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Connections on Distributional Bundles.

DANIEL CANARUTTO (*)

ABSTRACT - A general approach to the geometry of distributional bundles is presented. In particular, the notion of connection on these bundles is studied. A few examples, relevant to quantum field theory, are discussed.

Introduction.

The notion of smoothness introduced by Frölicher [Fr] provides a general setting for calculus in functional spaces [FK, KM] and differential geometry in functional bundles [JM, KM, CK, MK]. An important aspect of that approach is that the essential results can be formulated in terms of finite-dimensional spaces and maps, without heavy involvement in infinite-dimensional topology and other intricated questions. In particular, the notion of a smooth connection on a functional bundle has been applied in the context of the «covariant quantization» approach to Quantum Mechanics [JM, CJM].

In a previous paper [C00a] I applied these ideas to the differential geometry of certain bundles whose fibres are distributional spaces, more specifically scalar-valued generalized half-densities. The main purpose of the present paper is to extend those results to the general case of the bundle of generalized «tube» sections of a 2-fibred «classical» (i.e. finite dimensional) bundle; basic notions of standard differential geometry – such as tangent space, jet space, connection and curvature – are intro-

(*) Indirizzo dell'A.: Dipartimento di Matematica Applicata «G. Sansone», Via S. Marta 3, 50139 Firenze, Italia.

E-mail: canarutto@dma.unifi.it; http://www.dma.unifi.it/~canarutto

duced for this case; adjoint connections and tensor product connections are shown to exist. Furthermore, a suitable connection on the underlying classical bundle is shown to yield a connection on the corresponding distributional bundle; some particularly important cases are the vertical bundle and its tensor algebra, which turn out to be closely related to the notion of adjoint connection. Finally, I consider a few examples which are relevant in view of applications to quantum field theory: the «Dirac connection» on the bundle of 1-electron states for a given observer, and the connections induced on the phase-distributional bundles describing electron and photon fields.

1. Generalized sections.

Let $\mathsf{p} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ be a real or complex classical vector bundle, namely a finite-dimensional vector bundle over the Hausdorff paracompact smooth real manifold $\underline{\boldsymbol{Y}}$. Moreover assume that $\underline{\boldsymbol{Y}}$ is oriented, let $n := \dim \underline{\boldsymbol{Y}}$, and denote the positive component of $\wedge^n \mathrm{T}\underline{\boldsymbol{Y}}$ by $\mathbb{V}\underline{\boldsymbol{Y}} := (\wedge^n \mathrm{T}\underline{\boldsymbol{Y}})^+$. Then $\mathbb{V}\underline{\boldsymbol{Y}} \to \underline{\boldsymbol{Y}}$ is a *semi-vector* bundle [C98, C00a, C00b, CJM], as well as its dual bundle $\mathbb{V}^*\underline{\boldsymbol{Y}} \equiv (\wedge^n \mathrm{T}^*\underline{\boldsymbol{Y}})^+ \to \underline{\boldsymbol{Y}}$ which is called the bundle of *positive densities* on $\underline{\boldsymbol{Y}}$.

Let $\mathcal{Y}_0 \equiv \mathcal{D}_0(\underline{\boldsymbol{Y}}, \mathbb{V}^*\underline{\boldsymbol{Y}} \underset{\underline{\boldsymbol{Y}}}{\otimes} \boldsymbol{Y}^*)$ be the vector space of all smooth sections $\underline{\boldsymbol{Y}} \to \mathbb{V}^*\underline{\boldsymbol{Y}} \underset{\underline{\boldsymbol{Y}}}{\otimes} \boldsymbol{Y}^*$ which have compact support. A topology on this space can be introduced by a standard procedure [Sc]; its topological dual will be denoted as $\mathcal{Y} \equiv \mathcal{D}(\underline{\boldsymbol{Y}}, \boldsymbol{Y})$ and called the space of *generalized sections*, or *distribution-sections* of the given classical bundle, while $\mathcal{Y}_0$ is called the space of *test sections*. In particular, a sufficiently regular ordinary section $s : \underline{\boldsymbol{Y}} \to \boldsymbol{Y}$ can be seen as a generalized section by the rule

$$\langle s, u \rangle = \int_{\underline{\boldsymbol{Y}}} \langle s(\mathsf{y}), \mathsf{u}(\mathsf{y}) \rangle, \qquad u \in \mathcal{Y}_0 .$$

On turn, $\mathcal{Y}$ has a natural topology [Sc], and its subspace $\mathcal{Y}_0^* \equiv \mathcal{D}_0(\underline{\boldsymbol{Y}}, \boldsymbol{Y})$ of all smooth sections with compact support is dense in it. Some particular cases of generalized sections are that of *r-currents* ($\boldsymbol{Y} \equiv \wedge^r \mathrm{T}^*\underline{\boldsymbol{Y}}$, $r \in \mathbb{N}$) and that of *half-densities* [1] ($\boldsymbol{Y} \equiv (\mathbb{V}^*\underline{\boldsymbol{Y}})^{1/2}$).

