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Intersection differential forms

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ABSTRACT - We present a survey about the complex $I_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ of *intersection differential forms*, a sub complex of the de Rham complex of the regular locus $\mathscr{X}_{reg}$ of a controlled pseudomanifold $\mathscr{X}$ endowed with a perversity function $\boldsymbol{p}$, which computes the real intersection cohomology with respect to the fixed perversity.

Introduction.

The *intersection homology* theory developed by M. Goresky and R. MacPherson in [GM1] has proven to be an useful tool in studying singular spaces; the original geometric definition goes roughly as follow. Consider a PL-pseudomanifold $\mathscr{X}$ with a fixed stratification and its complex $C_{\bullet}(\mathscr{X};\mathbb{R})$ of PL-chains with real coefficients; if S were a stratum of $\mathscr{X}$ and ξ an i-chain in general position with S then their intersection would have dimension $\leq i+cod\,S-dim\,\mathscr{X}$. Thus one can assign to every codimension of the strata an integer $\boldsymbol{p}\,(cod\,S)$ that is the extra amount of this intersection dimension which is allowed. If the perversity function $\boldsymbol{p}$ is fixed then the chain ξ is called $\boldsymbol{p}$-perverse if for every stratum S, $dim|\xi|\cap S \leq i+cod\,S-dim\mathscr{X}+\boldsymbol{p}(cod\,S)$; the perversity measures how much a chain is free to intersect the singular part of the pseudomanifold. This leads to the definition of intersection homology as the homology of the complex $I_{\boldsymbol{p}}C_{\bullet}(\mathscr{X};\mathbb{R})$ of the $\boldsymbol{p}$-perverse chains in $\mathscr{X}$.

An alternative technical approach to intersection cohomology using sheaf theory is described in [GM2] where an axiomatic framework for IC is also proved (Deligne's axioms).

In this survey we describe another method originally due to Goresky and MacPherson to compute such cohomology using special differential

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forms which is more geometrical than the latter; in a smooth manifold a differential form can thought as a *shape evaluator* of chains (the flat cochain interpretation): every infinitesimal element of a smooth chain is evaluated using multilinear form which returns somehow its volume and then everything is integrated in some sense. In a singular space there is not a geometric definition of tangent bundle, so infinitesimal elements and differential forms are meaningless; however, if the space is good enough there is an additional structure (a control data) consisting in a family of tubular neighborhoods and retractions for the strata with high compatibility degree. This family can be used to define on the regular part of $\mathscr{X}$ the *direction toward the singularities*; using this extra informations the smooth part can remember the original singularities. Now, fix a stratum S with tube T_S, retraction π_S and associated perversity p: a smooth differential i-form ω in $\mathscr{X}_{reg}$ has perversity p with respect to S if for every $x \in \mathscr{X} \cap T_S$ evaluates as zero every «infinitesimal i-element» at x which contains more that p «infinitesimal directions» along the fibers of π_S which are smooth submanifolds. Thus a form is said to have low perversity if it is unable to detect an infinitesimal element when is too directed toward the singular locus. Fixing a perversity $\boldsymbol{p}$ and repeating this reasoning for every stratum lead to the complex of sheaves $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ of *differential intersection forms*.

The axiomatic framework is used to show that this complex has the same local properties of the IC complex and thus it computes intersection cohomology. Moreover, some well-known properties of classic forms as softness, Poincaré lemma, Mayer-Vietoris sequence and a partial Künneth formula are proved.

The definition of $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ here presented is not new: it has been introduced in [BRY] where it is attributed to Goresky and MacPherson (unpublished); such complex is reconsidered in [BHS] where the aim of the authors is to establish a pairing between a complex of singular intersection chains described by King [KIN] and the complex $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$: they show that intersection differential forms with some extra properties can be integrated on particular intersection homology chains, using some local computations typical in intersection homology to complete their program.

There also exists variations on this complex, for example the complex of Σ-regular forms of intersection introduced in the papers [BF] and also [SAR]; the definition is similar to the definition presented in this survey, but now forms are given on all strata (and not only the open one) with some gluing conditions on the neighborhood of the strata. Using the axiomatic theory of Deligne for intersection cohomology the authors also prove that their complex is quasi-isomorphic to the intersection cohomology sheaf.

The technical bulk of this article is a mixture of such local computations and axiomatic approach.

Notations.

– If $\mathscr{M}$ is a smooth manifold we denote with $tan\mathscr{M}$, $\Xi[\mathscr{M}]$ and $\Omega^j[\mathscr{M}]$ respectively the tangent bundle, the family of smooth vector fields and the differential j-forms.

– If $\mathscr{A}$ is a sheaf of $\mathbb{R}$-vector spaces over a topological space $\mathscr{X}$ and $S \subseteq \mathscr{X}$ we denote the section of $\mathscr{A}$ over S with $\Gamma(S;\mathscr{A})$ or $\mathscr{A}[S]$ and the stalk of $\mathscr{A}$ over a point $x \in \mathscr{X}$ as $\mathscr{A}_x$.

– If $\mathscr{A}^\bullet$ is a complex of sheaves then $\mathscr{H}^j(\mathscr{A}^\bullet)$ denote the j-th cohomology sheaf; if $M^\bullet$ is a complex of $\mathbb{R}$-vector spaces then $\mathrm{H}^j(M^\bullet)$ is j-th cohomology vector space.

1. Controlled pseudomanifolds.

We briefly collect here the main definitions and theorems about stratifications and control theory; a detailed and modern description of such topics can be found in [PFL] (chapters 1-3) which is the main reference. Other sources include the older [THO], [MA1], [MA2], [GWPL] and [VER].

Roughly speaking, a *controlled pseudomanifold* (also known as a *Thom-Mather stratified space*) is a nice topological space [1.1-A] with a good decomposition into smooth pieces [1.1-B] where a way is given to describe approaching to singular locus [1.1-C]; although the following formal definition seems to be very restrictive and unpleasant, almost every good singular space admits the structure of controlled pseudomanifold (see theorem [1.3] and [1.4]):

DEFINITION 1.1. (Controlled pseudomanifold) *A triple* $\mathscr{X} := (\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S \in \mathscr{S}})$ *is called a controlled pseudomanifold iff the following properties hold:*

A: *$\mathscr{X}$ is a locally compact Hausdorff space with countable topology;*

B: *$\mathscr{S}$ is a locally finite partition of $\mathscr{X}$ into subspaces (the strata of $\mathscr{X}$) that are smooth manifolds in the induced topology and satisfy the following conditions:*

  B1: *if $R, S \in \mathscr{S}$ and $R \cap \overline{S} \neq \emptyset$ then $R \subseteq \overline{S}$ (and we write $R \leq S$);*

  B2: *if the max dimension of the strata is $n \in \mathbb{N}$ then $\mathscr{X}$ has no* *stratum of dimension* $n-1$ *and the union of all* n*-strata is open-dense in* $\mathscr{X}$ *(if* $\mathscr{X}_k$ *is the* k*-skeleton of* $\mathscr{X}$*, i.e. the union of strata of dimension* $\leq k$*; then* $\mathscr{X}=\mathscr{X}_n$*,* $\mathscr{X}_{n-1}-\mathscr{X}_{n-2}=\emptyset$ *and* $\mathscr{X}-\mathscr{X}_{n-2}$ *is open-dense in* $\mathscr{X}$*);*

C: *for every* $S \in \mathscr{S}$ *the triple* (T_S, π_S, ρ_S) *(a tube for* S *in* $\mathscr{X}$*) satisfies the following relations:*

C1: $\pi_S : T_S \to S$ *is a continuous retraction of an open neighborhood* T_S *of* S *in* $\mathscr{X}$ *such that for every stratum* $R \geq S$ *the restriction* $\pi_S|_{T_S \cap R} : T_S \cap R \to S$ *is smooth;*

C2: $\rho_S : T_S \to \mathbb{R}^{\geq 0}$ *is a continuous map such that* $\rho^{-1}(0)=S$ *and for each stratum* $R \geq S$ *the restriction* $\rho_S|_{T_S \cap R} : T_S \cap R \to S$ *is smooth;*

C3: *for each pair* $R>S$ *of strata, for all* $x \in T_S \cap T_R \cap \pi_R^{-1}(T_S)$ *we have:*

$$\pi_S \pi_R(x) = \pi_S(x),$$
$$\rho_S \pi_R(x) = \rho_S(x);$$

C4: *for each pair* $R>S$ *of strata, the restriction*

$$(\pi_S, \rho_S)| : T_S \cap R \to S \times \mathbb{R}^{>0}$$

is submersive (in particular, $\pi_S| : T_S \cap R \to S$ *is submersive);*

The set $\mathscr{X}_{reg} := \mathscr{X}_n - \mathscr{X}_{n-2}$ *is called the regular part and* $\mathscr{X}_{sing} := \mathscr{X}_{n-2}$ *the singular part.*

A topological space with data satisfying the conditions of definition [1.1], except perhaps for property B2, is called a *controlled space*; such spaces are well behaved, however to achieve topological invariance of *intersection cohomology* (see [BOR] pp. 86-96, [GM1] and [GM2]) the pseudomanifold structure is fundamental (i.e. if $\mathscr{X}$ has dimension n then no stratum of dimension $n-1$ is allowed and the regular part must be open-dense).

Standard point-set topology shows that the topological space underlying a controlled space is *metrizable* and so *paracompact* (and *finite-dimensional* by definition); by the following lemma one can assume a good separation of tube-neighborhoods which will be used tacitly elsewhere ([PFL] Prop. 3.6.7(1)):

LEMMA 1.2. *By shrinking the tubes we can assume that:*

- *if* $S, R \in \mathscr{S}$ *and* $T_S \cap R \neq \emptyset$ *then* $R \geq S$*;*
- *if* $R, S \in \mathscr{S}$ *and* $T_S \cap T_R \neq \emptyset$ *then* R *and* S *are comparable*

(i.e. $R < S$*,* $R = S$ *or* $R > S$*).*

Every *open set* of a controlled pseudomanifold canonically inherits a structure of controlled pseudomanifold; also every *smooth manifold* with or without boundary has a natural controlled pseudomanifold structure (the latter requires a collar as tube for the boundary), but it is considerably more difficult to prove the following existence theorem (see [PFL] Th. 3.6.9):

THEOREM 1.3. *Every locally compact Whitney-stratified subset of a given smooth manifold admits a structure of controlled space.*

Consider as $\mathscr{X}$ a 2-torus with a circle collapsed and with the central hole filled with a disk as in figure 2; the stratification is $\mathscr{S} := \{S_0, S_1, S_2^I, S_2^{II}\}$, where S_2^I and S_2^{II} are diffeomorphic to an open 2-disk and S_1 to an open segment. It is easy to see that such space satisfies Whitney condition and thus by theorem [1.3] it admits a control structure; however condition [1.1-B2] is not satisfied since the singular set $\mathscr{X}_{sing} := S_0 \cup S_1$ has real codimension 1 and hence $\mathscr{X}$ is not a pseudomanifold. It can be shown that the intersection cohomology of a pseudomanifold is a topological invariant (see [BOR] Cor. 4.18, 4.19) thus not depending on the chosen stratification and this is generally false for a generic stratified space; however this simple example is useful to visualize what is going on.

