event $|H^{\langle g \rangle} : H|$ is finite and

$$|H : H \cap H^g| = |HH^g : H^g| \leq |H^{\langle g \rangle} : H|,$$

so $|H : H \cap H^g|$ is finite.

COROLLARY 3.5. *The intersection of finitely many permutable subgroups is inert.*

4. Inertial Groups.

In this section we collect some useful properties of inertial groups, mainly in the soluble case. The first result has already been noted in [3].

LEMMA 4.1. *FC-groups are inertial.*

PROOF. Let G be an FC-group and let $H \leq G$, $g \in G$. Then $C = C_G(g)$ has finite index in G, so $|H : H \cap C|$ is finite. Since $H \cap C \leq H \cap H^g$, it follows that $|H : H \cap H^g|$ is finite and H is inert in G.

REMARKS. 1. *Inertial groups need not be FC-groups.*

An obvious example is Dih$(\mathbb{Z})$, the infinite dihedral group.

2. *Finite extensions of FC-groups need not be inertial.*

In fact there are finitely generated abelian-by-finite groups which are not inertial. An example is the group

$$G = \langle x, a, b \mid [a, b] = 1 = x^3,\ a^x = b,\ b^x = a^{-1}b^{-1} \rangle,$$

as follows from Proposition 4.2 below. Another example is given in [3].

3. *The direct product of two inertial groups need not be inertial.*

For example, Dih$(\mathbb{Z}) \times \mathbb{Z}$ is not inertial, again by Proposition 4.2. (For another example see [3]).

4. *There are nilpotent p-groups of class 2 which are not inertial.*

For any prime p, let $G = \langle x_1, x_2, \ldots \rangle$ satisfy the relations $\gamma_3(G) = 1$ and $x_i^p = 1, (i = 1, 2, \ldots)$. Then $G' = Z(G) = \langle c_{ij} \mid i < j = 1, 2, \ldots \rangle = C$ say, where $c_{ij} = [x_i, y_j]$. Evidently G is a nilpotent p-group of class 2.

Define $H = \langle x_1, x_3, x_5, \ldots \rangle$. Then a simple calculation shows that

$$H \cap H^{x_2} = H \cap C = \langle c_{ij} \mid i,j \text{ odd} \rangle.$$

Since $|H : H \cap C|$ is infinite, H is not inert in G.

The next result is basic in the study of soluble inertial groups.

PROPOSITION 4.2. *Let A be a torsion-free abelian normal subgroup of a group G. Assume that every cyclic subgroup of G is inert. Then elements of G induce power automorphisms in A, i.e., if $g \in G$, then $a^g = a^\varepsilon$ for all $a \in A$, where $\varepsilon = \pm 1$. Hence $|G : C_G(A)| \leq 2$.*

PROOF. Let $1 \neq a \in A$ and $g \in G$. First of all we show that $[a, g^i] = 1$ for some $i > 0$, and for this purpose we may assume that gA has infinite order.

Since $\langle g \rangle$ is inert in G, we know that $\langle g \rangle / \langle g \rangle \cap \langle g \rangle^a$ is finite and hence $\langle g \rangle \cap \langle g \rangle^a \neq 1$. Let $1 \neq x \in \langle g \rangle \cap \langle g \rangle^a$; then $x = g^i = (g^j)^a = g^j[g^j, a]$ where $i, j \neq 0$. Since $|gA|$ is infinite, it follows that $i = j$ and $[a, g^i] = 1$, as claimed.

Next $\langle a \rangle$ is inert, so $\langle a \rangle \cap \langle a \rangle^g \neq 1$ and there exist $m, n \neq 0$ such that $a^m = (a^n)^g$. Now by induction on j

$$a^{m^j} = \left(a^{n^j}\right)^{g^j},$$

and by the previous paragraph we have $a = a^{g^i}$ for some $i > 0$, from which it follows that $a^{m^i} = a^{n^i}$. Hence $m = \pm n$ since a has infinite order. Consequently $(a^n)^g = a^{\varepsilon n}$ where $\varepsilon = \pm 1$, which implies that $a^g = a^\varepsilon$. Thus g induces a power automorphism in A (and the same ε will suffice for all $a \in A$).

The situation is less clear for abelian normal subgroups of inertial groups which are torsion. However the following result is sometimes useful.

PROPOSITION 4.3. *Let A be an abelian normal torsion subgroup of a group G. If every subgroup of A is inert in G, then $a^{\langle g \rangle}$ is finite for all $a \in A$, $g \in G$.*

PROOF. Suppose for the moment that $a \in A$ has prime order p and put

$$H = \left\langle a^{g^{2i}} \mid i \in \mathbb{Z} \right\rangle.$$

Then $a^{\langle g \rangle} = \langle H \cup H^g \rangle$, so we may suppose H to be infinite. Since $|H : H \cap H^g|$ is finite, $H \cap H^g \neq 1$. Let $1 \neq x \in H \cap H^g$; then we may assume that

$$x = a^{f(g^2)} = a^{h(g^2)g}$$

where $0 \neq f, g \in \mathbb{Z}_p[t]$, the polynomial ring. Then $a^{f(g^2) - h(g^2)g} = 1$ and $f(t^2) - h(t^2)t \neq 0$ in $\mathbb{Z}_p[t]$. This implies that $a^{\langle g \rangle}$ is finitely generated and hence finite.

