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Characterization of Abelian-by-Cyclic 3-Rewritable Groups.

A. ABDOLLAHI (*) - A. MOHAMMADI HASSANABADI (**)

ABSTRACT - Let n be an integer greater than 1. A group G is said to be n-rewritable (or a Q_n-group) if for every n elements $x_1, x_2, \ldots, x_n$ in G there exist distinct permutations σ and τ in S_n such that $x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} = x_{\tau(1)} x_{\tau(2)} \ldots x_{\tau(n)}$. In this paper we have completely characterized abelian-by-cyclic 3-rewritable groups: they turns out to have an abelian subgroup of index 2 or the size of derived subgroups is less than 6. In this paper, we also prove that $G/F(G)$ is an abelian group of finite exponent dividing 12, where $F(G)$ is the Fitting subgroup of G.

1. Introduction and results.

Let n be an integer greater than 1. A group G is said to be n-rewritable (or a Q_n-group) if for every n elements $x_1, x_2, \ldots, x_n$ in G there exist distinct permutations σ and τ in S_n such that

$$x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} = x_{\tau(1)} x_{\tau(2)} \ldots x_{\tau(n)}.$$

The class of 2-rewritable groups is precisely the class of abelian groups, while Q_3, Q_4, etc. are successively weaker properties.

In the above definition, if one of the permutations σ or τ can always be chosen to be the identity then the group G is said to

(*) Indirizzo dell'A.: Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran. E-mail: abdollahi@member.ams.org

(**) Indirizzo dell'A.: Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran. E-mail: aamohaha@yahoo.com

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be n-permutable (or a P_n-group). Thus $P_n \subseteq Q_n$ for all n. But for all $n>2$, $P_n \subsetneq Q_n$, (see Proposition 2.10 of [5]).

We define $P = \bigcup_{n>1} P_n$ and $Q = \bigcup_{n>1} Q_n$, so again $P \subseteq Q$. A complete classification of P-groups and Q-groups are given in [9] and [5] respectively; namely that the classes of P-groups and Q-groups both coincide with the class of finite-by-abelian-by-finite groups.

However there exist such a nice characterization for P-groups and Q-groups, the informations about P_n-groups and Q_n-groups for various n is very little.

Curzio, Longobardi and Maj [8] showed that a group G has the property P_3 if and only if $|G'| \leqslant 2$. Also Longobardi, Maj and Stonehewer [12, 13, 11, 10] proved that a group G has the property P_4 if and only if either $|G'| \leqslant 8$ or G has an abelian subgroup of index 2. Blyth [4] has shown that Q_4-groups are soluble and Maj [12] proved that Q_3-groups are metabelian.

Although Q_2-groups are just the abelian groups, there is no classification for Q_3-groups. Recently Blyth [3] has shown that a finite group of odd order is in Q_3 if and only if $|G'| \leqslant 5$. In view of this result and Lemma 2.1 of [1], to study finite, nilpotent Q_3-groups we need only consider finite nilpotent 2-groups. In [1, Theorem B], we characterized all finite 2-groups in Q_3, it is proved that a nilpotent 2-group G of class 2 lies in Q_3 if and only if $|\langle x, y, z\rangle'| \leqslant 4$ for all $x, y, z \in G$. Also a bound for the nilpotency class of certain finite 2-groups in Q_3 is given in [1, Theorem A]. The main aim of this paper is to characterize abelian-by-cyclic groups in Q_3. It is done as Theorem 1.1, below. It seems that toward having a complete characterization for all Q_3-groups, the first step is to characterize abelian-by-cyclic groups.

The main results of this paper are as follows.

THEOREM 1.1. *Let G be a finite abelian-by-cyclic group. Then G is a* 3*-rewritable group if and only if either G has an abelian subgroup of index* 2 *or its derived subgroup has order less than* 6.

As consequence of Theorem 1.1 and using similar methods as [2], we obtain the following result.

THEOREM 1.2. *Let G be a finite* 3*-rewritable group. Then $G/F(G)$ is an abelian group of finite exponent dividing* 12, *where $F(G)$ denotes the Fitting subgroup of G.*

2. Proofs.

LEMMA 2.1. *Suppose that G is a Q_3-group and A is an abelian subgroup of G containing G'. Let $a, b \in A$ and $x \in G$. Then one of the following holds.*

(*i*) $[x, b] = 1$,
(*ii*) $[a, x] = 1$,
(*iii*) $[b, x^2] = 1$,
(*iv*) $[a, x]^x = [x, b]$,
(*v*) $[x, a]^x = [x, b]$,
(*vi*) $[a, x] = [b, x]$,
(*vii*) $[a, x] = [x, b]$
(*viii*) $[x, b] = [b, x]^x [x, a]^x$,
(*ix*) $[x, b] = [b, x]^x [a, x]^x$.