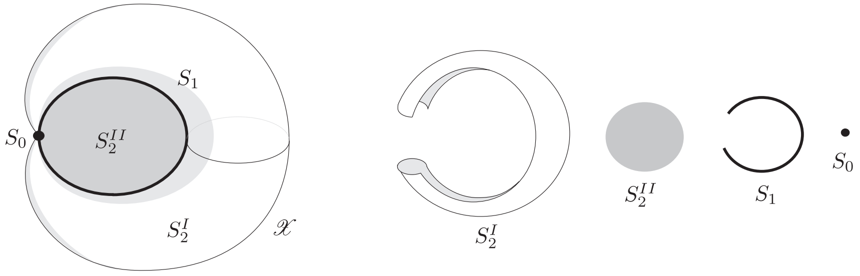

Fig. 1. – A stratification.

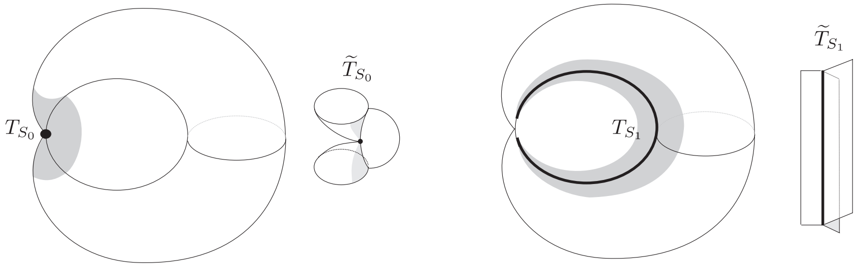

Fig. 2. – Control data.

The following theorem allows to construct control data for a variety of spaces ([BCR] pp. 184-214)) using theorem [1.3]:

THEOREM 1.4. *The following spaces can be Whitney-stratified:*
- *complex analytic varieties;*
- *real analytic varieties.*

In particular, complex analytic varieties and real analytic varieties with dense singular locus of real dimension ≥ 2 *admit a structure of controlled pseudomanifold.*

The following constructions are needed to describe the local structure of a controlled pseudomanifold:

EXAMPLE 1.5 (Cylinder). Given a controlled pseudomanifold $(\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ and a smooth manifold $\mathscr{M}$ without boundary (e.g $\mathbb{R}^l$) one can form a control structure over the cylinder $\mathscr{X}\times\mathscr{M}$ as follow: the strata set is $\{S\times M\}_{S\in\mathscr{S}}$ and control data for a stratum $S\times\mathscr{M}$ is given by $(T_S\times\mathscr{M}, \pi_S\times id_{\mathscr{M}}, \rho_S\times 0)$; it is easily seen that this datum verifies the conditions [1.1-A, B, C].

EXAMPLE 1.6 (Cone). Let $(\mathscr{L}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ be a *compact* controlled pseudomanifold and consider the open cone $con^\circ\mathscr{L} := \frac{\mathscr{L}\times[0,+\infty[}{\mathscr{L}\times\{0\}}$; the strata set is given by $\{S\times\,]0,+\infty[\}_{S\in\mathscr{S}}$ (lateral strata) plus the *vertex*, the unique 0-stratum. Control data for *lateral strata* is built using the cylinder structure as in example [1.5] and for the vertex we proceed as follow, taking the triple $(\frac{\mathscr{L}\times[0,\varepsilon[}{\mathscr{L}\times\{0\}}, \pi_{vert}, \rho_{vert})$ where $\varepsilon > 0$, the map π_{vert} collapses the neighborhood into the vertex and ρ_{vert} is the distance from vertex. As usual, the needed verifications are trivial, but it is important to remark that:

> *the only* 0*-stratum in the cone* $con^\circ\mathscr{L}$ *is its vertex and the associated retraction* π_{vert}*, being trivial, has just one fiber, namely the whole set* $\frac{\mathscr{L}\times[0,+\varepsilon[}{\mathscr{L}\times\{0\}}$.

DEFINITION 1.7 (Controlled isomorphism). *A controlled isomorphism between two controlled pseudomanifolds* $(\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ *and* $(\mathscr{Y}, \mathscr{T}, \{(T_R, \pi_R, \rho_R)\}_{R\in\mathscr{T}})$ *is an homeomorphism* $\phi\colon\mathscr{X}\to\mathscr{Y}$ *between the underlying topological spaces, mapping diffeomorphically strata of* $\mathscr{X}$ *into strata of* $\mathscr{Y}$ *and satisfying the following compatibility relations:*

if $S \in \mathcal{S}$ *is mapped by* ϕ *onto* $R \in \mathcal{T}$ *then there exists an open neighborhood* $U \subseteq T_S$ *for* S *in* $\mathcal{X}$ *such that* $\forall x \in U$*:*

- $f\pi_S(x) = \pi_R f(x)$,
- $\rho_S(x) = \rho_R f(x)$.

The following is the main result due to Thom and Mather of stratification theory (for a proof see [PFL] Th. 3.9.2 and Cor. 3.9.3); it roughly says that a controlled space is locally a cylinder over a cone:

THEOREM 1.8 (Local structure). *If* $\mathcal{X}$ *is a controlled pseudomanifold then it is topologically locally trivial with cones as typical fibers, i.e.: for every stratum* S *of* $\mathcal{X}$ *and for every* $x \in S$ *there exist an open neighborhood* U *of* x *in* $\mathcal{X}$*, a controlled compact pseudomanifold* $\mathcal{L}$ *of dimension* $dim\,\mathcal{X} - dim\,S - 1$ *and a controlled isomorphism:*

$$U \xrightarrow{\cong} \mathbb{R}^{dimS} \times con^{\circ}\mathcal{L}$$

mapping x *to* $(0, vertex)$ (*and then* $U \cap S$ *to* $\mathbb{R}^{dimS} \times \{vertex\}$).

Thus a controlled pseudomanifold is in particular a topological pseudomanifold in the sense of articles [BOR] pp. 1-2 and [GM2] and it can be shown that it also carries a structure of PL-pseudomanifold as required in [GM1].

REMARK 1.9. To construct the intersection differential form and work with intersection cohomology one really needs the good decomposition of $\mathcal{X}$ in strata, the tubular neighborhoods and retractions and the local topological triviality. The maps $\{\rho_S\}_{S \in \mathcal{S}}$ ([1.1-C2]) will not be explicitly used in what follows; however, they are fundamental to prove theorem [1.8] and to ensure that the link $\mathcal{L}$ is a controlled pseudomanifold too.

2. Deligne's axioms.

Here we briefly recall the formalism used by Deligne to give a sheaf theoretic definition of the *intersection cohomology* of Goresky & MacPherson; the main reference for derived categories and homological algebra are [KS] and [IVE].

In this section $\mathcal{X} = (\mathcal{X}, \mathcal{S}, \{(T_S, \pi_S, \rho_S)\}_{S \in \mathcal{S}})$ will denote a fixed *controlled pseudomanifold* with $dim\mathcal{X} = n$; denoting the k-skeleton of $\mathcal{X}$ by

$\mathscr{X}_k$ we get a filtration by *closed* sets:

$$\mathscr{X}_n \supseteq \mathscr{X}_{n-1} = \mathscr{X}_{n-2} \supseteq \mathscr{X}_{n-3} \supseteq \ldots\ldots \supseteq$$

$$\supseteq \mathscr{X}_k \supseteq \ldots\ldots \supseteq \mathscr{X}_1 \supseteq \mathscr{X}_0 \supseteq \mathscr{X}_{-1} := \emptyset$$

such that for each $k \in \{0, \ldots, n\}$ the set $S_k := \mathscr{X}_k - \mathscr{X}_{k-1}$ if not empty is a smooth k-manifold (S_k is again called the k-stratum, although in general the condition in definition [1.1-B1] is not satisfied). The closure of each $\mathscr{X}_k$ derives from local finiteness of strata [1.1-B] and the fact that if $R > S$ then $dim\, R > dim\, S$, as follows from the submersiveness condition in [1.1-C4].

Setting dually $U_k := \mathscr{X} - \mathscr{X}_{n-k}$ for each $k \in \{2, \ldots, n+1\}$ we obtain a sequence of *open sets*:

$$\mathscr{X}_{reg} = U_2 \subseteq U_3 \subseteq \ldots\ldots \subseteq U_k \subseteq U_{k+1} \subseteq \ldots\ldots \subseteq U_{n+1} = \mathscr{X}.$$

By construction, $U_{k+1} = U_k \overset{\circ}{\cup} S_{n-k}$ and is important to note that S_{n-k} is *closed* in U_{k+1}; in the sequel we will always denote by i_k the *open* inclusion $i_k : U_k \longrightarrow U_{k+1}$.