(1) The «square root» bundle $(\mathbb{V}^*)^{1/2}$ is characterized, up to isomorphism, by $(\mathbb{V}^*)^{1/2} \otimes (\mathbb{V}^*)^{1/2} \cong \mathbb{V}^*$.

The topological dual of $\mathcal{Y}_0^*$ is $\mathcal{Y}^* \equiv \mathcal{D}(\underline{Y}, \mathbb{V}^*\underline{Y} \underset{\underline{Y}}{\otimes} Y^*)$, that is the space of *generalized* Y^**-valued densities* on $\underline{Y}$, or the *adjoint* space of $\mathcal{Y}$.

REMARK. If $\theta \in \mathcal{Y}$ and $\phi \in \mathcal{Y}^*$ then, possibly, the contraction $\langle \theta, \phi \rangle$ may be defined even if neither one is a test section.

Generalized sections can be naturally restricted to any open subset $\check{\underline{Y}} \subset \underline{Y}$ of the base manifold, namely there is a natural linear projection $\mathcal{Y} \to \check{\mathcal{Y}} \equiv \mathcal{D}(\check{\underline{Y}}, \check{Y})$, where $\check{Y} := \mathsf{p}^{-1}(\check{\underline{Y}})$. Accordingly, if (b_i) is a local frame of Y, a generalized section $\zeta \in \mathcal{Y}$ has the local expression $\zeta = \zeta^i \mathsf{b}_i$ with $\zeta^i \in \mathcal{D}(\check{\underline{Y}}, \mathbb{C})$.

There is no inclusion $\check{\mathcal{Y}} \hookrightarrow \mathcal{Y}$, since elements in $\check{\mathcal{Y}}$ cannot be naturally extended to generalized sections on $\underline{Y}$ (such extension may not exist at all). However, a gluing property holds: if $\{\underline{Y}_i\}$ is a covering of $\underline{Y}$ and $\{\theta_i \in \mathcal{Y}_i\}$ is a family of generalized sections such that θ_i and θ_j coincide on $\underline{Y}_i \cap \underline{Y}_j$, then there exists a unique $\theta \in \mathcal{Y}$ whose restriction to $\underline{Y}_i$ coincides with $\theta_i \forall i$.

Let $\mathsf{p}' : Y' \to \underline{Y}'$ be another classical vector bundle and $\varphi : Y \to Y'$ a smooth fibred isomorphism over the diffeomorphism $\underline{\varphi}: \underline{Y} \to \underline{Y}'$; namely, $\mathsf{p}' \circ \varphi = \underline{\varphi} \circ \mathsf{p}$. Clearly, φ determines a natural isomorphism between the spaces of ordinary sections of the two bundles; one easily sees that this restricts to an isomorphism of the corresponding spaces of test sections, and extends to an isomorphism $\varphi_*: \mathcal{Y} \to \mathcal{Y}'$. One also has the adjoint construction. It is not difficult to see that φ_* turns out to be a continuous isomorphism (the proof is essentially the same as given in [C00a] for the particular case of scalar-valued half-densities).

2. F-smoothness in distributional spaces.

Let $\mathbb{I} \subset \mathbb{R}$ be an open interval. A curve $\alpha : \mathbb{I} \to \mathcal{Y}$ is said to be *F-smooth* (*Frölicher-smooth*) if the map

$$\langle \alpha, u \rangle : \mathbb{I} \to \mathbb{C} : t \mapsto \langle \alpha(t), u \rangle$$

is smooth for every $u \in \mathcal{Y}_0$. Accordingly, a function $\phi : \mathcal{Y} \to \mathbb{C}$ is called F-smooth if $\phi \circ \alpha : \mathbb{I} \to \mathbb{C}$ is smooth for all F-smooth curve α, and a map $\Phi : \mathcal{Y} \to \mathcal{W}$ between any two distributional spaces is called F-smooth if $\phi \circ \Phi \circ \alpha$ is smooth for all F-smooth $\alpha : \mathbb{I} \to \mathcal{Y}$ and $\phi : \mathcal{W} \to \mathbb{C}$.