PROOF. It is easy to see, applying the 3-rewritability property on the elements ax, bx and x, that the result follows. ■

LEMMA 2.2. *Let $G = A\langle x\rangle$ be a Q_3-group where A is a torsion abelian normal subgroup of G. Let p be a prime number and let a be a p-element of A such that $[a, x^2] \neq 1$. Then x centralises the p-complement of A.*

PROOF Let $b \in A$ be a p'-element. Then it follows from Lemma 2.1, using the elements a, b and x, that $[b, x] = 1$. ■

LEMMA 2.3. *Let G be a Q_3-group and let A be an abelian normal subgroup of G. Suppose also that $G = A\langle x\rangle$. If $[a, x^2] \neq 1$ for some element $a \in A$, then $G' = \langle [a, x]\rangle^G$.*

PROOF. Since $[a, x^2] \neq 1$, $[a, x] \neq 1$. Let $b \in A$ and use Lemma 2.1 and the fact that $G' = [A, x]$ to establish the lemma. ■

In view of Lemma 2.2, to characterize finite abelian-by-cyclic Q_3-groups, we need only consider groups $G = A\langle x\rangle$ where A is an abelian normal p-group for some prime p. In what follows G is a finite abelian-by-cyclic 3-rewritable group and p is a prime number, A a normal abelian p-subgroup of G and x an element of G such that $G = A\langle x\rangle$. To characterize groups $G = A\langle x\rangle$ where A is an abelian normal p-subgroup, we need the following.

LEMMA 2.4. *Let $G = A\langle x\rangle$ where A is an abelian normal p-subgroup. Let $a \in A$ be such that $[a, x^2] \neq 1$. Then one of the following holds*:

1) $[a, x]^{x^2} = [a, x]^{-1}$
2) $[a, x]^{x^2} = [a, x]$
3) $[a, x]^x = [a, x]$
4) $[a, x]^x = [a, x]^{-1}$
5) $[a, x]^{x^2} = [a, x]^x [a, x]$
6) $[a, x^3] = 1$.

PROOF. It follows easily from Lemma 2.1 by replacing a by a^x and b by a. ■

LEMMA 2.5. *Let $G = A\langle x\rangle$ where A is an abelian normal p-subgroup of G and $a \in A$ is such that $[a, x^2] \neq 1$. Then one of the following holds*:

I) $[a, x]^2 = 1$
II) $[a, x]^{2x} = [a, x]^{-1}$
III) $[a, x]^{2x} = [a, x]$
IV) $[a, x]^3 = 1$
V) $[a, x]^x = [a, x]$
VI) $[a, x]^{3x} = [a, x]^{-1}$.

PROOF. Replacing a by a^2 and b by a in Lemma 2.1 the result follows easily. ■

LEMMA 2.6. *Let $G = A\langle x\rangle$ where A is an abelian normal p-subgroup of G. Let $a \in A$ be such that $[a, x^2] \neq 1$, then one of the following holds*:

(A) $[a, x]^x = [a, x]$
(B) $[a, x]^{x^2} = [a, x]^x [a, x]^{-1}$
(C) $[a, x]^{2x} = [a, x]$
(D) $[a, x]^{x^2} = [a, x]^{-x} [a, x]$
(E) $[a, x]^{x^2} = [a, x]^{-1}$

PROOF. Applying the 3-rewritability property on the elements ax, a^x and xa one sees that the result follows. ■

LEMMA 2.7. *Let G be a Q_3-group, A a normal abelian p-subgroup of G and x an element of finite order in G such that $G=A\langle x\rangle$. If $[A,x^4]=1$, then either $[A,x^2]=1$ or $|G'|\leqslant 4$.*

PROOF. Applying the 3-rewritability property on the elements ax, ax^2 and ax^3, where a is any element in A, we have either $[a,x^2]=1$ or $[a,x]^{x^2}=[a,x]$.

Suppose that there exists $a\in A$ such that $[a,x^2]\neq 1$. Then $[a,x]^{x^2}= =[a,x]$ and by Lemma 2.3, $G'=\langle [a,x],[a,x]^x\rangle$. Let $x=yz$ where y is a 2-element, z is a $2'$-element and $[y,z]=1$. Then we have $G'=\langle [a,y],[a,y]^y\rangle$.