In the general case suppose that a has order $m > 1$ and let p be a prime dividing m. Then $B = \left(a^{\frac{m}{p}}\right)^{\langle g \rangle}$ is finite. Passing to the group $\langle a, g \rangle / B$ and using induction on m, we conclude that $a^{\langle g \rangle} B / B$ is finite, whence $a^{\langle g \rangle}$ is finite.

The final result in the section highlights the special role played by divisible abelian subgroups of an inertial group.

LEMMA 4.4. *Let G be an inertial group. Then G has a unique maximum divisible abelian torsion subgroup D and elements of G induce power automorphisms in D.*

PROOF. Let L be any p^∞-subgroup of G. If $g \in G$, then $|L : L \cap L^g|$ is finite, which shows that $L = L^g$ and $L \lhd G$. If M is any q^∞-subgroup, then $[L, M] = 1$ since M centralizes every finite subgroup of L. Consequently G contains a unique maximum divisible abelian torsion subgroup, D say. If d is a p-element in D, then d is contained in a p^∞-subgroup and hence $\langle d \rangle \lhd G$. It follows that elements of G induce power automorphisms in D.

5. Proofs of Theorems A and B.

We have developed enough of the theory of inertial groups to be able to prove the first two main results.

PROOF OF THEOREM A. Let G be a hyper-(abelian or finite) inertial group for which $\tau(G) = 1$. We show that G is abelian or dihedral.

Let N be a nilpotent normal subgroup of G and let M be a maximal abelian normal subgroup of N. If $a \in M$ and $x \in N$, then, since M is torsion-free, $a^x = a^\varepsilon$, where $\varepsilon = \pm 1$ by Proposition 4.2. Since $\langle x, a \rangle$ is nilpotent, $1 = [a, {}_n x] = a^{(\varepsilon-1)^n}$ for some $n > 0$, and it follows that $\varepsilon = 1$ and $[a, x] = 1$. However $M = C_N(M)$, so $M = N$ and N is abelian. Consequently the Fitting subgroup $F = \mathrm{Fit}(G)$ is abelian, and of course it is also torsion-free. Writing $C = C_G(F)$, we conclude that $|G : C| \leq 2$.

Assume that $F \neq C$. Then, since G is hyper-(abelian or finite), there is a non-trivial normal subgroup U/F of G/F such that $U \leq C$ and U/F is either finite or abelian. If U/F is finite, U is centre-by-finite, so U' is finite and thus U is abelian. This implies that $U = F$, a contradiction. Hence U/F is abelian, so that U is nilpotent and again $U = F$. Therefore $F = C$ and $|G : F| \leq 2$.

If $F = G$, then G is abelian. Otherwise $F \neq G$ and $G = \langle t, F \rangle$; thus $f^t = f^{-1}$, $(f \in F)$, and $t^2 \in F$. Hence $t^4 = 1$ and thus $t^2 = 1$ since F is torsion-free. Finally $G = \mathrm{Dih}\,(F)$. The converse is clearly true.

COROLLARY 5.1. *A torsion-free hyper-(abelian or finite) group which is inertial is abelian.*

PROOF OF THEOREM B. Let G be a finitely generated hyper-(abelian or finite) group which is inertial. It must be shown that G possesses a torsion-free abelian normal subgroup of finite index in which elements of G induce power automorphisms. Notice that finitely generated groups with this structure are finitely presented. Thus, arguing by contradiction, we may assume the statement in the theorem is true for all proper quotients of G, but false for G itself.

If $\tau(G) = 1$, the result is a consequence of Theorem A, so we may assume $\tau(G) \neq 1$. From this it follows that G has a non-trivial normal subgroup A which is either finite or an abelian torsion group. Suppose that A is finite. Then G/A satisfies the conclusion of the theorem and hence G is polycyclic-by-finite, which implies that it has a torsion-free normal subgroup B of finite index. By Corollary 5.1 the subgroup B is abelian, and by Proposition 4.2 elements of G induce power automorphisms in B. By this contradiction A is an abelian torsion group.

Let $1 \neq a \in A$ and $g \in G$. Then $a^{\langle g \rangle}$ is finite by Proposition 4.3. Since G/A is the product of finitely many cyclic groups, a^G is finite. The argument given above for A shows this to be impossible. The converse statement is clearly true.

6. Inertial $\mathfrak{S}_1\mathfrak{F}$-Groups.

We begin the study of inertial $\mathfrak{S}_1\mathfrak{F}$-groups with a discussion of the reduced groups. (For background on $\mathfrak{S}_1\mathfrak{F}$-groups see [4], §5).

Let G be an $\mathfrak{S}_1\mathfrak{F}$-group. Recall that the finite residual R of G is a divisible nilpotent group. Its torsion-subgroup D is a divisible abelian with finite total rank, i.e., it is a direct product of finitely many Prüfer p^∞-groups. Clearly D is the unique maximum divisible abelian torsion subgroup of G. If $D = 1$, then G will be called *torsion-reduced*.

The inertial $\mathfrak{S}_1\mathfrak{F}$-groups which are torsion-reduced are characterized in the following result.