It can be proved [Bo] that a function $f : \boldsymbol{M} \to \mathbb{R}$, where $\boldsymbol{M}$ is a classical manifold, is smooth (in the standard sense) iff the composition $f \circ c$ is a smooth function of one variable for any smooth curve $c : \mathbb{I} \to \boldsymbol{M}$. Thus one has a unique notion of smoothness based on smooth curves, including both classical manifolds and distributional spaces. This is convenient for dealing with smoothness relatively to product spaces such as $\boldsymbol{M} \times \mathcal{Y}$; moreover, one has a natural notion of smoothness for maps $\boldsymbol{M} \to \mathcal{Y}$ and $\mathcal{Y} \to \boldsymbol{M}$. Hence, one may simply write smooth for F-smooth.

Let $\mathcal{C}_{\mathcal{Y}}$ be the set of all F-smooth curves in $\mathcal{Y}$; take any $i \in \mathbb{N} \cup \{0\}$ and consider the following binary relation in $\mathbb{R} \times \mathcal{C}_{\mathcal{Y}}$:

$$(t, \alpha) \overset{i}{\sim} (s, \beta) \Leftrightarrow \mathrm{D}^k \langle \alpha, u \rangle (t) = \mathrm{D}^k \langle \beta, u \rangle (s) \quad \forall u \in \mathcal{Y}_0,\ k = 0, \dots, i\,.$$

Then clearly $\overset{i}{\sim}$ is an equivalence relation; the quotient

$$\mathrm{T}^i \mathcal{Y} := \mathcal{C}_{\mathcal{Y}} / \overset{i}{\sim}$$

will be called the *tangent space of order* i of $\mathcal{Y}$. The equivalence class of $(t, \alpha) \in \mathcal{C}_{\mathcal{Y}}$ will be denoted by $\partial^i \alpha(t)$. Obviously, $\mathrm{T}^i \mathcal{Y}$ is a fibred set over $\mathcal{Y}$; the fibre over some $\lambda \in \mathcal{Y}$ will be denoted by $\mathrm{T}^i_\lambda \mathcal{Y}$. In particular $\mathrm{T}^0 \mathcal{Y} = \mathcal{Y}$.

The set $\mathrm{T} \mathcal{Y} := \mathrm{T}^1 \mathcal{Y}$ is called simply the tangent space of $\mathcal{Y}$, and $\partial \alpha(t) := \partial^1 \alpha(t)$ is called the *tangent vector of* α *at* $\alpha(t)$. Any element in $\mathrm{T} \mathcal{Y}$ can be represented as $\partial \alpha(0)$, for a suitable curve α defined on a neighbourhood $\mathbb{I}$ of 0. It is not difficult to see that there is a natural isomorphism

$$\mathcal{Y} \times \mathcal{Y} \to \mathrm{T} \mathcal{Y} : (\lambda, \mu) \mapsto \partial [\lambda + t\mu]_{t=0}\,.$$

Proposition 2.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be smooth spaces (each one is either a classical manifold or a distributional space) and* $\Phi : \mathcal{A} \to \mathcal{B}$ *a smooth map. Then there exists a unique smooth map* $\mathrm{T}\Phi : \mathrm{T}\mathcal{A} \to T\mathcal{B}$, *called the* tangent prolongation *of* Φ, *such that for every smooth curve* $\alpha : \mathbb{I} \to \mathcal{A}$ *one has*

$$\partial [\Phi \circ \alpha](t) = \mathrm{T}\Phi \circ \partial \alpha(t), \qquad t \in \mathbb{I}\,.$$

The proof of this non-trivial statement is omitted because it is essentially similar to that of the particular case considered in [C00a]. It is not difficult to see that tangent prolongations behave naturally in terms of any compositions.

3. Distributional bundles.

The basic classical geometric setting underlying distributional bundles is the following. One considers a classical 2-fibred bundle

$$\boldsymbol{V}\xrightarrow{\mathsf{q}}\boldsymbol{E}\xrightarrow{\underline{\mathsf{q}}}\boldsymbol{B}\,,$$

where $\mathsf{q}:\boldsymbol{V}\to\boldsymbol{E}$ is a complex (or real) vector bundle, and the fibres of the bundle $\boldsymbol{E}\to\boldsymbol{B}$ are smoothly oriented. Moreover, one assumes that $\mathsf{q}\circ\underline{\mathsf{q}}:\boldsymbol{V}\to\boldsymbol{B}$ is also a bundle (not a vector bundle in general), and that for any sufficiently small open subset $\boldsymbol{X}\subset\boldsymbol{B}$ there are bundle trivializations

$$(\underline{\mathsf{q}},\,\underline{\mathsf{y}}):\boldsymbol{E_X}\to\boldsymbol{X}\times\underline{\boldsymbol{Y}}\,,\quad(\mathsf{q}\circ\underline{\mathsf{q}},\,\mathsf{y}):\boldsymbol{V_X}\to\boldsymbol{X}\times\boldsymbol{Y}$$