Now if $p=2$ then G is a finite 2-group and by [1, Lemma 2.4] $|G'|\leqslant \leqslant 4$. So suppose that $p\neq 2$. Then by exercise 6, page 282 of [14], $A= =C_A(y)\times[A,y]$ and so $y^2\in A$ which gives a contradiction. ■

LEMMA 2.8. *Let G be a finite Q_3-group, A a normal abelian subgroup of G and x an element of finite order such that $G=A\langle x\rangle$. If $[a,x^2]\neq 1$ for some element $a\in A$ and $[a,x]^x=[a,x]$, then $|G'|\leqslant 5$.*

PROOF. By Lemma 2.3, $G'=\langle [a,x]\rangle^{\langle x\rangle}$ and by the hypothesis $G'= =\langle [a,x]\rangle\leqslant Z(G)$ and so G is nilpotent of class at most 2. Thus $G=P\times Q$; where P is the Sylow 2-subgroup and Q is the Sylow $2'$-subgroup of G. By [1, Lemma 2.1], either P or Q is abelian. If P is abelian, then $G'=Q'$ and by the main Theorem of [3, Lemma 2.1], $|G'|\leqslant 5$. If Q is abelian, then $G'=P'$, and P is an abelian-by-cyclic 2-group in Q_3. In this case, Lemma 2.4 of [1] completes the proof. ■

LEMMA 2.9. *Let $G=A\langle x\rangle$ be in Q_3 where A is an abelian normal p-subgroup of G. If $a\in A$ is such that $[a,x^2]\neq 1$ and $[a,x]^7=1=[a,x^3]$, then G is abelian.*

PROOF. Let $K=\langle a\rangle^G\langle x\rangle$. Then x^3 and $a^7\in Z(K)$ so that $\frac{K}{Z(K)}$ is a $\{3,7\}$-group in Q_3 and is nilpotent. Thus K is nilpotent. Now $G'=K'$ and by lemma 2.3, $K'=G'=\langle [a,x]\,|\,[a,x^7]=1\rangle$. But since K is nilpotent $K=P\times Q$ where P is the Sylow 2-subgroup and Q is the 2-complement of K. Since Q is a finite group of odd order in Q_3, so by the main Theorem of [3], $|Q'|\leqslant 5$ and since $Q'\leqslant K'$ it follows that $|Q'|=1$. Also since P is a 2-group and $P'\leqslant K'$, we have $P'=1$ and the proof is complete. ■

LEMMA 2.10. *Every group containing an abelian subgroup of index 2 is in* Q_3.

PROOF. Let G be a group containing an abelian subgroup A of index 2. Thus $G = A\langle x \rangle$ for some element $x \in G$ such that $x^2 \in A$. Suppose that x_1, x_2 and x_3 are arbitrary elements in G. Then there are $a_1, a_2, a_3 \in A$ such that $x_i = a_i x$ for $i = 1, 2, 3$. Now it is easy to see that

$$(a_1 x)(a_2 x)(a_3 x) = (a_3 x)(a_2 x)(a_1 x) = x^3 a_1^x a_3^x a_2.$$

It follows that we have $x_1 x_2 x_3 = x_3 x_2 x_1$ for all elements x_1, x_2, x_3 in G. ■

PROOF OF THEOREM 1.1. Let $G = A\langle x \rangle$ where A is an abelian normal p-subgroup of G such that $[A, x^2] \neq 1$. If G contains an abelian subgroup of index 2, then Lemma 2.10 implies that $G \in Q_3$ and if $|G'| \leqslant 5$, then [5, Proposition 2.4] yields that $G \in Q_3$.

Now assume that $G \in Q_3$ and there is an element $a \in A$ such that $[a, x^2] \neq 1$. Suppose, for a contradiction, that $|G'| \geqslant 5$. Then by Lemma 2.7, $[a, x^4] \neq 1$. Considering the 36 cases arising from Lemmas 2.4 and 2.5 we see that one of the following must hold:

(a) $[a, x]^3 = 1$ and $[a, x]^{x^2} = [a, x]^{-1}$
(b) $[a, x]^3 = 1$ and $[a, x]^{x^2} = [a, x]$
(c) $[a, x]^{x^2} = [a, x]$ and $[a, x]^x = [a, x]$
(d) $[a, x]^{x^2} = [a, x]$ and $[a, x]^{3x} = [a, x]^{-1}$
(c) $[a, x]^x = [a, x]$
(f) $[a, x]^3 = 1$ and $[a, x]^{x^2} = [a, x]^x [a, x]$
(g) $[a, x^3] = 1$ and $[a, x]^{2x} = [a, x]$
(h) $[a, x^3] = 1$ and $[a, x]^3 = 1$
(i) $[a, x^3] = 1$ and $[a, x]^{3x} = [a, x]^{-1}$