Thus the space $\mathscr{X}$ is constructed beginning with the n-stratum $U_2 = S_n = \mathscr{X}_{reg}$, the regular part; then we attach the $(n-2)$-stratum S_{n-2} obtaining U_3, heuristically adding to $\mathscr{X}_{reg}$ a smooth $n-2$-stratum of singularities; such process continues until we reach U_{n-1} and after gluing S_0 (a discrete set of point) we complete the construction of $\mathscr{X}$. Starting from this, the main idea is to check and control object defined over the whole $\mathscr{X}$ (as chains, cochains or sheaves) and see how they change when a new stratum of singularities is added during the described construction. The control is imposed using an integer valued function (see [BOR] pp. 8-9):

DEFINITION 2.1 (Perversity). *A perversity associated to $\mathscr{X}$ is a function $\boldsymbol{p} : \{0, \ldots, dim \mathscr{X}\} \to \mathbb{Z}^{\geq 0}$ such that $\boldsymbol{p}_{(0)} = \boldsymbol{p}_{(1)} = \boldsymbol{p}_{(2)} = 0$ and $\boldsymbol{p}_{(k)} \leq \boldsymbol{p}_{(k+1)} \leq \boldsymbol{p}_{(k)} + 1$; the domain of $\boldsymbol{p}$ is the set of all the possible codimensions of strata of $\mathscr{X}$.*

In the case of complexes of sheaves the control process is achieved with the following: for each $k \in \{2, \ldots, n\}$ let $\boldsymbol{D}^B(U_{k+1})$ be the derived category of cohomologically bounded $\mathbb{R}$-sheaf complexes over U_{k+1}; the natural transformation $id_{\boldsymbol{D}^B(U_{k+1})} \to \mathrm{R}i_{k*} i_k^*$ induced by the adjunction gives for each sheaf complex $\mathscr{A}^\bullet$ over $\mathscr{X}$ the map:

$$att_k : \mathscr{A}^\bullet|_{U_{k+1}} \longrightarrow \mathrm{R}i_{k*} i_k^* (\mathscr{A}^\bullet|_{U_{k+1}}) = \mathrm{R}i_{k*}(\mathscr{A}^\bullet|_{U_k}).$$

which is usually called the *attaching map* since it describes the cohomological sheaf theoretic passage from U_k to U_{k+1} after the attaching of stratum S_{n-k}. Its role in intersection cohomology theory is described in the following (see [BOR] pp. 61-62 and [GM2]):

DEFINITION 2.2 (Deligne's axioms). *Let $\mathscr{X}$ be a controlled pseudomanifold of dimension n, and suppose $\boldsymbol{p}$ is a fixed perversity relative to the filtration of $\mathscr{X}$; a complex of sheaves $\mathscr{A}^\bullet$ belonging to $\boldsymbol{D}^B(\mathscr{X})$ satisfies Deligne's axioms (relative to the constant sheaf $\mathbb{R}_{\mathscr{X}_{reg}}$ and to perversity $\boldsymbol{p}$) if:*

– the complex $\mathscr{A}^\bullet$ is zero in negative degree and the restriction $\mathscr{A}^\bullet|_{\mathscr{X}_{reg}}$ is isomorphic in $\boldsymbol{D}^B(\mathscr{X}_{reg})$ to the constant sheaf $\mathbb{R}_{\mathscr{X}_{reg}}$ (seen as a complex concentrated in degree zero);

– for every $k \in \{2, \dots, n\}$, $x \in S_{n-k}$ and $j > \boldsymbol{p}_{(k)}$ we have $\mathscr{H}^j(\mathscr{A}^\bullet)_x = 0$;

– for every $k \in \{2, \dots, n\}$ and for each $j \leq \boldsymbol{p}_{(k)}$ the attaching-map att$_k$ induces a sheaf isomorphism:

$$\mathscr{H}^j(att_k) : \mathscr{H}^j(\mathscr{A}^\bullet|_{U_{k+1}}) \xrightarrow{\cong} \mathscr{H}^j(\mathrm{R}i_{k*}\mathscr{A}^\bullet|_{U_k}).$$

REMARK 2.3. The last axiom of Deligne is geometrically more transparent if we substitute to $\mathscr{A}^\bullet$ a *soft* resolution, which we still call $\mathscr{A}^\bullet$ for simplicity of notation; then the last axiom can be reformulated as follows (note that is fundamental that S_{n-k} is closed in U_{k+1}):

> $\forall k \in \{2, \dots, n\}, \forall j \leq \boldsymbol{p}_{(k)}$ and $\forall x \in S_{n-k}$ *the restriction maps induces an isomorphism (where V varies into a cofinal family of neighborhoods of x in U_{k+1}):*

$$\varinjlim_{V \ni x} \mathrm{H}^j(\Gamma(V; \mathscr{A}^\bullet)) \xrightarrow{\cong} \varinjlim_{V \ni x} \mathrm{H}^j(\Gamma(V - S_{n-k}; \mathscr{A}^\bullet))$$

Deligne's Axioms are patterned over the local properties of the *intersection cochain* complex of sheaves $\mathrm{I}_{\boldsymbol{p}}\mathrm{C}^\bullet_{\mathscr{X}}$ ([BOR] p. 34) and their role in this context is given by the following fundamental theorem by Goresky and MacPherson ([GM2]):

THEOREM 2.4. *Let $\mathscr{X}$ be a controlled pseudomanifold and $\boldsymbol{p}$ a fixed perversity; if $\mathscr{A}^\bullet$ is a complex in $\boldsymbol{D}^B(\mathscr{X})$ satisfying Deligne's axioms then its hypercohomology is naturally isomorphic to intersection cohomology.*

Again, if $\mathscr{A}^\bullet$ is a complex of soft sheaves, the previous theorem implies that it computes intersection cohomology via its *global section cohomology*, i.e. $\mathrm{H}^\bullet(\Gamma(\mathscr{X}; \mathscr{A}^\bullet)) \cong \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^\bullet(\mathscr{X}; \mathbb{R})$.

Since all axioms are «local» to prove that a complex calculates intersection cohomology is enough to do some local computations; however, absolutely no hint is given about the isomorphism between the cohomology modules.

3. The complex of intersection differential forms.

Every *submersion* $\pi\colon\mathscr{M}\longrightarrow\mathscr{N}$ between smooth manifold induces the following well-known filtration:

$\bigwedge^k_{\leq 0} tan^*\mathscr{M} \subseteq \bigwedge^k_{\leq 1} tan^*\mathscr{M} \subseteq \ldots\ldots \subseteq \bigwedge^k_{\leq dim\mathscr{M}-dim\mathscr{N}} tan^*\mathscr{M} = \bigwedge^k tan^*\mathscr{M}$

where $\bigwedge^k_{\leq p} tan^*\mathscr{M} := \bigoplus_{j=0\ldots p} \bigwedge^j (ker\ \mathrm{d}\pi)^* \otimes \bigwedge^{k-j} (\pi^* tan\,\mathscr{N})^*$; if the map π is represented locally as a projection $(x_1,\ldots,x_n,t_1,\ldots,t_{m-n})\longmapsto(x_1,\ldots x_n)$ then for every point $a\in\mathscr{M}$ we have $(\bigwedge^k_{\leq p} tan^*\mathscr{M})_a = span_{|I|\leq p}\left\{\mathrm{d}x_J\wedge\mathrm{d}t_I\right\}$;

Using the contraction operator $\bullet$ between a vector field and a differential form one obtains a coordinate-free description of the subbundle $\bigwedge^k_{\leq p} tan^*\mathscr{M}$ as follow:

$$\bigwedge^k_{\leq p} tan^*\mathscr{M} = \left\{\omega\in\bigwedge^k tan^*\mathscr{M} \,\middle|\, \begin{array}{l}\text{for all } \boldsymbol{v}_0,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p\in sect(ker\ \mathrm{d}\pi)\\ \boldsymbol{v}_0\bullet\boldsymbol{v}_1\bullet\ldots\bullet\boldsymbol{v}_p\bullet\omega=0\end{array}\right\}$$

Note that the tangent bundle of each submanifold-fiber $\pi^{-1}(y)$ is exactly the restriction $(ker\mathrm{d}\pi)|_{\pi^{-1}(y)}$, thus to discriminate smooth forms in $\mathscr{M}$ one looks in local trivialization of the submersion π and counts how much terms $\mathrm{d}t_*$ directed *along the fibers* are present in wedge product. Thus if one considers the fibers of π as *particular directions* (if $R>S$ are strata of a pseudomanifold then by condition [1.1-C4] the map $\pi|:T_S\cap R\to S$ is submersive and its fibers can be thought as a description of how the stratum R approaches S) one is lead to the following definition:

DEFINITION 3.1 (Perversity condition). *Let $\pi\colon\mathscr{M}\to\mathscr{N}$ be a submersion between two smooth manifolds; a smooth j-form $\omega\in\Omega^j[\mathscr{M}]$ over $\mathscr{M}$ has perversity $p\in\{0,\ldots,dim\mathscr{M}-dim\,\mathscr{N}\}$ with respect to π if for every $(p+1)$-uple of smooth vector fields $\boldsymbol{v}_0,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p\in\Xi[\mathscr{M}]$ tangent to fibers of π (i.e. smooth sections of ker $\mathrm{d}\pi$) we have:*

$$\boldsymbol{v}_0\bullet\boldsymbol{v}_1\bullet\ldots\bullet\boldsymbol{v}_p\bullet\omega=0.$$

For example consider the projection $\pi(x,t)=x$ as in figure 3; every 0-perverse form with respect to π is of the form $f(x,t)\mathrm{d}x$. Heuristically, if c is an embedded smooth 1-chain then a 0-perverse form can only detect

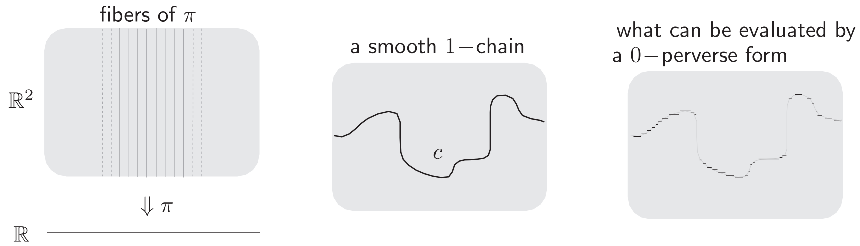

Fig. 3. – Perversity condition.

horizontal infinitesimal elements of c (i.e. elements not parallel to the fibers of π):

Now let $(\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ be an n-dimensional *controlled pseudomanifold* as in definition [1.1] and $\boldsymbol{p}$ a fixed perversity for $\mathscr{X}$; by hypothesis [1.1-B2], the set $\mathscr{X}-\mathscr{X}_{n-2}=\mathscr{X}_{reg}$ is dense in $\mathscr{X}$ and this implies by [1.1-B1] that the relation $S\leq\mathscr{X}_{reg}$ is true for every $S\in\mathscr{S}$. Then, again by definition [1.1-C4], the map $\pi_S|: T_S\cap\mathscr{X}_{reg}\longrightarrow S$ is a smooth *submersion* and this heuristically justifies the following:

DEFINITION 3.2 (Perverse and Intersection differential forms; [BRY] def. 1.2.5, [BHS] chap. B). *Let $(\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ be a controlled pseudomanifold of dimension n, $\boldsymbol{p}$ a fixed perversity for $\mathscr{X}$ and let $j\in\{0,\ldots,n\}$; a smooth differential j-form $\omega\in\Omega^j[\mathscr{X}_{reg}]$ is called $\boldsymbol{p}$-perverse for $\mathscr{X}$ iff $\forall S\in\mathscr{S}$ and $\forall x\in S$ there exists an open neighborhood U for x in T_S such that ω has perversity $\boldsymbol{p}_{(\mathrm{cod}S)}$ with respect to the submersion $\pi_S|:U\cap\mathscr{X}_{reg}\longrightarrow S$ (see definition* [3.1]*). The $\mathbb{R}$-vector space of the intersection j-forms over $\mathscr{X}$ relative to perversity $\boldsymbol{p}$ is defined as:*

$$\mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}] := \left\{\omega\in\Omega^j[\mathscr{X}_{reg}]\,\middle|\, \textit{both } \omega \textit{ and } \mathrm{d}\omega \textit{ are } \boldsymbol{p}\textit{-perverse for } \mathscr{X}\right\}.$$

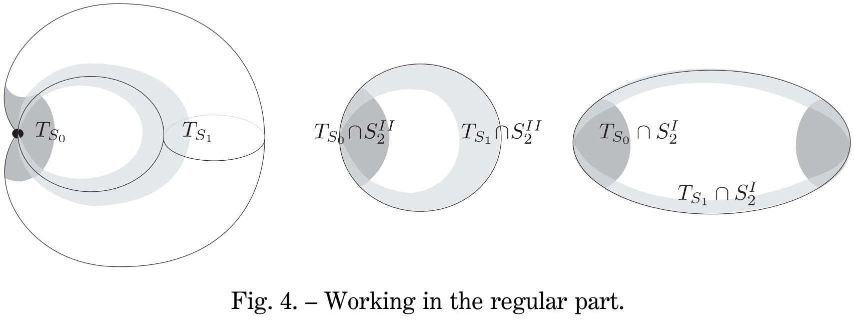

Fig. 4. – Working in the regular part.