(here $\boldsymbol{E_X}:=\underline{\mathsf{q}}^{-1}(\boldsymbol{X})$ and the like) with the following projectability property: there exists a surjective submersion $\mathsf{p}:\boldsymbol{Y}\to\underline{\boldsymbol{Y}}$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{V_X} & \xrightarrow{(\mathsf{q}\circ\underline{\mathsf{q}},\,\mathsf{y})} & \boldsymbol{X}\times\boldsymbol{Y} \\ {\scriptstyle\mathsf{q}}\big\downarrow & & \big\downarrow{\scriptstyle 1_{\boldsymbol{X}}\times\mathsf{p}} \\ \boldsymbol{E_X} & \xrightarrow[(\underline{\mathsf{q}},\,\underline{\mathsf{y}})]{} & \boldsymbol{X}\times\underline{\boldsymbol{Y}} \end{array}$$

commutes; this implies that $\boldsymbol{Y}\to\underline{\boldsymbol{Y}}$ is a vector bundle, not trivial in general.

The above conditions are easily checked to hold in many cases which are relevant for physical applications (as in the cases considered in § 11 and § 12). In particular, the above conditions hold if $\boldsymbol{V}=\boldsymbol{E}\underset{\boldsymbol{B}}{\times}\boldsymbol{W}$ where $\boldsymbol{W}\to\boldsymbol{B}$ is a vector bundle, if $\boldsymbol{V}=\mathrm{V}\boldsymbol{E}$ (the vertical bundle of $\boldsymbol{E}\to\boldsymbol{B}$) and if $\boldsymbol{V}$ is any component of the tensor algebra of $\mathrm{V}\boldsymbol{E}\to\boldsymbol{E}$.

Let n be the dimension of the fibres of $\boldsymbol{E}\to\boldsymbol{B}$. The orientation requirement implies that $\wedge^n\mathrm{V}\boldsymbol{E}\to\boldsymbol{E}$ is a trivializable bundle with smoothly oriented fibres, and one has the smooth bundle $\mathbb{V}^*\boldsymbol{E}:=(\wedge^n\mathrm{V}^*\boldsymbol{E})^+\to$ $\to\boldsymbol{E}$. Then for each $x\in\boldsymbol{B}$ one may consider the distributional space $\boldsymbol{\mathcal{V}}_x:=$ $=\boldsymbol{\mathcal{D}}(\boldsymbol{E}_x,\,\boldsymbol{V}_x)$, and obtains the fibred set

$$\wp:\boldsymbol{\mathcal{V}}\equiv\boldsymbol{\mathcal{D}}_{\boldsymbol{B}}(\boldsymbol{E},\,\boldsymbol{V}):=\coprod_{x\in\boldsymbol{B}}\boldsymbol{\mathcal{V}}_x\to\boldsymbol{B}\,.$$

For any two classical local bundle trivializations $(\underline{\mathsf{q}},\,\underline{\mathsf{y}})$ and $(\mathsf{q}\circ\underline{\mathsf{q}},\,\mathsf{y})$ as above, let

$$\mathsf{Y} : \boldsymbol{\mathcal{V}}_{\boldsymbol{X}} \equiv \wp^{-1}(\boldsymbol{X}) \to \boldsymbol{\mathcal{Y}} \equiv \boldsymbol{\mathcal{D}}(\underline{\boldsymbol{Y}}, \boldsymbol{Y}),$$

$$\mathsf{Y}_x := (\mathsf{y}_x)_*, \quad x \in \boldsymbol{X}.$$

Then $(\wp, \mathsf{Y}) : \boldsymbol{\mathcal{V}}_{\boldsymbol{X}} \to \boldsymbol{X} \times \boldsymbol{\mathcal{Y}}$ is a local bundle trivialization of $\boldsymbol{\mathcal{V}} \to \boldsymbol{B}$. Moreover, if $(\underline{\mathsf{q}}, \underline{\mathsf{y}}') : \boldsymbol{E}_{\boldsymbol{X}'} \to \boldsymbol{X}' \times \underline{\boldsymbol{Y}}'$ and $(\mathsf{q} \circ \underline{\mathsf{q}}, \mathsf{y}') : \boldsymbol{V}_{\boldsymbol{X}'} \to \boldsymbol{X}' \times \boldsymbol{Y}'$ are two other classical bundle trivializations related by the same projectability property, then $(\wp, \mathsf{Y}') \circ (\wp, \mathsf{Y})^{-1} : \boldsymbol{X} \cap \boldsymbol{X}' \times \boldsymbol{\mathcal{Y}} \to \boldsymbol{X} \cap \boldsymbol{X}' \times \boldsymbol{\mathcal{Y}}'$ is F-smooth and linear. Hence, suitable classical bundle atlases on $\boldsymbol{V} \to \boldsymbol{B}$ and $\boldsymbol{E} \to \boldsymbol{B}$ determine a linear F-smooth bundle atlas on $\boldsymbol{\mathcal{V}} \to \boldsymbol{B}$, which is said to be an *F-smooth distributional bundle* (2). Clearly, $\boldsymbol{\mathcal{V}}$ turns out to be an F-smooth space in a natural way: a curve $\alpha : \mathbb{I} \to \boldsymbol{\mathcal{V}}$ is defined to be F-smooth if $(\wp, \mathsf{Y}) \circ \alpha$ is such for any local F-smooth trivialization; in general, the F-smoothness of any map from or to $\boldsymbol{\mathcal{V}}$ is equivalent to the F-smoothness of its local trivialized expressions.