Proposition 6.1. *Let G be a torsion-reduced $\mathfrak{S}_1\mathfrak{F}$-group. Then the following are equivalent:*

(i) *G is inertial;*

(ii) *G is of elementary type;*

(iii) *there is a torsion-free abelian subgroup A of finite index in which elements of G induce power automorphisms;*

(iv) *G is finite-by-F or finite-by-*$\mathrm{Dih}\,(F)$ *where F is torsion-free abelian.*

PROOF. It is either obvious or already known that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). We show next that (i) $\Rightarrow$ (iii). Let G be inertial. Since G is torsion-reduced, its finite residual R is torsion-free. Also G/R is reduced, so it has a normal torsion-free subgroup A/R with finite index in G/R. By Corollary 5.1 the subgroup A is abelian and by Proposition 4.2 elements of G induce power automorphisms in A. Thus (iii) holds.

To complete the proof we show that (i) implies (iv). Since G is inertial and $\tau(G)$ is finite, we may suppose that the latter is trivial; the result now follows from Theorem A.

Primary groups

In the light of Proposition 6.1, we may confine our analysis to inertial $\mathfrak{S}_1\mathfrak{F}$-groups which are not torsion-reduced. Let G be an $\mathfrak{S}_1\mathfrak{F}$-group and let D be its maximum divisible abelian torsion subgroup. Should D happen to be a p-group, G will be called *p-primary*. We show next that it is sufficient to describe the inertial $\mathfrak{S}_1\mathfrak{F}$-groups which are p-primary.

LEMMA 6.2. *Let G be an $\mathfrak{S}_1\mathfrak{F}$-group with maximum divisible abelian torsion subgroup D. Then G is inertial if and only if $G/D_{p'}$ is inertial for all p in $\pi(D)$. Moreover $G/D_{p'}$ is p-primary.*

PROOF. Since the condition is certainly necessary, we assume that it holds for G. The group $G/D_{p'}$ is inertial, so by Lemma 4.4 elements of G induce power automorphisms in $D/D_{p'}$ and hence in D_p. Thus every subgroup of D is normal in G.

Let $H \leq G$: then $H \cap D \triangleleft G$ and in showing that H is inert in G we may suppose that $H \cap D = 1$. Let $g \in G$ and $p \in \pi(D)$. Then $|HD_{p'} : HD_{p'} \cap H^g D_{p'}|$ is finite and thus so is $|H : H \cap H^g D_{p'}|$. Since $\pi(D)$ is finite, it follows that $|H : H \cap K|$ is finite, where

$$K = \bigcap_{p \in \pi(D)} \left(H^g D_{p'}\right).$$

If $k \in K$, then $k = h_p^g d_p$ where $h_p \in H$ and $d_p \in D_{p'}$. Therefore, if $p \neq q$, we have

$$\left(h_q^g\right)^{-1} h_p^g = d_q d_p^{-1} \in H^g \cap D = 1,$$

which implies that $d_p = d_q$ and $h_p = h_q$. Hence $K = H^g$ and in consequence $|H : H \cap H^g|$ is finite, so that G is inertial.

7. Primary Inertial $\mathfrak{S}_1\mathfrak{F}$-Groups.

In this section we begin to analyze the structure of primary inertial $\mathfrak{S}_1\mathfrak{F}$-groups. Let G be an inertial $\mathfrak{S}_1\mathfrak{F}$-group which is not of elementary type and denote by

$$D$$

its maximum divisible abelian torsion subgroup. Since G/D is of elementary type by Proposition 6.1, we have $D \neq 1$. By Lemma 6.2 we can assume that D is a p-group, i.e., G is p-primary.

There are three normal subgroups which play a prominent role in the analysis. By Proposition 6.1 there is a torsion-free abelian subgroup A/D with finite index in G/D in which elements of G induce power automorphisms. Also put

$$C = C_A(D) \quad \text{and} \quad L = C_G(A/D).$$

Then A, C and L are normal in G,

$$D \leq C \leq A \leq L \quad \text{and} \quad |G : L| \leq 2.$$

Since by Lemma 4.4 elements of G induce power automorphisms in D, there are two possible situations:

(i) $[D, A] = 1$, the *central case*;
(ii) $[D, A] = A$, the *non-central case.*

There is a further dichotomy arising from the location of the subgroup L:

(iii) $L = G$, i.e., A/D is G-central;
(iv) $|G : L| = 2$.

It is easy to deduce from Proposition 6.1 that the group G/D is finite-by-torsion-free abelian in case (iii) and finite-by-dihedral (on some torsion-free abelian group) in case (iv). Cases (iii) and (iv) are referred to as the *non-dihedral case* and the *dihedral case* respectively. Thus in all there are four cases to be dealt with.

The above notation with D, C, A, L will be maintained throughout this and the following two sections.

A reduction

In establishing the necessity of the conditions in Theorem C one very useful observation is that it is always possible to factor out by a finite normal subgroup without disturbing the standard subgroups D, C, A, L. The basis for this reduction is the next lemma, which assures us that in factoring out by a finite normal subgroup we remain in the same central or non-central and dihedral or non-dihedral cases.

LEMMA 7.1. *Let F be a finite normal subgroup of the inertial $\mathfrak{S}_1\mathfrak{F}$-group G. Then:*

(i) *DF/F is the maximum divisible abelian torsion subgroup of G/F;*
(ii) *$C_G(DF/F) = C_G(D)$.*
(iii) *$AF/DF \overset{G}{\simeq} A/D$, so AF/DF is torsion-free abelian;*
(iv) *$C_G(AF/DF) = C_G(A/D)$.*

PROOF. (i) Suppose that E/F is a divisible abelian torsion subgroup of G/F. Since G is an $\mathfrak{S}_1\mathfrak{F}$-group, E is a Černikov group and therefore ED/D is finite. Thus $E \leq DF$ and DF/F is the maximum divisible abelian torsion subgroup of G/F.