Figure 4 shows what is happening in the singular torus of figures 1 and 2; we work only in the regular part using the additional data of the control structure. The restrictions of the retractions $\pi_{S_0}|_{T_{S_0}\cap\mathscr{X}_{reg}}$ and $\pi_{S_1}|_{T_{S_1}\cap\mathscr{X}_{reg}}$ originates the *directions* toward S_0 and S_1 that are smooth submanifold by submersiveness (since S_0 is a point such direction is a whole neighborhood, and for S_1 they are represented by immersed lines, the «normal directions» toward S_1). Thus the geometric discrimination of definition [3.2] can be applied as in figure 3.

Since definition [3.2] is purely local in $\mathscr{X}$ and each open U of $\mathscr{X}$ canonically inherits a controlled pseudomanifold structure we are lead to the following:

DEFINITION 3.3 (Intersection differential forms complex). *The complex of sheaves of intersection forms of perversity* $\boldsymbol{p}$ *on the controlled pseudomanifold* $\mathscr{X}$ *is the subcomplex of* $a_*\Omega^\bullet_{\mathscr{X}_{reg}}$ $(a:\mathscr{X}_{reg}\hookrightarrow\mathscr{X})$ *defined by the assignment* $U\mapsto \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[U]$ *for every* U *open in* $\mathscr{M}$.

REMARK 3.4. We are working with special differential forms defined over the regular part of the pseudomanifold and not with stratified forms (i.e. a smooth form on every stratum); see in particular [PFL] pp. 68-69 and the paper [BF] for a complete analysis of the latter approach.

REMARK 3.5 (Invariance). The complex $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet_{\mathscr{X}}$ is invariant under controlled isomorphisms [1.7] of $\mathscr{X}$ due to the fact that such maps are diffeomorphism over the regular part $\mathscr{X}_{reg}$ and preserves tube retraction and fibers; if a smooth manifold $\mathscr{M}$ is controlled trivially then $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{M}]$ clearly coincides with the de Rham complex $\Omega^\bullet[\mathscr{M}]$. Theorem [7.1] and theorem [2.4] implies that for every pseudomanifold $\mathscr{X}$ the cohomology of $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{X}]$ is naturally isomorphic to the intersection cohomology and hence it is a topological invariant of $\mathscr{X}$ (see [BOR]), i.e. it depends only on the underlying topological structure and not on stratification or control data. This is highly not trivial even for a manifold since it can be unnaturally stratified in a very complex way.

4. The softness of the complex $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet_{\mathscr{X}}$.

The complex of smooth differential forms $\Omega^\bullet_{\mathscr{M}}$ over a smooth manifold is the main example of a *complex of soft sheaves*; this is usually proved using the fact that every sheaf of modules over a *soft sheaf of* *rings with units* is soft, that $\Omega^{\bullet}_{\mathcal{M}}$ is a $\mathcal{C}^{\infty}_{\mathcal{M}}$-module and that $\mathcal{C}^{\infty}_{\mathcal{M}}$ is a soft sheaf; the latter requires the existence of smooth partitions of unity and in order to mimic the same procedure in singular spaces we need ad hoc partitions.

Over a controlled pseudomanifold $\left(\mathcal{X}, \mathcal{S}, \{(T_S, \pi_S, \rho_S)\}_{S \in \mathcal{S}}\right)$ one can work with *real controlled function* (see [PFL]):

$$\mathcal{C}^{cnt}_{\mathcal{X}} :=$$

$$\left\{ f : \mathcal{X} \to \mathbb{R} \,\middle|\, \begin{array}{l} f \text{ is continuous, } f|_S \text{ is smooth for each } S \in \mathcal{S} \text{ and there exists} \\ \text{an open neighborhood } U \text{ for } S : \forall x \in T_S \cap U, f\pi_S(x) = f(x) \end{array} \right\}$$

Thus a controlled function must be constant along the retraction-fibers of every germ of tubular neighborhood; dealing with germs is fundamental to prove theorem [4.1].

To show that $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathcal{X}}$ is a *complex of* $\mathcal{C}^{cnt}_{\mathcal{X}}$*-modules* It is an open set U in $\mathcal{X}$, an intersection j-form $\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U]$ and a controlled function $f \in \mathcal{C}^{cnt}_{U}$; ω can be seen as a particular smooth form in the differential manifold $U \cap \mathcal{X}_{reg}$ and thus multiplied by the smooth function $f|_{U \cap \mathcal{X}_{reg}}$: it is easy to show that $f\omega := f|_{U \cap \mathcal{X}_{reg}}\omega$ is indeed $\boldsymbol{p}$-perverse for U (it respects all $\boldsymbol{p}$-perversity condition even without any hypothesis about f). The exterior derivative $\mathrm{d}(f\omega) := \mathrm{d}(f|_{U \cap \mathcal{X}_{reg}}\omega) = \mathrm{d}(f|_{U \cap \mathcal{X}_{reg}}) \wedge \omega + f|_{U \cap \mathcal{X}_{reg}}\mathrm{d}\omega$ is $\boldsymbol{p}$-perverse due to the fact that the latter is true by definition for $\mathrm{d}\omega$ and that by controlledness $\mathrm{d}(f|_{U \cap \mathcal{X}_{reg}}) = \mathrm{d}(f|_{U \cap \mathcal{X}_{reg}}) \circ \mathrm{d}\pi_S$ for every stratum S in U; thus $\boldsymbol{v} \bullet \mathrm{d}(f|_{U \cap \mathcal{X}_{reg}}) \wedge \omega$ is automatically zero for every $\boldsymbol{v}$ vector field over $U \cap \mathcal{X}_{reg}$ and tangent to π_S-fibers. This implies that $f\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U]$ and, since the restriction maps of such sheaves are indeed restrictions of forms and function, $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathcal{X}}$ is a $\mathcal{C}^{cnt}_{\mathcal{X}}$*-module*; to continue, we recall the following result due to Verona ([VER] pp.8-9):

THEOREM 4.1. *Let $\mathcal{X}$ be a controlled pseudomanifold; then every open cover of $\mathcal{X}$ has a subordinated partition of unit composed of controlled functions.*

Let K be a compact of $\mathcal{X}$ and $f \in \mathcal{C}^{cnt}_{\mathcal{X}}[K]$; by *metrizability* of $\mathcal{X}$ the section f over K can be extended to a controlled function $\widetilde{f} \in \mathcal{C}^{cnt}_{\mathcal{X}}[U]$ over an open set $U \supseteq K$. So if $\{\psi_U, \psi_{\mathcal{X}-K}\}$ is a controlled partition of unit subordinated to the open cover $\{U, \mathcal{X} - K\}$ the controlled function $\widetilde{f}\psi_U$ is a well-defined element of $\mathcal{C}^{cnt}_{\mathcal{X}}[\mathcal{X}]$ extending f. By this we can conclude that:

COROLLARY 4.2. *The sheaf $\mathcal{C}^{cnt}_{\mathcal{X}}$ of controlled functions is soft.*

Corollary 4.3 (Softness). *If $\mathscr{X}$ is a controlled pseudomanifold and $\boldsymbol{p}$ is a perversity then the complex $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ is a complex of soft sheaves.*

The softness of the complex of intersection forms allows to make local computation required in Deligne's axioms without explicit reference to total derived functors (in particular without using injective or flabby-type resolutions).

5. Intersection homotopy operator.

In order to accomplish easily computations over cylinders and cones one needs to develop a Poincaré operator for singular case; let $(\mathscr{X}, \mathscr{S}, \{(T_S, \pi_S, \rho_S)\}_{S\in\mathscr{S}})$ be a controlled pseudomanifold of dimension n and an *open* segment $I =]a, b[$ where a generic point $t_0 \in I$ is fixed; we denote the t_0-section and projection respectively with $s : \mathscr{X} \to \mathscr{X} \times I$ and $\pi : \mathscr{X} \times I \to \mathscr{X}$.

A perversity $\boldsymbol{p}$ associated to the canonical filtration $\{\mathscr{X}_j\}_{j=0\dots n}$ induces a perversity for $\mathscr{X} \times I$ with respect to $\{\mathscr{X}_j \times I\}_{j=0\dots n}$:

$$\begin{array}{lllllllll}
\mathscr{X}_n \times I & (\mathscr{X}_{n-1} \times I = \mathscr{X}_{n-2} \times I) & \mathscr{X}_{n-3} \times I & \dots\dots & \mathscr{X}_{n-k} \times I & \dots\dots & \mathscr{X}_0 \times I \\
\mathscr{X}_n & (\mathscr{X}_{n-1} = \mathscr{X}_{n-2}) & \mathscr{X}_{n-3} & \dots\dots & \mathscr{X}_{n-k} & \dots\dots & \mathscr{X}_0 \\
\boldsymbol{p}_{(0)} & \boldsymbol{p}_{(1)} = \boldsymbol{p}_{(2)} & \boldsymbol{p}_{(3)} & \dots\dots & \boldsymbol{p}_{(k)} & \dots\dots & \boldsymbol{p}_{(n)}
\end{array}$$

The latter is due to the fact that $\mathscr{X} \times I$, decomposed canonically as in [1.5], has no 0-dimensional stratum, so it is no use to assign $\boldsymbol{p}_{(n+1)}$ for codimension $n+1$ (just recall that codimensions of $\mathscr{X}_{n-k}$ in $\mathscr{X}$ and of $\mathscr{X}_{n-k} \times I$ in $\mathscr{X} \times I$ are the same); such argument can be inverted and I can be easily replaced with any boundaryless smooth manifold (e.g $\mathbb{R}^l$).