If α is F-smooth then it is natural to set

$$\mathrm{T}((\wp, \mathsf{Y}) \circ \alpha) = \big(\mathrm{T}(\wp \circ \alpha), \mathrm{T}(\mathsf{Y} \circ \alpha)\big) : \mathbb{I} \times \mathbb{R} \to \mathrm{T}\boldsymbol{X} \times \mathrm{T}\boldsymbol{\mathcal{Y}}.$$

One says that two F-smooth curves are first-order equivalent at some point if their trivialized expressions are such; in this way one obtains the definition of the tangent space $\mathrm{T}\boldsymbol{\mathcal{V}}$. Obviously, this is a fibred set over $\boldsymbol{\mathcal{V}}$; a local bundle trivialization $(\wp, \mathsf{Y})$ of $\boldsymbol{\mathcal{V}}$ yields the local bundle trivialization

$$\mathrm{T}(\wp, \mathsf{Y}) : \mathrm{T}\boldsymbol{\mathcal{V}}_{\boldsymbol{X}} \to \mathrm{T}(\boldsymbol{X} \times \boldsymbol{\mathcal{Y}}) \equiv \mathrm{T}\boldsymbol{X} \times \mathrm{T}\boldsymbol{\mathcal{Y}},$$

and the transition maps between two induced trivializations are F-smooth and linear. Hence $\pi_{\boldsymbol{\mathcal{V}}} : \mathrm{T}\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$, the tangent bundle of $\boldsymbol{\mathcal{V}}$, is an F-smooth vector bundle. One has another F-smooth bundle with the same total F-smooth space, namely

$$\mathrm{T}\wp : \mathrm{T}\boldsymbol{\mathcal{V}} \to \mathrm{T}\boldsymbol{B} : \partial\alpha \mapsto \partial(\underline{\mathsf{q}} \circ \alpha).$$

Moreover one has the *vertical subbundle*

$$\mathrm{V}\boldsymbol{\mathcal{V}} := \mathrm{Ker}\,\mathrm{T}\wp \subset \mathrm{T}\boldsymbol{\mathcal{V}},$$

the natural identification $\mathrm{V}\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}} \underset{\boldsymbol{B}}{\times} \boldsymbol{\mathcal{V}}$ and the exact sequence over $\boldsymbol{\mathcal{V}}$

$$0 \to \mathrm{V}\boldsymbol{\mathcal{V}} \to \mathrm{T}\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}} \underset{\boldsymbol{B}}{\times} \mathrm{T}\boldsymbol{B} \to 0.$$

(2) Not every trivialization of a distributional bundle derives from trivializations of the underlying classical 2-fibred bundle.

The subbundle of $\mathrm{T}^*\boldsymbol{B} \underset{\boldsymbol{\mathfrak{V}}}{\otimes} \mathrm{T}\boldsymbol{\mathfrak{V}}$ which projects over the identity of $\mathrm{T}\boldsymbol{B}$ is called the *first jet bundle*, denoted by $\mathrm{J}\boldsymbol{\mathfrak{V}} \to \boldsymbol{\mathfrak{V}}$. This is an affine bundle over $\boldsymbol{\mathfrak{V}}$, with «derived» vector bundle $\mathrm{T}^*\boldsymbol{B} \underset{\boldsymbol{\mathfrak{V}}}{\otimes} \mathrm{V}\boldsymbol{\mathfrak{V}}$. The restriction of $\mathrm{T}^* \wp \otimes \mathrm{T}(\wp, \mathsf{Y})$ is a local bundle trivialization which is denoted by

$$\mathrm{J}(\wp, \mathsf{Y}) : \mathrm{J}\boldsymbol{\mathfrak{V}}_{\boldsymbol{X}} \to \mathrm{J}(\boldsymbol{X} \times \boldsymbol{\mathcal{Y}}) \cong \boldsymbol{\mathcal{Y}} \times (\mathrm{T}^*\boldsymbol{X} \otimes \boldsymbol{\mathcal{Y}}) .$$