(ii) If $g \in C_G(DF/F)$, then $[D,g]$ is finite; however it is also divisible, so $[D,g] = 1$.

(iii) Since A/D is torsion-free, $A \cap F \leq D$ and hence

$$AF/DF \overset{G}{\simeq} A/A \cap (DF) = A/D.$$

(iv) This follows from (iii).

COROLLARY 7.2. (i) *The central case applies to G/F if and only if it applies to G.*

(ii) *The dihedral case applies to G/F if and only if it applies to G.*

As a consequence of the corollary we can make a first reduction.

LEMMA 7.3. *It may be assumed (by factoring out by a finite normal subgroup) that L/D is abelian.*

PROOF. Since L/D is centre-by-finite, $L'D/D$ is finite. Now elements of G induce power automorphisms in D, so we have $[D,G'] = 1$ and therefore $L'D$ is centre-by-finite. Hence $(L'D)'$ is finite and may be factored out. This makes $L'D$ an abelian Černikov group. Factoring out by a further finite G-invariant subgroup of $L'D$, we can make this subgroup divisible and hence $L' \leq D$.

8. The Central Case.

In the notation established in §7, we are faced with the situation where

$$[D, A] = 1,$$

and hence $G/C_G(D)$ is finite. There are two sub-cases that must be treated separately.

The non-dihedral case

In this case we have $L = G$ and thus we can assume that G/D is abelian by Lemma 7.3. The case where in addition $[D, G] = D$ is disposed of by the next result.

LEMMA 8.1. *Assume that G/D is abelian and $[D, G] = D$. Then, modulo a finite normal subgroup, G has the form $X \ltimes D$ where X is abelian, its elements induce power automorphisms in D and $p \notin \mathrm{sp}\,(C_X(D))$.*

From this we deduce:

COROLLARY 8.2. *The group G is of split extension type (I).*

For the subgroup X is abelian and torsion-reduced, so its torsion-subgroup is finite.

PROOF OF LEMMA 8.1. Since $D = [D, G]$ and G/D is abelian, it follows from [4, Theorem H] or [5, 10.3.6] that $H^2(G/D, D)$ is a torsion group. By [5, 10.1.15] G splits over D modulo some finite G-invariant subgroup of D. This subgroup may be factored out without affecting the hypotheses on G, by Corollary 7.2. Therefore we may assume that

$$G = XD \quad \text{and} \quad X \cap D = 1$$

where X is abelian. Put

$$Y = C_X(D);$$

it remains to show that $p \notin \mathrm{sp}\,(Y)$. Assume that this is false.

Since D is a non-trivial divisible abelian p-group and $p \in \mathrm{sp}\,(Y)$, there exists a homomorphism $\theta : Y \to D$ with infinite image. Put

$$E = Y^{1+\theta}$$

and notice that $Y \times D = E \times D$.

Since $[D,G]\neq 1$, there exists a g in G such that $[D,g]\neq 1$; then $d^g=d^a$, $(d\in D)$, for some p-adic integer $a\neq 1$. Now G is inertial, so $|E:E\cap E^g|$ is finite. Let $u\in E\cap E^g$; thus $u=e_1=e_2^g=e_2[e_2,g]$ where $e_i\in E$. Since $[E,g]\leq D$ and $E\cap D=1$, it follows that $u=e_1=e_2\in C_E(g)$, and consequently

$$E\cap E^g=C_E(g).$$

Therefore $|[E,g]|=|E:C_E(g)|$ is finite.

Next let $y\in Y$ and write $e=y^{1+\theta}\in E$. Since $Y\leq Z(G)$, we have

$$[e,g]=(yy^\theta)^{-1}(yy^\theta)^g=(yy^\theta)^{-1}y(y^\theta)^a=(y^\theta)^{a-1}.$$

Since $[E,g]$ is finite, it follows that $(Y^\theta)^{a-1}$ is finite. But $a\neq 1$, so this gives the contradiction that Y^θ is finite, which completes the proof of Lemma 8.1.

We are left with the case where $[D,G]=1$. Let T/D be the torsion-subgroup of the abelian group G/D. Then T/D is finite, so T is centre-by-finite and thus T' is finite. Factoring out by T', we can assume that T is abelian. Since T is a Černikov group, we can make it divisible by factoring out another finite normal subgroup. Hence $T=D$. Therefore we can suppose that *G/D is a torsion-free abelian group.*

At this point it is convenient to have the following technical result at our disposal.

LEMMA 8.3. *Assume that $[D,G]=1$ and G/D is abelian. Suppose further that $D\leq B\leq G$ where B is abelian. Then $[B,g]$ is finite for all g in G.*

PROOF. Since D is divisible, $B=D\times F$ where F is abelian. Now F is inert in G, so $|F:F\cap F^g|$ is finite for all $g\in G$. Next $F\cap F^g=C_F(g)$ by an argument in the proof of Lemma 8.1. Therefore $|[F,g]|=|F:C_F(g)|$ is finite. Finally, $[B,g]=[F,g]$ since $[D,G]=1$.