The morphisms s e π induce two *pullbacks* over the complex of differential forms defined over the regular part:

$$\Omega^{\bullet}_{\mathscr{X}_{reg}} \xrightarrow{\pi^*} \Omega^{\bullet}_{\mathscr{X}_{reg} \times I} \xrightarrow{s^*} \Omega^{\bullet}_{\mathscr{X}_{reg}};$$

next goal is the extension of these morphisms to the complex of intersection forms:

Lemma 5.1. *The pullback operations for smooth forms over the regular part of $\mathscr{X}$ and $\mathscr{X} \times I$ are compatible with every perversity $\boldsymbol{p}$ and induce* *two pullbacks on the intersection forms complex:*

$$\pi^* : \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}} \longrightarrow \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}\times I},$$
$$s^* : \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}\times I} \longrightarrow \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}.$$

PROOF. By construction, each k-codimension stratum S of $\mathscr{X}$ with tube $\pi_S : T_S \to S$ corresponds to the k-codimension stratum $S \times I$ of $\mathscr{X} \times I$ with tube $\pi_S \times id_I : T_S \times I \to S \times I$. So, after choosing a point $(x,t) \in S \times I$ and an open neighborhood $U \times J \subseteq T_S \times I$ for (x,t) the only vector fields over the regular part, parallel to fibers of $\pi_S \times id_I$, are the following:

$$(\boldsymbol{v}, 0) : (U \times J) \cap (\mathscr{X}_{reg} \times I) \to tan(U \cap \mathscr{X}_{reg}) \oplus tan\, I$$

where the field $\boldsymbol{v} : U \cap \mathscr{X}_{reg} \longrightarrow tan(U \cap \mathscr{X}_{reg})$ is parallel to fibers of $\pi_S|_{U\cap\mathscr{X}_{reg}}$; in other words, *every vector in a smooth point of the composed tube must have a component tangent to the fiber of original tube and the other must be zero.*

Suppose now that $\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}]$; ω is $\boldsymbol{p}$-perverse and if $(\boldsymbol{v}_0, 0)$, $(\boldsymbol{v}_1, 0), \ldots, (\boldsymbol{v}_{\boldsymbol{p}(codS\times I)}, 0)$ are vector fields parallel to fibers of the tube $T_S \times I$ of a stratum $S \times I \ni (x,t)$ evaluating inner products and the latter observation gives:

$$(\pi^*\omega)_{[(x,t)]}((\boldsymbol{v}_0(x), 0), \ldots, (\boldsymbol{v}_{\boldsymbol{p}(codS\times I)}(x), 0), -) = \omega_{[x]}(\boldsymbol{v}_0(x), \ldots, \boldsymbol{v}_{\boldsymbol{p}(codS\times I)}(x), -) = 0.$$

In the same way, if $\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X} \times I]$ and $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{p}(codS)}$ are vector fields parallel to fibers of the tube T_S for a stratum $S \ni x$ then:

$$(s^*\omega)_{[x]}(\boldsymbol{v}_0(x), \ldots, \boldsymbol{v}_{\boldsymbol{p}(codS)}(x), -) = \omega_{[(x,t_0)]}((\boldsymbol{v}_0(x), 0), \ldots, (\boldsymbol{v}_{\boldsymbol{p}(codS)}(x), 0), -) = 0$$

Since by hypothesis even $\mathrm{d}\omega$ is $\boldsymbol{p}$-perverse one obtains $\mathrm{d}s^* = s^*\mathrm{d}$ and $\mathrm{d}\pi^* = \pi^*\mathrm{d}$; the same reasoning can be done for the exterior derivatives and consequently s^* and π^* are well defined over intersection forms. ☐

It is possible to define an *integration operator* over vertical fibers of $\mathscr{X} \times I$, following mutatis mutandis the standard construction over smooth manifolds; we begin to work with smooth forms defined over $\mathscr{X}_{reg} \times I$, the regular part of the pseudomanifold $\mathscr{X} \times I$ (recall that I is an *open segment* in $\mathbb{R}$ and t_0 a chosen base point). So if $\omega \in \Omega^j[\mathscr{X}_{reg} \times I]$ one defines $\theta^j\omega \in \Omega^{j-1}[\mathscr{X}_{reg} \times I]$ as:

$$(\theta^j\omega)_{[x,t]}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}) := \int\limits_{[t_o,t]} \omega_{[x,\tau]}\left(\frac{\partial}{\partial t}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}\right) \mathrm{d}\tau$$

for every point $(x,t)\in\mathscr{X}_{reg}\times I$ and every $(j-1)$-uple of tangent vectors $\boldsymbol{v}_1,\dots,\boldsymbol{v}_{j-1}\in tan_x\mathscr{X}_{reg}\oplus tan_t I$ (as usual θ^0 is by definition zero over 0-form) where $\frac{\partial}{\partial t}$ is the standard tangent vector of $I\subset\mathbb{R}$. This is the *homotopy operator* of Poincaré and the following result is also a classic one (see [BT] pp. 33-35), adapted in this case to the smooth manifold $\mathscr{X}_{reg}\times I$:

THEOREM 5.2. *The following homotopic formula holds for every differential j-form ω over $\mathscr{X}_{reg}\times I$, the regular part of $\mathscr{X}\times I$:*

$$\mathrm{d}\theta^j\omega-\theta^{j+1}\mathrm{d}\omega=(-1)^j\big((s\pi)^*\omega-\omega\big).$$

In what follows, we describe how the latter operator naturally extends to intersection forms;

THEOREM 5.3 (Intersection homotopy operator). *The Poincaré homotopy operator $\theta^\bullet$ is compatible with $\boldsymbol{p}$, i.e. the following relation continues to hold over $\mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}\times I]$ for each intersection j-form ω:*

$$\mathrm{d}\theta^j\omega-\theta^{j+1}\mathrm{d}\omega=(-1)^j\big((s\pi)^*\omega-\omega\big);$$

in particular, $\theta^\bullet$ realizes an algebraic homotopy between the endomorphisms $(s\pi)^*$ and $\mathrm{id}_{\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{X}\times I]}$ in $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{X}\times I]$:

$$\begin{tikzcd}
& \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}\times I] \arrow[r, "\mathrm{d}"] \arrow[dl, bend right, "\theta^j"'] \arrow[d, shift right, "(s\pi)^*"'] \arrow[d, shift left, "id"] & \mathrm{I}_{\boldsymbol{p}}\Omega^{j+1}[\mathscr{X}\times I] \arrow[dl, bend left, "\theta^{j+1}"] \\
\mathrm{I}_{\boldsymbol{p}}\Omega^{j-1}[\mathscr{X}\times I] \arrow[r, "\mathrm{d}"] & \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}\times I] &
\end{tikzcd}$$

PROOF. Since every intersection form is a differential form over the regular part (recall that $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet_{\mathscr{X}\times I}\subseteq a_*\Omega^\bullet_{\mathscr{X}_{reg}\times I}$), by [5.2] it suffices to show that θ^j respects perversity; we proceed with a *decreasing induction* starting from $n+1$ (the dimension of $\mathscr{X}\times I$):

$$\begin{tikzcd}
& {} \arrow[r, dotted] & \mathrm{I}_{\boldsymbol{p}}\Omega^n[\mathscr{X}\times I] \arrow[r] \arrow[dl, "\theta^n"'] & \mathrm{I}_{\boldsymbol{p}}\Omega^{n+1}[\mathscr{X}\times I] \arrow[r, dotted] \arrow[dl, "\theta^{n+1}"'] & 0 \\
{} \arrow[r, dotted] & \mathrm{I}_{\boldsymbol{p}}\Omega^{n-1}[\mathscr{X}\times I] \arrow[r] & \mathrm{I}_{\boldsymbol{p}}\Omega^n[\mathscr{X}\times I] \arrow[r] & \mathrm{I}_{\boldsymbol{p}}\Omega^{n+1}[\mathscr{X}\times I] \arrow[r, dotted] & 0
\end{tikzcd}$$

It must be shown that $\theta^j\big(\mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}\times I]\big)\subseteq\mathrm{I}_{\boldsymbol{p}}\Omega^{j-1}[\mathscr{X}\times I]$; suppose then $\omega\in\mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{X}\times I]$ and consequently that ω is $\boldsymbol{p}$-perverse. It is easily seen that

$\theta^j\omega$ is $\boldsymbol{p}$-perverse (just recall how inner product works; in a simpler way note that if ω was $f(x,t,l)\mathrm{d}x_{j_1}\wedge\ldots\wedge\mathrm{d}x_{j_a}\wedge\mathrm{d}t_{i_1}\wedge\ldots\wedge\mathrm{d}t_{i_b}\wedge\mathrm{d}l$ in $\mathbb{R}^{a+b}\times\mathbb{R}$ then $\theta^{a+b+1}\omega$ would be $\left(\int f(x,t,L)\mathrm{d}L\right)\mathrm{d}x_{j_1}\wedge\ldots\wedge\mathrm{d}x_{j_a}\wedge\mathrm{d}t_{i_1}\wedge\ldots\wedge\mathrm{d}t_{i_b}$: integration cannot increase perversity).

Induction is needed to show that $\mathrm{d}\theta^j\omega$ is $\boldsymbol{p}$-perverse and one can proceed as follow:

- if $j=n+1$ then the smooth homotopy relation gives $\mathrm{d}\theta(n+1)\,\omega = = (-1)^{n+1}\left((s\pi)^*\omega-\omega\right)$; the inductive start point is settled noting that $(s\pi)^*$ respects the perversity of ω since $\mathrm{d}(s\pi)_{[x,t]}(\boldsymbol{v},l)=(\boldsymbol{v},0)$ and ω is $\boldsymbol{p}$-perverse;
- if $j<n+1$ then $\mathrm{d}\omega$ is $\boldsymbol{p}$-perverse and by inductive hypothesis θ^{j+1} respects perversity; smooth homotopy formula allows to conclude.

This ends the proof of the theorem.

6. Local computations.

In this section we perform some basic computations on cones and cylinders built from a given pseudomanifold (which can be found also in the paper [BHS], Chapitre C: Calculs locaux, page 230); if $\mathcal{X}$ is a controlled pseudomanifold of dimension n by local structure theorem [1.8] every point x belonging to a k-dimensional stratum S of $\mathcal{X}$ admits an open neighborhood U in $\mathcal{X}$ isomorphic (as controlled space, [1.7]) to $\mathbb{R}^{n-k}\times con^\circ\mathcal{L}$ (where $\mathcal{L}$ is a *compact* pseudomanifold of dimension $k-1$) such that the pair $(U\cap S,x)$ is mapped to $(\mathbb{R}^{n-k}, \mathit{vertex})$. Since as noted in remark [3.5] the complex of intersection forms is invariant under controlled isomorphism this will suffice.