If $\mathsf{x} \equiv (\mathsf{x}^a) : \boldsymbol{X} \to \mathbb{R}^m$ is a coordinate chart then one has the fibred charts

$$(\mathsf{x}, \mathsf{Y}) : \boldsymbol{\mathfrak{V}} \to \mathbb{R}^m \times \boldsymbol{\mathcal{Y}},$$

$$(\mathsf{x}^a, \mathsf{Y}, \dot{\mathsf{x}}^a, \dot{\mathsf{Y}}) := \mathrm{T}(\mathsf{x}, \mathsf{Y}) : \mathrm{T}\boldsymbol{\mathfrak{V}} \to \mathbb{R}^m \times \boldsymbol{\mathcal{Y}} \times \mathbb{R}^m \times \boldsymbol{\mathcal{Y}},$$

$$(\mathsf{x}^a, \mathsf{Y}, \mathsf{Y}_a) := \mathrm{J}(\mathsf{x}, \mathsf{Y}) : \mathrm{J}\boldsymbol{\mathfrak{V}} \to \mathbb{R}^m \times \boldsymbol{\mathcal{Y}} \times (\mathbb{R}^m \otimes \boldsymbol{\mathcal{Y}}) .$$

Tangent prolongations of F-smooth maps involving $\boldsymbol{\mathfrak{V}}$ can be expressed through local trivializations; in particular, if $\sigma : \boldsymbol{B} \to \boldsymbol{\mathfrak{V}}$ is an F-smooth section, then $\mathrm{T}\sigma : \mathrm{T}\boldsymbol{B} \to \mathrm{T}\boldsymbol{\mathfrak{V}}$ projects over the identity of $\mathrm{T}\boldsymbol{B}$, so that it can be viewed as a section $\mathrm{j}\sigma : \boldsymbol{B} \to \mathrm{J}\boldsymbol{\mathfrak{V}}$. Setting $\sigma^{\mathsf{Y}} := \mathsf{Y} \circ \sigma : \boldsymbol{B} \to \boldsymbol{\mathcal{Y}}$ one has

$$(\mathsf{x}^a, \mathsf{Y}, \dot{\mathsf{x}}^a, \dot{\mathsf{Y}}) \circ \mathrm{T}\sigma = \mathrm{T}\sigma^{\mathsf{Y}} = (\mathsf{x}^a, \sigma^{\mathsf{Y}}, \dot{\mathsf{x}}^{\mathsf{a}}, \dot{\mathsf{x}}^a \partial_a \sigma^{\mathsf{Y}}) ,$$

$$(\mathsf{x}^a, \mathsf{Y}, \mathsf{Y}_a) \circ \mathrm{j}\sigma = \mathrm{J}\sigma^{\mathsf{Y}} = (\mathsf{x}^a, \sigma^{\mathsf{Y}}, \partial_a \sigma^{\mathsf{Y}}) .$$

For maps $f : \boldsymbol{\mathfrak{V}} \to \mathbb{R}$ one introduces the notation

$$\partial_{\mathsf{Y}} f := \mathrm{V}f \circ \big(\mathbf{1}_{\boldsymbol{\mathfrak{V}}} \times (\wp, \mathsf{Y})^{-1}\big) : \boldsymbol{\mathfrak{V}} \times \boldsymbol{\mathcal{Y}} \to \mathbb{R} ,$$

and obtains the local coordinate expression

$$\mathrm{d}f := \mathrm{pr}_1 \circ \mathrm{T}f = \partial_a f \mathrm{d}\mathsf{x}^a + (\partial_{\mathsf{Y}} f) \circ \mathrm{d}\mathsf{Y} .$$

REMARK. If $\underline{\breve{\boldsymbol{Y}}} \subset \underline{\boldsymbol{Y}}$ is an open subset such that $\breve{\boldsymbol{Y}} := \mathsf{p}^{-1}(\underline{\breve{\boldsymbol{Y}}})$ is trivializable, and $(\mathsf{y}^i, \mathsf{y}^A) : \breve{\boldsymbol{Y}} \to \mathbb{R}^n \times \mathbb{R}^p$ is a linear bundle chart, then σ^{Y} has a coordinate expression whose components are scalar-valued distributions $\sigma^A \in \boldsymbol{\mathcal{D}}_{\boldsymbol{X}}(\underline{\breve{\boldsymbol{Y}}}, \mathbb{R})$.