We can now complete the analysis of the central/non-dihedral case by analyzing the central extension

$$D\rightarrowtail G\twoheadrightarrow F,$$

where $F=G/D$, a torsion-free abelian group. Since

$$H^2(F,D)\simeq \mathrm{Hom}\,(M(F),D)$$

and $M(F)=F\wedge F$ (the exterior square), the group G is determined by an element

$$\delta\in\mathrm{Hom}\,(F\wedge F,D).$$

Now $\mathrm{Im}(\delta) = G' \cap D = G'$ and this must be infinite – for otherwise G would be of elementary type. We will show that δ satisfies the conditions for the central extension type IIa.

Let $F_0 \leq F$ and $D_0 < D$, and suppose that $\delta_{D_0,F_0} = 0$. This means that, if B/D_0 is the preimage of F_0 under the natural map $G/D_0 \twoheadrightarrow F$, then B/D_0 is abelian. By Lemma 8.3 applied to the group G/D_0, we conclude that $[B,g]D_0/D_0$ is finite for all $g \in G$, which translates into $(F_0 \wedge f)^{\delta_{D_0}}$ being finite where $f = gD \in F$. Therefore G is a group of central extension type IIa, which concludes our discussion of the central/non-dihedral case.

The dihedral case

We have now to deal with the situation where

$$[D, A] = 1 \quad \text{and} \quad |G : L| = 2,$$

i.e., the central/dihedral case. Thus elements in $G \backslash L$ induce the inverting automorphism in A/D. An essential role in our analysis of this case is played by the following technical result.

LEMMA 8.4. *Assume that* $[D,A] = 1$ *and* $|G : L| = 2$. *Suppose that* B *is an abelian subgroup satisfying* $D \leq B \leq A$. *If* $p \in \mathrm{sp}\,(B/D)$, *then each element of* $G \backslash L$ *induces the inverting automorphism in* D, *and in addition* $[D, L] = 1$.

PROOF. Since B is abelian, we have $B = D \times F$ where F is torsion-free. By hypothesis $p \in \mathrm{sp}\,(B/D)$, so there is a homomorphism $\theta : F \to D$ with infinite image. Write $E = F^{1+\theta}$, noting that $B = D \times E$.

Choose any g in $G \backslash L$; then we claim that

$$E \cap E^g = \{e \in E \mid e^g = e^{-1}\}.$$

To see this, suppose that $u \in E \cap E^g$; then $u = e_1 = e_2^g = e_2^{-1}d$ where $e_i \in E$ and $d = e_2^{1+g} \in D$. Since $E \cap D = 1$, we obtain $e_1 e_2 = d = 1$, whence $u = e_1$ and $e_1^g = e_1^{-1}$, as required. Since E is inert in G, it follows that $|E^{1+g}| = |E : E \cap E^g|$ is finite. By the same argument $|F^{1+g}|$ is finite.

Next let $f \in F$ and put $e = f^{1+\theta} \in E$. Therefore, writing $d = f^{1+g} \in D$, we have

$$e^{1+g} = ff^\theta (ff^\theta)^g = ff^\theta f^{-1} d (f^\theta)^a = d(f^\theta)^{1+a},$$

where g induces the automorphism $x \mapsto x^a$ in D, with a a p-adic integer. Since E^{1+g} and F^{1+g} are both finite, so is $(F^\theta)^{1+a}$. However F^θ is infinite, so this can only mean that $a = -1$ and g induces inversion in D.

Finally, if $\ell \in L$, then $g\ell$ also induces inversion in D, which implies that $[D,\ell]=1$ and $[D,L]=1$. This completes the proof of Lemma 8.4.

We return to our analysis of the dihedral case. Let us assume for the present that in addition $[D,L]\neq 1$, so that

$$D=[D,L].$$

Since L/D is abelian, it follows from [4, Theorem H] or [5, 10.3.6] that $H^2(L/D,D)$ is a torsion group, and because $|G:L|$ is finite, $H^2(G/D,D)$ is also a torsion group. Therefore, modulo some finite subgroup of D, the group G splits over D, and by Corollary 7.2 we can assume that

$$G=Q\ltimes D$$

where Q is abelian-by-finite and inertial. Thus Q is finite-by-Dih (A/D) by Proposition 6.1.

Next $A=A\cap(QD)=(A\cap Q)D$, so A is abelian since $[D,A]=1$. Applying Lemma 8.4 with $B=A$ and using the fact that $[D,L]\neq 1$, we see that $p\notin \mathrm{sp}\,(A/D)$. Since $A/D\simeq A\cap Q\leq C_Q(D)$ and $|G:A|$ is finite, it follows that $p\notin C_Q(D)$ and therefore G is of split extension type I.

For the rest of the section we assume that

$$[D,L]=1.$$

Let T/D be the torsion-subgroup of the abelian group L/D. Then T is centre-by-finite, so, factoring out T', we can assume T to be abelian. Of course T is Černikov, so factoring out one more time, we may assume that T is divisible, i.e., $T=D$. Thus L/D is torsion-free abelian. This allows us to replace A by L, which means that we now have the situation

$$[D,A]=1,\quad |G:A|=2$$

and

$$G/D=\mathrm{Dih}\,(A/D).$$

There are two further sub-cases to be considered.