The following theorem is well-known in the smooth case ([BT] p. 35) and its proof in the singular case proceeds along the same line:

THEOREM 6.1 (Cylinders). *Let $\mathcal{X}$ be a controlled pseudomanifold and $\boldsymbol{p}$ an associated perversity; then the pullback $s^*: \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}\times I]\to \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}]$ induces an isomorphism for each cohomologic $\mathbb{R}$-vector space:*

$$\mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}\times I]) \xrightarrow{\cong} \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}]) \qquad \forall j\in\{0,\ldots,\mathit{dim}\mathcal{X}\}.$$

PROOF. As in the smooth case $\pi s = id_{\mathcal{X}}$ and consequently $s^*\pi^* = = id_{\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}]}$; moreover, by theorem [5.3], $\theta^\bullet$ realizes an algebraic homotopy between π^*s^* and $\mathrm{id}_{\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathcal{X}\times I]}$. Thus s^* is a quasi-isomorphism between the two given complexes. □

Roughly speaking, theorem [6.1] allows to represent a cohomology class of the cylinder $\mathscr{X} \times I$ as a class of one of its horizontal slice $\mathscr{X} \times \{t\}$; it will be useful in theorem [6.4] to *move* classes between different slices.

To deal with the cone $con^\circ \mathscr{L}$ is essential to recall that, as described in [1.6], its control structure makes heavy use of the control over $\mathscr{L} \times]0, +\infty[$ and that:

> *the vertex is the only stratum of* $con^\circ \mathscr{L}$ *with max codimension (i.e.* $dim \mathscr{L} + 1$*) and the corresponding retraction has only one fiber, namely* $T_{vert} := \mathscr{L} \times [0, \varepsilon[$.

Note that every perversity $\boldsymbol{p}$ over a cone $con^\circ \mathscr{L}$ (stratified and controlled via its base) naturally induces a perversity $\boldsymbol{p}$ over its cylinder $\mathscr{L} \times]0, +\infty[$ and its base $\mathscr{L}$ (just discard the perversity for the vertex $\boldsymbol{p}_{(dim \mathscr{L}+1)} = \boldsymbol{p}_{(cod\ vertex)}$); the contrary is obviously false.

The crucial ingredient is the following immediately proved lemma:

Lemma 6.2. *Let* $\mathscr{M}$ *be a smooth manifold,* $\omega \in \Omega^j[\mathscr{M}]$ *a differential form and fix an integer* $l \leq j$*; then if the inner product of* ω *with any l-uple of vector fields over* $\mathscr{M}$ *is always zero we can conclude that* ω *vanished. In other words:*

$$(\forall \boldsymbol{v}_1, \ldots, \boldsymbol{v}_l \in \Xi[\mathscr{M}] \quad \boldsymbol{v}_1 \bullet \ldots \bullet \boldsymbol{v}_l \bullet \omega = 0) \Longrightarrow \quad \omega = 0.$$

Remark 6.3. It is worthy to remark that if we insert via inner product $l > j$ fields into a j-form we automatically obtain zero by definition, so the latter relation doesn't give any hint about the nullity of the original form.

As in the paper [BOR] the key step to apply the axiomatization of Deligne is to understand the cohomological relation between a cone and its base as follow:

Theorem 6.4 (Cones). *Let* $\mathscr{L}$ *be a compact controlled pseudomanifold of dimension k; endow the cone* $con^\circ \mathscr{L}$ *with the canonical control structure from its base* $\mathscr{L}$ *as in* [1.6] *and fix an associated perversity* $\boldsymbol{p}$ *for* $con^\circ \mathscr{L}$*. Then the morphism* s^* *induces the following isomorphisms:*

$$\mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ \mathscr{L}]) \cong \begin{cases} \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{L}]) & \text{if } j \leq \boldsymbol{p}_{(k+1)}, \\ 0 & \text{if } j > \boldsymbol{p}_{(k+1)}. \end{cases}$$

Proof. Using the previous remarks and lemma [6.2] we obtain the following relation between intersection forms over the cone $con^\circ \mathscr{L}$ and forms over the associated cylinder $\mathscr{L}_{reg} \times \mathbb{R}^{>0}$:

$$\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[con^{\circ}\mathscr{L}]=\left\{\omega\in\Omega^{j}[\mathscr{L}_{reg}\times\mathbb{R}^{>0}]\;\middle|\;\begin{array}{l}\omega\in\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[\mathscr{L}\times\mathbb{R}^{>0}]\text{ and:}\\ \text{if } j>\boldsymbol{p}_{(k+1)}\text{ then }\exists\varepsilon>0:\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}=0\\ \text{if } j=\boldsymbol{p}_{(k+1)}\text{ then }\exists\varepsilon>0:\mathrm{d}\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}=0\end{array}\right\}$$

To prove this, we begin to note that ω and $\mathrm{d}\omega$ are in particular $\boldsymbol{p}$-perverse for $\mathscr{L}\times\mathbb{R}^{>0}$ (i.e. $\omega\in\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[\mathscr{L}\times\mathbb{R}^{>0}]$) since the control structure of the cone is an extension of the one of its cylinder (examples [1.5] and [1.6]); this is not enough, since there are the perversity conditions over the tube for the vertex of the cone $T_{vert}:=\mathscr{L}\times[0,\varepsilon[$ with $\varepsilon\geq 0$.

This means that for every $(\boldsymbol{p}_{(k+1)}+1)$-uple of vector fields $\boldsymbol{v}_0,\boldsymbol{v}_1,\dots,\boldsymbol{v}_{\boldsymbol{p}_{(k+1)}}$ defined in $T_{vert}\cap\mathscr{L}_{reg}\times]0,\varepsilon[$ and tangent to fibers of π_{vert} the following relations must be satisfied:

$$\begin{cases}\boldsymbol{v}_0\bullet\boldsymbol{v}_1\bullet\dots\bullet\boldsymbol{v}_{\boldsymbol{p}_{(k+1)}}\bullet\omega=0,\\ \boldsymbol{v}_0\bullet\boldsymbol{v}_1\bullet\dots\bullet\boldsymbol{v}_{\boldsymbol{p}_{(k+1)}}\bullet\mathrm{d}\omega=0.\end{cases}$$

Next, recall that since the fiber of π_{vert} is by construction the regular part of the whole retracting neighborhood (see the remark in [1.6] and figure 5), the latter conditions must be verified for *every* vector field; now lemma [6.2] allows to conclude: if $\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}$ is a smooth j-form that becomes zero after inserting $\boldsymbol{p}_{(k+1)}+1$ vector fields over $\mathscr{L}_{reg}\times]0,\varepsilon[$ and $j\geq\boldsymbol{p}_{(k+1)}+1$ then $\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}=0$ and consequently $\mathrm{d}\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}=0$. On the contrary, one has just to impose a condition over $\mathrm{d}\omega$ only if $j+1=\boldsymbol{p}_{(k+1)}+1$, requiring that $\mathrm{d}\omega|_{\mathscr{L}_{reg}\times]0,\varepsilon[}=0$; this is enough taking into account remark [6.3].

Concluding, one is able to deal with forms over a cone just working over its cylinder plus some nullity condition nearby the vertex since the vertex is the only $k+1$-codimensional stratum and its associated retraction fiber is

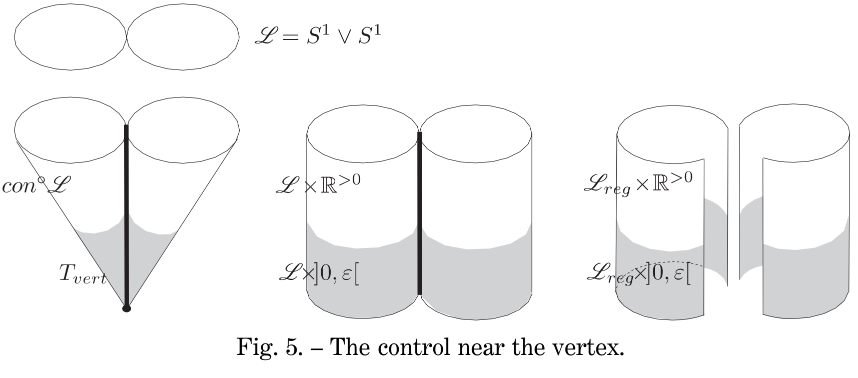

Fig. 5. – The control near the vertex.

the whole neighborhood $T_{vert} \cap \mathscr{L}_{reg} \times \mathbb{R}^{>0}$: every vector field approaches the singularity.

We now use this characterization to prove the theorem, analyzing separately the two cases:

– the $j > \boldsymbol{p}_{(k+1)}$ case –

Let $\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^j[con^\circ\mathscr{L}]$ a *cocycle*; by the latter, the following relations holds:

$$\begin{cases} \omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{L} \times \mathbb{R}^{>0}] \\ \omega = 0 \ \text{ over } \ \mathscr{L}_{reg} \times]0, \varepsilon[\ \text{ (for some } \varepsilon > 0) \end{cases}$$

So ω is just a closed intersection form over the cylinder, zero nearby the vertex; we are going to show that ω is indeed a coboundary moving the cohomology class via theorem [6.1] nearby the vertex, where some nullity condition holds. For this purpose, choose a point $a_0 \in]0, \varepsilon[$ and denote with s^*, π^* and $\theta^\bullet$ the *a_0-section, projection and homotopy* operator related to a_0 and $\mathscr{L} \times \mathbb{R}^{>0}$ as in section [5]. The homotopy relation $\mathrm{d}\theta^j\omega = (-1)^{(j+1)}\omega$ is true in $\mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{L} \times \mathbb{R}^{>0}]$ due to the fact that ω is a cocycle and $\pi^* s^* \omega = 0$ since ω can be pushed in a slice where it is zero; thus to obtain a primitive for ω in $\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ\mathscr{L}]$ it is enough to show that $\theta^j\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^{j-1}[con^\circ\mathscr{L}]$. By construction $\theta^j\omega$ «lives» over the cylinder, but it is actually an intersection form belonging to the cone due to the fact that $j > \boldsymbol{p}_{(k+1)}$ and so $\omega = 0$ over $\mathscr{L}_{reg} \times]0, \varepsilon[$. Finally $\theta^j\omega$ is a primitive of our cocycle that becomes a coboundary, trivializing the associate cohomology:

$$\mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ\mathscr{L}]) \cong 0 \qquad \forall j > \boldsymbol{p}_{(k+1)}.$$