4. F-smooth fibred morphisms.

Let $\boldsymbol{V}' \to \boldsymbol{E}' \to \boldsymbol{B}'$ another 2-fibred bundle with the same properties, and $\wp' : \boldsymbol{\mathfrak{V}}' \to \boldsymbol{B}'$ the induced distributional bundle. Let moreover

$\Phi : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}'$ be a fibred F-smooth map over the smooth map $\phi : \boldsymbol{B} \to \boldsymbol{B}'$. Then, similarly to the classical case, the tangent prolongation

$$\mathrm{T}\Phi : \mathrm{T}\boldsymbol{\mathcal{V}} \to \mathrm{T}\boldsymbol{\mathcal{V}}'$$

is a linear fibred morphism over Φ and a fibred morphism over $\mathrm{T}\phi : \boldsymbol{TB} \to \boldsymbol{TB}'$. setting $\Phi^{\mathsf{Y}'} := \mathsf{Y}' \circ \Phi : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{Y}}'$ one has (3)

$$(\mathsf{x}', \mathsf{Y}', \dot{\mathsf{x}}', \dot{\mathsf{Y}}') \circ \mathrm{T}\Phi = \Big(\phi^{a'}, \Phi^{\mathsf{Y}'}, \dot{\mathsf{x}}^a \partial_a \phi^{\mathsf{x}'}, \dot{\mathsf{x}}^a \partial_a \Phi^{\mathsf{Y}'} + \partial_{\mathsf{Y}} \Phi^{\mathsf{Y}'} \circ \dot{\mathsf{Y}}\Big).$$

If moreover ϕ is a diffeomorphism, then the restriction of $\phi_* \otimes T\Phi$ determines a fibred morphism $\mathrm{J}\Phi : \mathrm{J}\boldsymbol{\mathcal{V}} \to \mathrm{J}\boldsymbol{\mathcal{V}}'$ over Φ.

If Φ is linear over ϕ, then one writes

$$\Phi^{\mathsf{Y}'}{}_{\mathsf{Y}} := \partial_{\mathsf{Y}} \Phi^{\mathsf{Y}'} = \Phi^{\mathsf{Y}'} \circ (\wp, \mathsf{Y})^{-1} : \boldsymbol{X} \to \mathrm{Lin}(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Y}}'),$$

which is analogous to the matrix expression of a linear morphism in finite-dimensional case.

Let now $\varphi : \boldsymbol{V} \to \boldsymbol{V}'$ be a classical linear isomorphism over the fibred diffeomorphism $\underline{\varphi} : \boldsymbol{E} \to \boldsymbol{E}'$, which on turn is projectable over the diffeomorphism $\phi : \boldsymbol{B} \to \boldsymbol{B}'$. Then one has the induced linear isomorphism $\Phi := \varphi_* : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}'$ over $\boldsymbol{B}$. In the domain of a local coordinate chart one has (4)

$$(\Phi\lambda)^{A'} = (\Phi^{\mathsf{Y}'}{}_{\mathsf{Y}} \lambda^{\mathsf{Y}})^{A'} = (\varphi^{A'}{}_A \lambda^A) \circ \overleftarrow{\underline{\varphi}}, \qquad \lambda \in \boldsymbol{\mathcal{V}},$$

$$(\partial_a \Phi^{\mathsf{Y}'}{}_{\mathsf{Y}} \lambda^{\mathsf{Y}})^{A'} = (\partial_a \varphi^{A'}{}_A \circ \overleftarrow{\underline{\varphi}})(\lambda^A \circ \overleftarrow{\underline{\varphi}}) + [\partial_i(\varphi^{A'}{}_A \lambda^A) \circ \overleftarrow{\underline{\varphi}}]\, \partial_{a'} \overleftarrow{\underline{\varphi}}^i (\partial_a \phi^{a'} \circ \overleftarrow{\phi}),$$

where back pointing arrows indicate the inverse maps. By using these formulas one can write down the coordinate expressions of $\mathrm{T}\Phi$ and $\mathrm{J}\Phi$. As a special case, one also gets the transformation formulas in $\mathrm{T}\boldsymbol{\mathcal{V}}$ and $\mathrm{J}\boldsymbol{\mathcal{V}}$ between any two charts induced by classical charts; a detailed treatment of these aspects lies outside the scope of a short paper and will be exposed in a future survey paper.

When $\boldsymbol{V} = \mathrm{V}\boldsymbol{E}$, $\boldsymbol{V}' = \mathrm{V}\boldsymbol{E}'$ and $\underline{\varphi}$ is a fibred diffeomorphism over ϕ, then one has the special case $\varphi = V\underline{\varphi}$, which extends to any component of the tensor algebra of $\mathrm{V}\boldsymbol{E} \to \boldsymbol{E}$. In particular, one is interested in the bun-

(3) These partial derivatives are naturally defined as a consequence of proposition 2.1.