Case: A' is finite

As usual we may assume that A is abelian. Write $G=\langle t,A\rangle$; then t induces inversion in A/D. Write $A=D\times F$; then $|F:F\cap F^t|$ is finite, which by the usual argument shows that F^{t+1} is finite. Now $F^{t+1}\lhd G$ since $F^{t+1}\leq D$, so we may factor out by F^{t+1} and assume that t induces inversion in F.

Next we have $t^2\in A$, which shows that $t^2\equiv t^{-2}(\bmod D)$ and $t^4\in D$. Since A/D is torsion-free, $t^2\in D$ and $\langle t^2\rangle\lhd G$. Factor out by $\langle t^2\rangle$ to get

$t^2 = 1$ and $G = \langle t \rangle \ltimes D$. Now $d^t = d^a$, $(d \in D)$, for some p-adic integer a. If $p \in \mathrm{sp}\,(A/D)$, then Lemma 8.4 shows that $a = -1$ and $G \simeq \mathrm{Dih}\,(A)$, which is of elementary type. Therefore $p \notin \mathrm{sp}\,(A/D)$.

We now have

$$G = \big\langle t, A \mid t^2 = 1,\ d^t = d^a,\ f^t = f^{-1},\ f \in F \big\rangle$$

and $G = Q \ltimes D$ where $Q = \langle t, F \rangle = \mathrm{Dih}\,(F)$. Since $p \notin \mathrm{sp}\,(A/D) = \mathrm{sp}\,(Q)$, the group G is of split extension type I.

Case: A' is infinite

In order to handle this case we need another lemma of a technical nature.

Lemma 8.5. *Assume that* $[D, A] = 1$, $|G : A| = 2$ *and* A' *is infinite. Then* $D \leq Z(G)$.

Proof. Let $g \in G$ and $a_1, a_2 \in A$. Then $a_i^g = a_i^\varepsilon d_i$ where $\varepsilon = \pm 1$ and $d_i \in D$. Hence

$$[a_1, a_2]^g = [a_1^\varepsilon d_1, a_2^\varepsilon d_2] = [a_1^\varepsilon, a_2^\varepsilon] = [a_1, a_2],$$

since $[D, A] = 1$ and $A' \leq D$, which shows that $A' \leq Z(G)$. Since A' is infinite and elements of G induce power automorphisms in D, it follows that $D \leq Z(G)$.

By Lemma 8.5 we have a *central extension*

$$D \rightarrowtail G \twoheadrightarrow Q = G/D,$$

say with cohomology class $\delta \in \mathrm{Hom}\,(M(Q), D)$. Also $Q = \mathrm{Dih}\,(A/D)$. Notice that $\mathrm{Im}(\delta) = G' \cap D$ is infinite since it contains A'. We show that δ satisfies the conditions of the central extension type IIb.

Let $F_0 \leq F = A/D$ and put $D_0 = \mathrm{Im}(\delta_{F_0})$; assume that $D_0 < D$. Then $F_0 = B/D_0$ is abelian. If $p \in \mathrm{sp}(F_0) = \mathrm{sp}(B/D)$, then Lemma 8.4 shows that elements of $G \backslash L$ induce the inverting automorphism in D/D_0. This is impossible since $D \leq Z(G)$ by Lemma 8.5; hence $p \notin \mathrm{sp}\,(F_0)$, as required.

9. The Non-Central Case.

We now deal with the case where

$$D = [D, A].$$

Also, of course, A/D is torsion-free abelian and G/A is finite.

Suppose first that we are in the dihedral case; thus $|G:L| = 2$ and $a^g \equiv a^{-1} (\bmod D)$ for $a \in A$, $g \in G \backslash L$. It follows that $A^2 \leq [A, g]D \leq G'D$. Since elements of G induce power automorphisms in D, we have $[D, G'] = 1$ and hence $[D, A^2D] = 1$. Also, A^2D/D is torsion-free abelian and $|G : A^2D|$ is finite. Consequently, if we replace A by A^2D, we are again in the situation $[D, A] = 1$, i.e., we are in the central/dihedral case, which was dealt with in §8: either G is of split extension type I or of central extension type IIb.

Now assume we have the non-central/non-dihedral case: thus $L = G$ and G/D is abelian, while $D = [D, G]$. It follows directly from Lemma 8.1 that, modulo a finite normal subgroup, G is of split extension type I.

10. Sufficiency.

In order to complete the proof of the main result, Theorem C, we have to demonstrate that a group of type I, IIa or IIb is inertial. This is done in two lemmas.

LEMMA 10.1. *Let G be an extension of a finite group by a group of split extension type I. Then G is an inertial $\mathfrak{S}_1\mathfrak{F}$-group.*

PROOF. We can assume that G is of split extension type, so that

$$G = Q \ltimes D :$$

here D is a divisible abelian p-group of finite rank, Q is finite-by-F or finite-by-Dih (F), with F a torsion-free abelian group of finite rank such that $p \notin \mathrm{sp}\,(C_Q(D))$ and Q acts as a group of power automorphisms on D. Clearly G is an $\mathfrak{S}_1\mathfrak{F}$-group—what needs to be shown is that G is inertial.

Let $H \leq G$ and put $D_0 = H \cap D \lhd G$. If $D_0 = D$, then H is inert since G/D is inertial: assume that $D_0 < D$. Then $C_Q(D/D_0) = C_Q(D)$ and as a consequence we can assume that

$$H \cap D = 1.$$

By Proposition 6.1 there is a normal subgroup A of finite index such that A/D is torsion-free abelian with elements of G inducing power automorphisms in it. Since $|H : H \cap A|$ is finite, it is enough to show that $H \cap A$ is inert in G. Thus it is permissible to assume that $H \leq A$, and hence HD/D is abelian.