– the $j \leq \boldsymbol{p}_{(k+1)}$ case –

Again, using the initial characterization, we get:

$$\begin{cases} \mathrm{I}_{\boldsymbol{p}}\Omega^j[con^\circ\mathscr{L}] = \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{L} \times \mathbb{R}^{>0}] & \text{if } j < \boldsymbol{p}_{(k+1)} \\ \mathrm{I}_{\boldsymbol{p}}\Omega^j[con^\circ\mathscr{L}] \cap \mathit{kernel}\ \mathrm{d} = \mathrm{I}_{\boldsymbol{p}}\Omega^j[\mathscr{L} \times \mathbb{R}^{>0}] \cap \mathit{kernel}\ \mathrm{d} & \text{if } j = \boldsymbol{p}_{(k+1)} \end{cases}$$

The former relation is trivial and to achieve the latter it is enough to observe that if $j = \boldsymbol{p}_{(k+1)}$ the compatibility between ω and vertex tube retraction requires the vanishing of its exterior derivative nearby; this is trivially satisfied if $\mathrm{d}\omega = 0$. Since only closed forms are needed to compute cohomology up to level $j \leq \boldsymbol{p}_{(k+1)}$ the vertex gives no additional constraint and the cylindric computation [6.1] allows to conclude:

$$\mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ\mathscr{L}]) = \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{L} \times \mathbb{R}^{>0}]) \xrightarrow{\cong} \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{L}]) \quad \forall j \leq \boldsymbol{p}_{(k+1)}.$$

$\square$

Note that cohomology done with intersection forms is not homotopy invariant; for example, consider the contractible space $con^{\circ}(S^1 \times S^1)$ and fix a perversity $\boldsymbol{p}$ such that $\boldsymbol{p}(3)=1$ (this is enough since such space has only a singular point with codimension 3, the vertex). Theorem [6.4] implies that:

$$\mathrm{H}^1\big(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[con^{\circ}(S^1 \times S^1)]\big) \cong \mathrm{H}^1\big(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[S^1 \times S^1]\big) \cong \mathrm{H}^1_{\mathrm{DR}}\big(S^1 \times S^1\big) \cong \mathbb{R}^1 \times \mathbb{R}^1$$

7. Verification of Deligne's axioms.

We are now ready to show that $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ satisfies Deligne's Axioms ([2.2]) and then by theorem [2.4] it has the same local cohomological properties of $\mathrm{I}_{\boldsymbol{p}}\mathrm{C}^{\bullet}_{\mathscr{X}}$; this will follow the results of the previous sections about the *local structure* of a pseudomanifold ([1.8]), *softness* of $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ ([4.3]) and *cylindric-conic computations* ([6.1], [6.4]).

THEOREM 7.1. *Let $\mathscr{X}$ be a controlled pseudomanifold and $\boldsymbol{p}$ be an associated perversity; then the complex of sheaves $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ of intersection forms satisfies Deligne's axioms relative to perversity $\boldsymbol{p}$ and constant sheaf $\mathbb{R}_{\mathscr{X}_{reg}}$.*

PROOF. Let n be the dimension of $\mathscr{X}$; boundedness and triviality in negative degree are obvious by definition; the first axiom follows easily from the fact that $\mathscr{X}_{reg}$ is a smooth n-manifold and so that an intersection form is just a differential form on $\mathscr{X}_{reg}$; Poincaré Lemma allows to conclude.

To simplify the remaining verifications, we set $S_j := \bigcup\{S \in \mathscr{S} \mid dim\, S = j\}$ and we use the notation introduced in section [2]; it is necessary to show that:

$$\forall k, \forall x \in S_{n-k} \begin{cases} \mathscr{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}})_x = 0 & \text{if } j > \boldsymbol{p}(k) \\ \mathscr{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{U_{k+1}})_x \xrightarrow{\approx} \mathscr{H}^j(\mathrm{R}i_{k*}\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{U_k})_x & \text{if } j \leq \boldsymbol{p}(k). \end{cases}$$

Taking into account the remark [2.3] and the *softness* result of [4.3] for the complex $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}$ the diagram $\mathscr{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{U_{k+1}})_x \xrightarrow{\approx} \mathscr{H}^j(\mathrm{R}i_{k*}\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{U_k})_x$ can be replaced by the following equivalent one:

$$\varinjlim_{x\in V\subseteq\mathscr{X}} \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[V]) \xrightarrow{\approx} \varinjlim_{x\in V\subseteq\mathscr{X}} \mathrm{H}^j(\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[V-S_{n-k}])$$

where the direct limit can be done over the directed set of *trivializing open neighborhoods* for $\mathscr{X}$ over x and *the arrow is simply the restriction map*; this simple observation allows to avoid any explicit reference to total derived functors and consequently to apply theorems [6.1] and [6.4] with ease. Let V be a trivializing neighborhood of x and let $V\xrightarrow{\approx}\mathbb{R}^{n-k}\times con^\circ\mathscr{L}$ be a *controlled isomorphism* (by the local structure theorem [1.8]), where $\mathscr{L}$ denotes the controlled compact *link* for S_{n-k} in x (S_{n-k} has dimension $k-1$); we therefore can form the following commutative diagram:

$$\begin{tikzcd}
\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[V] \arrow[r, "\approx"] \arrow[d, Rightarrow] & \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathbb{R}^{n-k}\times con^\circ\mathscr{L}] \arrow[r, "\approx"] \arrow[d] & \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ\mathscr{L}] \arrow[d] \arrow[dr, dotted, "\approx\ \text{if } \bullet\leq\boldsymbol{p}(k)"] & \\
\mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[V-S_{n-k}] \arrow[r, "\approx"] & \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathbb{R}^{n-k}\times(con^\circ\mathscr{L}-\bullet)] \arrow[r, "\approx"] & \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[con^\circ\mathscr{L}-\bullet] \arrow[r, "\approx"] & \mathrm{I}_{\boldsymbol{p}}\Omega^\bullet[\mathscr{L}]
\end{tikzcd}$$

(we need here to know that any perversity works with *codimension*, and this implies that a $\boldsymbol{p}$ relative to $\mathbb{R}^{n-k}\times con^\circ\mathscr{L}$ induces a perversity for $con^\circ\mathscr{L}$). The commutativity of the diagram is implied by the fact that vertical arrows are restrictions and the remaining ones are induced by inclusions and every *horizontal arrow* is a quasi-isomorphism of complexes of $\mathbb{R}$-vector spaces due to the computation over cylinders; the following reasoning allows to conclude:

- if $j>\boldsymbol{p}(k)$ then the first row gives the result after a direct limit;
- if $j\leq\boldsymbol{p}(k)$ then the *dashed* arrow is a quasi-isomorphism of $\mathbb{R}$-modules complexes till order $\boldsymbol{p}(k)$ (by theorem [6.4]) and we deduce that the *double arrow* is a quasi-isomorphism up to level $\boldsymbol{p}(k)$.

Note that the isomorphism [2.4] is obtained in an abstract way and absolutely no hint is given on its concrete definition; however in the paper [BHS] pp. 223-224 a procedure to perform integration of perverse forms over perverse chains (with some additional conditions) is discussed in detail.

8. Mayer-Vietoris sequence and examples.

Here we collect some lemmas that help in computation of intersection cohomology via differential forms.

Definition 8.1 (de Rham Intersection Cohomology). *Let* $\mathscr{X}$ *be a controlled pseudomanifold and* $\boldsymbol{p}$ *a perversity for* $\mathscr{X}$*; the j-th module of de Rham Intersection Cohomology of* $\mathscr{X}$ *relative to* $\boldsymbol{p}$ *is:*

$$\mathrm{I}_{\boldsymbol{p}}H^{j}_{\mathrm{DR}}(\mathscr{X}) := \mathbb{H}^{j}(\mathscr{X};\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}) = \mathrm{H}^{j}\mathrm{R}\Gamma(\mathscr{X};\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}_{\mathscr{X}}) = \mathrm{H}^{j}(\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[\mathscr{X}]).$$

To identify the 0-th cohomology module note that an intersection 0-form $f \in \mathrm{I}_{\boldsymbol{p}}\Omega^{0}[\mathscr{X}]$ is a smooth map defined over $\mathscr{X}_{reg}$ such that its differential df is a $\boldsymbol{p}$-perverse 1-form; if f is a cocycle then $\mathrm{d}f = 0$ and thus $\mathrm{I}_{\boldsymbol{p}}H^{0}_{\mathrm{DR}}(\mathscr{X})$ is the vector space of the *locally constant maps* defined in $\mathscr{X}_{reg}$:

Lemma 8.2. *Let* $\mathscr{X}$ *be a controlled pseudomanifold and* $\boldsymbol{p}$ *a fixed perversity; then the* 0*-th module of the de Rham intersection cohomology computes the number of connected components of the regular part of* $\mathscr{X}$*, i.e.:*

$$\mathrm{I}_{\boldsymbol{p}}H^{0}_{\mathrm{DR}}(\mathscr{X}) \cong \prod_{j\in J}\mathbb{R} \qquad \text{with } J := \{\text{connected components of } \mathscr{X}_{reg}\}$$

Theorem 8.3 (Mayer-Vetoris sequence). *Let* $\mathscr{X}$ *be a controlled pseudomanifold,* $\boldsymbol{p}$ *a perversity and* U, V *open subsets of* $\mathscr{X}$ *such that* $\mathscr{X} = U \cup V$*; then the following diagram of complex of* $\mathbb{R}$*-vector spaces is exact:*

$$0 \longrightarrow \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[\mathscr{X}] \xrightarrow{\alpha} \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[U] \oplus \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[V] \xrightarrow{\beta} \mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[U\cap V] \longrightarrow 0$$

where $\alpha := \begin{bmatrix} res_{\mathscr{X},U} \\ res_{\mathscr{X},V} \end{bmatrix}$ *and* $\beta := [res_{U,U\cap V} - res_{V,U\cap V}]$ *are induced by the restriction morphisms.*

Proof ([BT] pp. 22-23). The exactness at first two nodes is easily proved as in the smooth case; to check the surjectivity of β one can proceed as follow: let $\omega \in \mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U\cap V]$ and using theorem [4.1] choose a *controlled partition of unity* $\{\psi_U, \psi_V\}$ subordinated to the open cover $\{U, V\}$ of $\mathscr{X}$. As shown in section [4], by controlledness, the forms $(\psi_V)|_{U\cap V}\omega$ and $(\psi_U)|_{U\cap V}\omega$ belong to $\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U\cap V]$ and as in the smooth case they can be extended by zero to elements ω_U and ω_V of $\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U]$ and $\mathrm{I}_{\boldsymbol{p}}\Omega^{j}[V]$ respectively (note that to obtain a form in U one must multiply for ψ_V and viceversa). It is clear that β maps $(\omega_U, -\omega_V) \in \mathrm{I}_{\boldsymbol{p}}\Omega^{j}[U] \oplus \mathrm{I}_{\boldsymbol{p}}\Omega^{j}[V]$ to ω.