Now comparing each one of the cases (A)-(E) from Lemma 2.6 with each one of the cases (a)-(i) above we have either $[a, x]^x = [a, x]$ or $[a, x]^7 = 1$, which cannot happen by Lemmas 2.8 and 2.9, respectively. ■

PROOF OF THEOREM 1.2. First we prove that $G/F(G)$ is an abelian $\{2, 3\}$-group. By [12], G is metabelian and so it is clear that $G/F(G)$ must be abelian. Now assume G is a counterexample of minimum order with respect to the property that $G/F(G)$ is not a $\{2, 3\}$-group. It follows from [7, Satz 2.9] that G is a semidirect product of a group N by a cyclic group Q of order p, where p is a prime other than 2 and 3. Since p is odd it follow from the main Theorem of [3, Satz 2.9] that N is an elementary abelian 2-group. Furthermore $G' = N = C_G(N) = F(G)$ is a minimal normal subgroup of G. Now by Theorem 1.1, $|N| \leqslant 4$. Therefore G itself is a $\{2, 3\}$-group, which is a contradiction.

Hence $G/F(G)$ is an abelian $\{2, 3\}$-group. Now suppose, for a contradiction, that G is a counterexample of least possible order subject to the property that the Sylow 2-subgroup of $G/F(G)$ does not have exponent dividing 4. Thus by [7, Satz 2.9] G is a semidirect product of a group N by a cyclic group $Q = \langle b \rangle$ of order 8 and N is an elementary abelian $2'$-group. Moreover $C_N(b^4) = 1$ and $N = G'$. It follows from Theorem 1.1, that $|N| = 3$ or 5. But then the order of $Aut(N)$ divides 4 and so $b^4 = 1$, a contradiction. Therefore the exponent of the Sylow 2-subgroup of $G/F(G)$ divides 4. By a similar argument one can prove that the Sylow 3-subgroup of $G/F(G)$ is elementary abelian. It completes the proof. ■

REFERENCES

[1] A. ABDOLLAHI - A. MOHAMMADI HASSANABADI, 3-*rewritable nilpotent* 2-*groups of class* 2, to appear in Comm. Algebra, **32** (2004).

[2] M. BIANCHI - R. BRANDL - A. GILLIO BERTA MAURI, *On the* 4-*permutational property*, Arch. Math. (Basel), **48**, No. 4, (1987), pp. 281-285.

[3] R. D. BLYTH, *Odd order groups with the rewriting property* Q_3, Arch. Math. (Basel), **78**, No. 5 (2002), pp. 337-344.

[4] R. D. BLYTH, *Rewriting products of group elements*-II, J. Algebra, **119** (1988), pp. 246-259.

[5] R. D. BLYTH, *Rewriting products of group elements*-I, J. Algebra, **116** (1988), pp. 506-521.

[6] R. D. BLYTH - D. J. S. ROBINSON, *Semisimple groups with the rewriting property* Q_5, Comm. Algebra, **23**, No. 6 (1995), pp. 2171-2180.

[7] R. BRANDL, *Zur theorie der untergruppenabgeshlossenen formationen: Endlich varietäten*, J. Algebra, **73** (1981), pp. 1-22.

[8] M. CURZIO - P. LONGOBARDI - M. MAJ, *Su di un problema combinatorio in teoria dei gruppi*, Atti Acc. Lincei Rend. Sem. Mat. Fis. Nat., **74** (1983), pp. 136-142.

[9] M. CURZIO - P. LONGOBARDI - M. MAJ - D. J. S. ROBINSON, *On a permutational properties of groups*, Arch. Math. (basel), **44** (1985), pp. 385-389.

[10] P. LONGOBARDI - M. MAJ - S. STONEHEWER, *The classification of groups in which every product of four elements can be reordered*, Rend. Semin. Mat. Univ. Padova, **93** (1995), pp. 7-26.

[11] P. LONGOBARDI - S. E. STONEHEWER, *Finite 2-groups of class 2 in which every product of four elements can be reordered*, Illinois Journal of Mathematics, **35**, No. 2 (1991), pp. 198-219.
[12] M. MAJ, *On the derived length of groups with some permutational property*, J. Algebra **136**, No. 1 (1991), pp. 86-91.
[13] M. MAJ - S. E. STONEHEWER, *Non-nilpotent groups in which every product of four elements can be reordered*, Can. J. Math., **42**, No. 6 (1990), pp. 1053-1066.
[14] D. J. S. ROBINSON, *A course in the theory of groups*, 2nd ed., Berlin-New York, 1995.

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