Suppose that $[D, H] \neq 1$. Then $[D, H] = D$ and it follows from [5, 10.3.6] and [5, 10.1.10] that, modulo a finite subgroup of D, the complements of D in

HD are conjugate to H. Now $HD = (HD \cap Q)D$, so H and $HD \cap Q$ are conjugate, which shows that we may assume that $H \leq Q$. Hence H is inert in Q.

Let $x \in Q$ and $d \in D$. Then $H \cap H^d = C_H(d)$ and $|H : H \cap H^d| \leq \leq |[H, d]| \leq |d^H|$, which is finite. Thus $|H : H \cap H^d|$ is finite. Also $|H : H \cap H^x|$ is finite, as is $|H^d : H^d \cap H^{xd}|$, from which we deduce that $|H \cap H^d : H \cap H^d \cap H^{xd}|$ is finite. Hence $|H : H \cap H^d \cap H^{xd}|$ is finite, as must be $|H : H \cap H^{xd}|$. Thus H is inert in G.

Now assume that $[D, H] = 1$, so that $HD = H \times D$, which is abelian. If $g \in G$, then $a^g \equiv a^{\varepsilon} (\bmod D)$ for $a \in A$, where $\varepsilon = \pm 1$. Therefore

$$H \cap H^g = \{h \in H \mid h^g = h^{\varepsilon}\}$$

and

$$|H : H \cap H^g| \leq |H^{g-\varepsilon}|,$$

since $h \mapsto h^{g-\varepsilon}$ is a homomorphism from H to D. Finally,

$$\mathrm{sp}\,(H) = \mathrm{sp}\,(HD/D) \subseteq \mathrm{sp}\,(C_Q(D)),$$

so $p \notin \mathrm{sp}\,(H)$. Consequently $H^{g-\varepsilon}$ is finite and again H is inert.

LEMMA 10.2. *Let G be an extension of a finite group by a group of central type IIa or IIb. Then G is an inertial $\mathfrak{S}_1\mathfrak{F}$-group.*

PROOF. We may assume G is of type IIa or IIb, so there is a central extension

$$D \rightarrowtail G \twoheadrightarrow Q$$

with D a divisible abelian p-group of finite rank and $Q = F$ or $\mathrm{Dih}\,(F)$, where F is a torsion-free abelian group of finite rank. Let the cohomology class of the extension be

$$\delta \in \mathrm{Hom}\,(M(Q), D);$$

here δ satisfies the conditions for the type IIa or IIb.

Let $H \leq G$ and write $D_0 = H \cap D \lhd G$. It suffices to show that H/D_0 is inert in G/D_0. Suppose first that $\mathrm{Im}(\delta_{D_0})$ is infinite; then we can pass at once to G/D_0, i.e., we may assume that $H \cap D = 1$. In addition we may suppose that HD/D is abelian, so $HD = H \times D$ is abelian. If $g \in G$, the usual argument shows that $|H : H \cap H^g| = |H^{g-\varepsilon}|$ where $h^g \equiv h^{\varepsilon} (\bmod D)$, $h \in H$, and $\varepsilon = \pm 1$. Also $h \mapsto h^{g-\varepsilon}$ is a homomorphism from H to D. Let $F_0 = HD/D$. If Q is dihedral, then, since HD is abelian, $\delta_{F_0} = 0$ and $p \notin \mathrm{sp}\,(F_0) = \mathrm{sp}\,(H)$ and thus $|H^{g-\varepsilon}|$ is finite.

If, on the other hand, Q is abelian, then $\varepsilon = 1$ and $\delta_{F_0} = 0$ where $F_0 = HD/D$. Therefore for any $g \in G$ the group $((HD/D) \wedge (gD))^{\delta}$ is finite, i.e., $[H, g] = H^{g-1}$ is finite. Thus $|H : H \cap H^g|$ is finite in both cases.

Now suppose that $\mathrm{Im}(\delta_{D_0})$ is finite, i.e., $(G' \cap D)D_0/D_0$ is finite. Passing to the quotient group modulo $(G' \cap D)D_0$, we may assume that $D_0 = 1 = G' \cap D$. Thus $\delta = 0$ and G splits over D. We now have $G = D \times Q$ and $H \cap D = 1$. Also we may assume that $Q = \mathrm{Dih}\,(F)$ and $H \leq D \times F$. If $t \in Q \backslash F$, then $H \cap H^t = \{h \in H \mid h^t = h^{-1}\}$ and $|H : H \cap H^t| = |H^{t+1}|$. Now $p \notin \mathrm{sp}\,(H)$ since $\delta = 0$, so the image of the homomorphism $h \mapsto h^{t+1}$, $(h \in H)$, is finite, i.e., $|H^{t+1}|$ is finite and again $|H : H \cap H^t|$ is finite. Therefore H is inert in G.

This completes the proof of Lemma 10.2, and thereby Theorem C.

