On a Generalization of Groups with All Subgroups Subnormal.

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THEOREM. Let $G$ be a Fitting $p$-group. If for every proper subgroup $H$ of $G$, $H^G \neq G$ and $H^{(n)}$ is hypercentral for a non-negative integer $n$, then $G' \neq G$.

2. Proper normal closures.

Let $\phi_0(x) = x$. Define
\[
\phi_i(x_1, \ldots, x_{2^i}) = [\phi_{i-1}(x_1, \ldots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})].
\]
Now a group $G$ is soluble if and only if there is a positive integer $n$ such that $\phi_n(G) = 1$.

Our results are closely related to the class of groups with the following property: for every given sequence $x_1, x_2, \ldots$ of elements in the group there exists a natural number $d$ such that
\[
\phi_d(x_1, \ldots, x_{2^d}) = 1.
\]

Let denote the class of groups having the above property by $\rho$. Now we give some properties of this class of groups.

LEMMA 2.1. Let $G$ be a group. If every given sequence $x_1, x_2, \ldots$ of elements in $G$ there exists a natural number $d$ such that
\[
\phi_d(x_1, \ldots, x_{2^d}) = 1,
\]
then $G$ is hyperabelian and $G' \neq G$.

PROOF. It is enough to prove that $G$ contains a nontrivial normal abelian subgroup for the first part of the assertion. Suppose that this is not the case. Clearly there exist elements $x_1, x_2 \in G$ such that $\phi_1(x_1, x_2) = [x_1, x_2] \neq 1$. Assume that we have found elements $x_1, x_2, \ldots, x_{2^d}$ in $G$ such that
\[
y = \phi_d(x_1, \ldots, x_{2^d}) \neq 1.
\]
Now by the assumption $\langle y \rangle^G$ cannot be soluble. This implies that
\[
[\langle y \rangle, (\langle y \rangle^G)^{(i)}] \neq 1.
\]
Hence there exist elements $x_{2^{i+1}}, \ldots, x_{2^{i+1}}$ in $\langle y \rangle^G$ such that
\[
\phi_{i+1}(x_1, \ldots, x_{2^{i+1}}) = [\phi_i(x_1, \ldots, x_{2^i}), \phi_i(x_{2^i+1}, \ldots, x_{2^{i+1}})] \neq 1.
\]
This means that there exists a sequence of elements $x_1, x_2, \ldots$ in $G$ such
that \( \phi_i(x_1, \ldots, x_2) \neq 1 \) for every natural number \( i \). But this is a contradiction.

Now assume that \( G' = G \), then \( Z_2(G) = Z_1(G) \) by Grün’s Lemma, i.e., \( Z(G/Z(G)) = 1 \). Since the property given in the hypothesis is inherited to homomorphic images we may assume that \( Z(G) = 1 \). Let \( x_1, x_2 \) be elements in \( G \) such that \( \phi_1(x_1, x_2) = [x_1, x_2] \neq 1 \), as above. Assume that we have found elements \( x_1, x_2, \ldots, x_2^n \) in \( G \) such that

\[
y = \phi_i(x_1, \ldots, x_2) \neq 1.
\]

Now we can find an element \( x \) in \( G \) such that \( [x, y] \neq 1 \). Since \( G \) is perfect, \( x \in G^{(i+1)} \) and hence we can find elements \( x_2^{i+1}, \ldots, x_2^{i+1} \) in \( G \) such that

\[
\phi_{i+1}(x_1, \ldots, x_2^{i+1}) = [y, \phi_i(x_2^{i+1}, \ldots, x_2^{i+1})] = [\phi_i(x_1, \ldots, x_2), \phi_i(x_2^{i+1}, \ldots, x_2^{i+1})] \neq 1.
\]

Thus we see that there exist elements \( x_1, x_2, \ldots \) in \( G \) such that \( \phi_i(x_1, \ldots, x_2) \neq 1 \) for every natural number \( i \), a contradiction.

Let \( G \) be the direct product of a non-soluble hypercentral \( p \)-group (Example 6.12 of [8] for example) and a non-hypercentral soluble \( p \)-group of derived length \( n \) (see Example 6.10 of [8]). Then \( G^{(n)} \) is hypercentral and proof of the theorem shows that \( G \) has the above property. This example shows that the class of all soluble groups and the class of all hypercentral groups are proper subclasses of \( \gamma_\infty \) and \( \gamma_\infty \) is a proper subclass of the class of all hyperabelian groups, since McLain’s group \( M(Q, F) \) where \( F \) is a field of characteristic \( p > 0 \) (see 12.1.9 of [6]) is an example of a perfect hyperabelian group.

Now we give some auxiliary lemmas to prove the theorem.

**Lemma 2.2.** Let \( G \) be a locally nilpotent perfect \( p \)-group. Suppose that for every proper subgroup \( H \) of \( G \) and for every given sequence of elements \( x_1, x_2, \ldots \) in \( H \) there exists a natural number \( d \) such that

\[
\phi_d(x_1, \ldots, x_2) = 1
\]

and \( H^0 \neq G \). Then there exist a proper normal subgroup \( N \), a finite subgroup \( U \) of \( G \) such that

\[
\bigcap_{y \in G \setminus N} \langle U, y \rangle \neq U
\]