For a non trivial example of computations consider the topological space $S^1 \times \sum T^3$ where $T^3 = S^1 \times S^1 \times S^1$ is the 3-torus and $\sum T^3 := := \frac{[-1,1]\times T^3}{<\{-1\}\times T^3, \{1\}\times T^3>}$ is the suspension of T^3 (see also [BOR] pp. 35-39 for a similar computation using intersection homology); such space is a controlled pseudomanifold since S^1 and T^3 are smooth manifolds trivially controlled and a suspension can be controlled doubling the cone construction. The singular part is the disjoint union of two S^1 and it is enough to fix a $p \in \{0,1,2\}$ to assign a perversity relative to the 1-strata; by theorems [6.1], [6.4] and remark [3.5] the following relations hold:

$$\begin{aligned}
&\mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(]-1,1[\times T^3\big) \cong \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(T^3\big) = \mathrm{H}^{\bullet}_{\mathrm{DR}}\big(T^3\big) \cong \big(\mathbb{R},\mathbb{R}^3,\mathbb{R}^3,\mathbb{R},0,0,\dots\big)\\
&\mathrm{I}_0\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(con^{\circ}T^3\big) \cong \big(\mathbb{R},0,0,0,0,\dots\big)\\
&\mathrm{I}_1\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(con^{\circ}T^3\big) \cong \big(\mathbb{R},\mathbb{R}^3,0,0,0,\dots\big)\\
&\mathrm{I}_2\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(con^{\circ}T^3\big) \cong \big(\mathbb{R},\mathbb{R}^3,\mathbb{R}^3,0,0,\dots\big)
\end{aligned}$$

Now we can use Mayer Vietoris to compute for example $\mathrm{I}_2\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(\sum T^3\big)$ as follow: consider $\sum T^3$ as $con^{+}T^3 \cup con^{-}T^3$ where $con^{+}T^3 := \frac{]-1,1]\times T^3}{<\{1\}\times T^3>}$, $con^{-}T^3 := \frac{[-1,1[\times T^3}{<\{-1\}\times T^3>}$ and $con^{+}T^3 \cap con^{-}T^3 = \,]-1,1[\,\times T^3$; by Mayer-Vietoris theorem the following diagram is exact:

$$\begin{tikzcd}
 & & 0 \arrow[dll, bend right] \\
\mathrm{I}_2\mathrm{H}^{0}_{\mathrm{DR}}\big(\sum T^3\big) \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r, Rightarrow] & \mathbb{R} \arrow[dll, bend right] \\
\mathrm{I}_2\mathrm{H}^{1}_{\mathrm{DR}}\big(\sum T^3\big) \arrow[r] & \mathbb{R}^3\oplus\mathbb{R}^3 \arrow[r, Rightarrow] & \mathbb{R}^3 \arrow[dll, bend right] \\
\mathrm{I}_2\mathrm{H}^{2}_{\mathrm{DR}}\big(\sum T^3\big) \arrow[r] & \mathbb{R}^3\oplus\mathbb{R}^3 \arrow[r, Rightarrow] & \mathbb{R}^3 \arrow[dll, bend right] \\
\mathrm{I}_2\mathrm{H}^{3}_{\mathrm{DR}}\big(\sum T^3\big) \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[dll, bend right] \\
\mathrm{I}_2\mathrm{H}^{4}_{\mathrm{DR}}\big(\sum T^3\big) \arrow[r] & 0 &
\end{tikzcd}$$

A simple dimensional count is not enough to identify the cohomology of $\sum T^3$, however in this case every double arrow $\Rightarrow$ is clearly an *epimorphism*: in fact the map $\mathrm{I}_2\mathrm{H}^{k}_{\mathrm{DR}}\big(con^{+}T^3\big) \to \mathrm{I}_2\mathrm{H}^{k}_{\mathrm{DR}}\big(]-1,1[\times T^3\big)$ is an isomorphism if $k \in \{0,1,2\}$ as shown in the proof of theorem [6.4] and hence by construction of the Mayer-Vietoris sequence from [8.3] the maps $\Rightarrow$ are epimorphism. Thus by exactness the dimension of every unknown module can be computed and a similar reasoning can be made for the other perversity giving the following results:

$$\begin{aligned}
&\mathrm{I}_0\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(\sum T^3\big) \cong \big(\mathbb{R},0,\mathbb{R}^3,\mathbb{R}^3,\mathbb{R},0,0,\dots\big)\\
&\mathrm{I}_1\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(\sum T^3\big) \cong \big(\mathbb{R},\mathbb{R}^3,0,\mathbb{R}^3,\mathbb{R},0,0,\dots\big)\\
&\mathrm{I}_2\mathrm{H}^{\bullet}_{\mathrm{DR}}\big(\sum T^3\big) \cong \big(\mathbb{R},\mathbb{R}^3,\mathbb{R}^3,0,\mathbb{R},0,0,\dots\big)
\end{aligned}$$

Note that since the complex $\mathrm{I}_{\boldsymbol{p}}\Omega^{\bullet}[\sum T^3]$ computes the intersection cohomology the Poincaré duality for complementary perversities must hold.

Instead of using again Mayer-Vietoris sequence to finally compute $\mathrm{I}_p\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \sum T^3)$ one can use the following theorem:

THEOREM 8.4 (Partial Künneth formula). *Let $\mathscr{M}$ be a compact smooth manifold without boundary, $\mathscr{X}$ a controlled pseudomanifold and $\boldsymbol{p}$ a perversity; then there is an isomorphism:*

$$\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathscr{M}) \otimes \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathscr{X}) \xrightarrow{\cong} \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathscr{M} \times \mathscr{X})$$

PROOF. The manifold $\mathscr{M}$ has a finite *good cover* (i.e. every intersection of opens of the cover is empty or diffeomorphic to $\mathbb{R}^m$; see [BT] pp. 42-43) since is compact; moreover, being $\mathscr{M}$ controlled trivially, the space $\mathscr{M} \times \mathscr{X}$ can be controlled as in example [1.5]. Consider the projections $\mathscr{M} \xleftarrow{\pi} \mathscr{M} \times \mathscr{X} \xrightarrow{\rho} \mathscr{X}$ and the following map:

$$\Psi : \mathrm{H}^{a}_{\mathrm{DR}}(\mathscr{M}) \times \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{b}_{\mathrm{DR}}(\mathscr{X}) \longrightarrow \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{a+b}_{\mathrm{DR}}(\mathscr{M} \times \mathscr{X})$$

$$(\omega, \mu) \mapsto \pi^*\omega \wedge \rho^*\mu$$

To show that $\pi^*\omega \wedge \rho^*\mu$ is $\boldsymbol{p}$-perverse recall that the retraction of a tube $\mathscr{M} \times T_S$ of a stratum $\mathscr{M} \times S$ is $id_{\mathscr{M}} \times \pi_S$; hence if $(m_1, \ldots, m_i, x_1, \ldots, x_j, t_1, \ldots t_k)$ are local coordinates for $\mathscr{M} \times (T_S \cap \mathscr{X}_{reg})$ with $(t_1, \ldots t_k)$ as fiber coordinates for the submersion $\pi_S|$ then $\pi^*\omega = f\,\mathrm{dm}_{l_1} \wedge \ldots \mathrm{dm}_{l_a}$ for some smooth function f and hence the perversity condition is clearly depending only on the perversity of μ due to the definition of $\wedge$. In a similar manner, $\mathrm{d}(\pi^*\omega \wedge \rho^*\mu) = \pi^*\mathrm{d}\omega \wedge \rho^*\mu + (-1)^a \pi^*\omega \wedge \rho^*\mathrm{d}\mu$ is $\boldsymbol{p}$-perverse since the same is true for $\mathrm{d}\mu$ and this implies that $\pi^*\omega \wedge \rho^*\mu$ is indeed an intersection form.

One has to check that if ω is a smooth closed a-form, α is a smooth $(a-1)$-form, μ is a closed intersection b-form and β is an intersection $(b-1)$-form then:

$$\pi^*(\omega + \mathrm{d}\alpha) \wedge \rho^*(\mu + \mathrm{d}\beta) - \pi^*\omega \wedge \rho^*\mu = \mathrm{d}\tau \qquad \text{for some } \tau \in \mathrm{I}_{\boldsymbol{p}}\Omega^{a+b-1}(\mathscr{M} \times \mathscr{X})$$

This holds since for example $\mathrm{d}\pi^*\alpha \wedge \rho^*\mu = \mathrm{d}(\pi^*\alpha \wedge \rho^*\mu)$ (μ is by hypothesis closed) and $\pi^*\alpha \wedge \rho^*\mu$ is an intersection form being μ of the same type; hence Ψ is a well defined bilinear map inducing consequently a linear map in tensor product.

To show that Ψ is an isomorphism we follow the proof in Bott-Tu ([BT] pp. 47-50) with the same Mayer-Vietoris technique working with a good cover of the manifold $\mathscr{M}$:

– if $\mathscr{M} = \mathbb{R}^m$ then the Künneth formula is true since by theorem [6.1]:

$$\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathbb{R}^m) \otimes \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathscr{X}) = \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathscr{X}) \xrightarrow{\cong} \mathrm{I}_{\boldsymbol{p}}\mathrm{H}^{\bullet}_{\mathrm{DR}}(\mathbb{R}^{\mathrm{m}} \times \mathscr{X})$$

– if it is true for U, V and $U \cap V$ then it is true for $U \cup V$; just use controlled partition of unity and Mayer-Vietoris for intersection cohomology;

– conclude with an induction on the cardinality of the finite good cover.

□

The partial Künneth formula allows to achieve the following results:

$$\begin{aligned}
\mathrm{I}_0\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \textstyle\sum T^3) &\cong (\mathbb{R}, \mathbb{R}, \mathbb{R}^3, \mathbb{R}^6, \mathbb{R}^4, \mathbb{R}, 0, 0, \dots) \\
\mathrm{I}_1\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \textstyle\sum T^3) &\cong (\mathbb{R}, \mathbb{R}^4, \mathbb{R}^3, \mathbb{R}^3, \mathbb{R}^4, \mathbb{R}, 0, 0, \dots) \\
\mathrm{I}_2\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \textstyle\sum T^3) &\cong (\mathbb{R}, \mathbb{R}^4, \mathbb{R}^6, \mathbb{R}^3, \mathbb{R}, \mathbb{R}, 0, 0, \dots)
\end{aligned}$$

Note again the Poincaré duality between $\mathrm{I}_0\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \sum T^3)$ and $\mathrm{I}_2\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \sum T^3)$ with respect to complementary perversities and the autoduality of $\mathrm{I}_1\mathrm{H}^{\bullet}_{\mathrm{DR}}(S^1 \times \sum T^3)$.

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