11. Existence Questions.

In this section we discuss the existence of the non-elementary inertial $\mathfrak{S}_1\mathfrak{F}$-groups. There is little difficulty in constructing examples of groups of split extension type I for any given prime p, in both the dihedral and non-dihedral cases. On the other hand, a group of central type IIa or IIb involves a torsion-free abelian group of finite rank which must satisfy some complex conditions. It is essential to produce explicit examples showing that groups of both types exist.

Constructing groups of central extension type

Let p be an arbitrary prime and D a group of type p^{∞}. Denote by $\mathbb{Q}_p$ the additive group of p-adic rationals $\left\{\frac{m}{p^n} \mid m, n \in \mathbb{Z}\right\}$. The torsion-free abelian group F that we require is to be an extension of $\mathbb{Z}$ by $\mathbb{Q}_p$ with extension class ε of infinite order,

$$\varepsilon : \; C \hookrightarrow F \twoheadrightarrow \mathbb{Q}_p$$

where C is infinite cyclic.

In fact such groups abound. For by standard homological calculations it can be shown that

$$\mathrm{Ext}\,(\mathbb{Q}_p, \mathbb{Z}) \simeq \hat{\mathbb{Z}}_p/\mathbb{Z}$$

where $\hat{\mathbb{Z}}_p$ is the additive group of p-adic integers. Since ε has infinite order, the extension does not nearly split over C, i.e., there is no subgroup of finite index in F which splits over C.

A group of the above type may be constructed explicitly as follows. Let V be a 2-dimensional rational vector space with basis $\{u, v\}$. Let a be an irrational p-adic integer, written $a = a_0 + a_1 p + a_2 p^2 + \cdots$, $(0 \leq a_i < p)$. Define elements v_i in V by

$$v_0 = v \text{ and } v_i = p^{-i}(v_0 + (a_0 + a_1 p + \cdots + a_{i-1}p^{i-1})\, u).$$

Thus $pv_{i+1} = v_i + a_i u$. Let

$$F = \langle u, v_i \mid i = 0, 1, 2, \ldots \rangle.$$

Then $F/\langle u \rangle \simeq \mathbb{Q}_p$ and it is not hard to show that F does not nearly split over $C = \langle u \rangle$, so F is an example of the kind of group required.

We note some properties of abelian groups of this sort.

Lemma 11.1. *Let $C \hookrightarrow F \twoheadrightarrow \mathbb{Q}_p$ be an abelian extension with extension class of infinite order, where $C \simeq \mathbb{Z}$. Then the following hold.*

(i) $M(F) = F \wedge F \simeq \mathbb{Q}_p$.
(ii) If $H \leq F$, then H is either finitely generated or of finite index.
(iii) The group F has no subgroups isomorphic with $\mathbb{Q}_p$.

Proof. Let $F/C \simeq \mathbb{Q}_p$ with C infinite cyclic. Then $M(F)$ can be computed using the Lyndon-Hochschild-Serre spectral sequence for $C \hookrightarrow F \twoheadrightarrow \mathbb{Q}_p$. Clearly $E^2_{20} = E^2_{02} = E^2_{30} = 0$; therefore $M(F) \simeq E^2_{11} \simeq \simeq \mathbb{Q}_p \otimes \mathbb{Z} \simeq \mathbb{Q}_p$.

Next let $H \leq F$. Suppose first that $|F : HC|$ is finite. Then $H \cap C \neq 1$, otherwise F would nearly split over C. Hence $|HC : H|$ is finite and $|F : H|$ is finite. If $|F : HC|$ is infinite, then HC/C is cyclic since $F/C \simeq \mathbb{Q}_p$. Hence H is finitely generated.

Finally, (iii) follows directly from (ii).

The group F will now be used to construct examples of groups of types IIa and IIb. Since $M(F) \simeq \mathbb{Q}_p$, there is a surjective homomorphism

$$\delta \in \operatorname{Hom}(M(F), D),$$

where D is of type p^∞. This determines a central extension

$$\delta : \; D \rightarrowtail G_1 \twoheadrightarrow F.$$

Next the inversion automorphism of F acts trivially on $M(F) = F \wedge F$, so it lifts to an automorphism τ of G_1 acting trivially on D. Now define

$$G_2 = \langle \tau \rangle \ltimes G_1.$$

Regarding the groups G_1, G_2 we prove:

LEMMA 11.2. *The group G_1 is an inertial $\mathfrak{S}_1\mathfrak{F}$-group of type IIa and G_2 is an inertial $\mathfrak{S}_1\mathfrak{F}$-group of type IIb.*

PROOF. It is sufficient to check the validity of the conditions on F and δ in the definitions of types IIa and IIb.

Suppose that $Q = G_2/D = \mathrm{Dih}(F)$. If $F_0 \leq F$ and δ_{F_0} is not surjective, then δ_{F_0} has finite order and F_0 cannot have finite index in F. Hence F_0 is finitely generated by Lemma 11.1, which shows that $\mathrm{sp}(F_0)$ is empty. Thus G_2 is of type IIb.

Now assume that $Q = G_1/D = F$, and let $D_0 < D$ and $F_0 \leq F$; then D_0 is finite. Assume that $\delta_{D_0,F_0} = 0$. Then δ_{F_0} has finite order and $|F : F_0|$ must be infinite. Hence F_0 is finitely generated and therefore $(F_0 \wedge f)^{\delta_{D_0}}$ is finite for all $f \in F$. It follows that G_1 is of type IIa.

That the groups G_1 and G_2 are inertial follows from Lemma 10.2.

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