and \( Z(G/N) = 1 \).
PROOF. Assume that the assertion is false. By the first part of the proof of Lemma 4 of [4], for every finite subgroup $U$ of $G$, element $a \in G \setminus U$, proper subgroup $T$ of $G$ and outer commutator word $\phi(x_1, \ldots, x_n)$ (see [4] for the definition) there exist $y_1, \ldots, y_n$ in $G$ such that $\phi(y_1, \ldots, y_n) \notin T$ and $a \notin \langle U, y_1, \ldots, y_n \rangle$. Let $K$ be a proper subgroup of $G$. If we put $Z(G/K) = Z/K^G$ then $Z \neq G$ and $Z(G/Z) = 1$, since $G$ is perfect. Now

$$\bigcap_{z \in G \setminus Z} \langle z \rangle = 1$$

by hypothesis. If $a$ is a non-trivial element in $G$ then there exists an element $z_1$ in $G \setminus Z$ such that $a \notin \langle z_1 \rangle$. Assume that we have found elements $z_2, \ldots, z_w$ in $G$ such that $\phi_a(z_1, \ldots, z_w) \notin Z$ and $a \notin \langle z_1, \ldots, z_w \rangle$. Now there exist elements $z_{2^w+1}, \ldots, z_{2^{w+1}}$ in $G$ such that

$$\phi_a(z_{2^w+1}, \ldots, z_{2^{w+1}}) \notin C_G(\phi_a(z_1, \ldots, z_w)Z)$$

and $a \notin \langle z_1, \ldots, z_w, z_{2^w+1}, \ldots, z_{2^{w+1}} \rangle$, since $Z(G/Z) = 1$. This implies that

$$\phi_{a+1}(z_1, \ldots, z_w, z_{2^w+1}, \ldots, z_{2^{w+1}}) =$$

$$[\phi_a(z_1, \ldots, z_w), \phi_a(z_{2^w+1}, \ldots, z_{2^{w+1}})] \notin Z.$$

Put $X = \langle z_i : i = 1, 2, \ldots \rangle$. Then $X$ is a proper subgroup of $G$, since $a \notin X$ and $\phi_a(z_1, \ldots, z_w) \neq 1$ for all natural number $d$. But this is a contradiction.

The following lemma is a version of (5) Lemma of [4].

**Lemma 2.3.** Let $G$ be a Fitting $p$-group. If $H^G \neq G$ and for every given sequence of elements $x_1, x_2, \ldots$ in $H$ and for every proper subgroup $H$ of $G$ there exists a natural number $d$ such that

$$\phi_d(x_1, \ldots, x_w) = 1,$$

then $G' \neq G$.

**Proof.** Assume that $G' = G$. By Lemma 2.2

$$\bigcap_{y \in G \setminus N} \langle U, y \rangle \neq U$$

for a finite subgroup $U$ of $G$ and for a proper normal subgroup $N$ of $G$
such that \( Z(G/N) = 1 \). Let 
\[
a \in \left( \bigcap_{y \in G/N} \langle U, y \rangle \right) \setminus U
\]
and put \( M = \langle U, a \rangle N \) and \( Z(G/M) = Z/M \). Now \( Z(G/Z) = 1 \). Since \( G/Z \)
is hyperabelian there exists an element \( y \) in \( G \) such that \( \langle y \rangle Z/Z \) is an infinite elementary abelian \( p \)-group. Now \( \langle y, a, U \rangle \) is nilpotent and 
\( \langle y, a, U \rangle \cap (\langle y, a, U \rangle \cap Z) \) is an infinite elementary abelian \( p \)-group. Put \( R = \langle y, a, U \rangle \cap Z \). By (6) Satz of [3] there exists a subgroup \( V \) of 
\( \langle y, a, U \rangle \) such that \( U < V, a \notin V \) and \( VR/R \) is infinite. But then there exists an element \( v \in V \setminus N \) such that \( a \notin \langle U, v \rangle \), a contradiction.

**Proof of the Theorem.** Since \( H^{(n)} \) is hypercentral, there exists an ordinal \( \beta \) such that 
\[
1 = Z_0(H) \triangleleft Z_1(H^{(n)}) \triangleleft \cdots \triangleleft Z_\beta(H^{(n)}) = H^{(n)} \triangleleft H.
\]
Put \( Z_\gamma(H^{(n)}) = L_\gamma \) for all ordinals \( \gamma \leq \beta \) and \( L_{\beta+1} = H \). Let \( x_1, x_2, \ldots \) be a given sequence of elements in \( H \). Assume that \( \phi_n(x_1, \ldots, x_{2^n}) \neq 1 \) for all \( n \). Put \( y_1 = \phi_n(x_1, \ldots, x_{2^n}) \). Then there exists an ordinal \( \alpha_1 \) such that \( y_1 \in L_{\alpha_1+1} \setminus L_{\alpha_1} \). Now we have 
\[
y_2 = \phi_{n+1}(x_1, \ldots, x_{2^{n+1}}) = [y_1, \phi_n(x_{2^n+1}, \ldots, x_{2^{n+1}})] \in L_{\alpha_1},
\]
and thus there exists an ordinal \( \alpha_2 \) such that \( \alpha_1 > \alpha_2 \). By continuing in this way we see that \( \alpha_1 > \alpha_2 > \cdots \) is an infinite descending chain of ordinals, a contradiction. Now we have that \( \phi_n(x_1, \ldots, x_{2^n}) = 1 \) for a natural number \( d \). By Lemma 2.3 \( G' \neq G \).

**Corollary.** Let \( G \) be a Fitting \( p \)-group and let for every proper subgroup \( H \) of \( G \), \( H^G \neq G \). Then,

(i) if every proper subgroup of \( G \) is soluble then \( G \) is soluble.

(ii) if every proper subgroup of \( G \) is hypercentral then \( G' \neq G \).

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**References**


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