(4) The proof of the second formula is not difficult but somewhat delicate, as one must take carefully into account the various involved compositions.

dles of scalar q-densities, where q is a rational number, namely in the distributional bundles $\boldsymbol{\mathcal{D}}_{\boldsymbol{B}}(\boldsymbol{E}, \mathbb{C}\otimes\mathbb{V}^{-q}\boldsymbol{E})$ where $\mathbb{V}^{-q}\boldsymbol{E}\equiv(\mathbb{V}^{*}\boldsymbol{E})^{q}$ and the like. One gets

$$\partial_a \Phi^{\mathsf{Y}'}(\lambda) = (\partial_i \lambda^{\mathsf{Y}} \circ \underline{\overleftarrow{\varphi}})\, \partial_{a'} \underline{\overleftarrow{\varphi}}^i (\partial_a \phi^{a'} \circ \overleftarrow{\phi})\, |V\underline{\overleftarrow{\varphi}}|^q +$$

$$+ q(\lambda^{\mathsf{Y}} \circ \underline{\overleftarrow{\varphi}}) \cdot |V\underline{\overleftarrow{\varphi}}|^q (\partial_i \varphi^{i'} \circ \underline{\overleftarrow{\varphi}}\varphi)\, \partial_{a'} \partial_{i'} \underline{\overleftarrow{\varphi}}^i (\partial_a \phi^{a'} \circ \overleftarrow{\phi}),$$

where $|\mathrm{V}\underline{\overleftarrow{\varphi}}|$ denotes the vertical Jacobian determinant of $\underline{\overleftarrow{\varphi}}$.

5. Distributional connections.

Similarly to the standard finite-dimensional case, a *connection* on the distributional bundle $\boldsymbol{\mathfrak{D}}$ is defined to be an F-smooth section

$$\mathfrak{C} : \boldsymbol{\mathfrak{D}} \rightarrow \mathrm{J}\,\boldsymbol{\mathfrak{D}}.$$

In the domain $\boldsymbol{X}\subset\boldsymbol{B}$ of a local bundle chart $(\mathsf{x}, \mathsf{Y}) : \boldsymbol{\mathfrak{D}}_{\boldsymbol{X}} \rightarrow \mathbb{R}^m \times \boldsymbol{\mathcal{Y}}$ one has the local expression

$$\mathfrak{C}_a^{\mathsf{Y}} := \mathsf{Y}_a \circ \mathfrak{C} : \boldsymbol{\mathfrak{D}} \rightarrow \boldsymbol{\mathcal{Y}}.$$

The existence of global connections then follows from standard arguments, using the paracompactness of $\boldsymbol{B}$.

Basically, one deals with *linear* connections, that is connections $\mathfrak{C}$ which are linear morphisms over $\boldsymbol{B}$. Then one writes

$$\mathfrak{C}_a^{\mathsf{Y}} = \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}} \circ \mathsf{Y}\,, \qquad \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}} : \boldsymbol{X} \rightarrow \mathrm{End}\,(\boldsymbol{\mathcal{Y}}).$$

If $\mathfrak{C}_a{}^{\mathsf{Y}'}{}_{\mathsf{Y}'}$ is the expression of $\mathfrak{C}$ in a different fibred chart $(\mathsf{x}', \mathsf{Y}')$ over the same domain $\boldsymbol{X}$, then

$$\mathfrak{C}_{a'}{}^{\mathsf{Y}'}{}_{\mathsf{Y}'} = \partial_{a'} \overleftarrow{\mathsf{k}}^a \cdot (\partial_a \mathcal{K}^{\mathsf{Y}'}{}_{\mathsf{Y}} + \mathcal{K}^{\mathsf{Y}'}{}_{\mathsf{Y}} \circ \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}}) \circ \mathcal{K}^{\mathsf{Y}}{}_{\mathsf{Y}'},$$

where

$$\mathcal{K} \equiv (\mathsf{k}, \mathcal{K}^{\mathsf{Y}'}{}_{\mathsf{Y}}) := (\mathsf{x}', \mathsf{Y}') \circ (\mathsf{x}, \mathsf{Y})^{-1} : \mathbb{R}^m \times \boldsymbol{\mathcal{Y}} \rightarrow \mathbb{R}^m \times \boldsymbol{\mathcal{Y}}'$$

denotes the transition map.

As in the finite-dimensional case, a connection yields a number of structures (whose assignment is actually equivalent to that of the connection itself). First, $\mathfrak{C}$ can be viewed as a linear map $\boldsymbol{\mathfrak{D}} \underset{\boldsymbol{B}}{\times} \mathrm{T}\boldsymbol{B} \rightarrow \mathrm{T}\,\boldsymbol{\mathfrak{D}}$, and $(\pi_{\boldsymbol{\mathfrak{V}}}, \mathrm{T}\,\wp) \circ \mathfrak{C}$ is the identity of $\boldsymbol{\mathfrak{V}} \underset{\boldsymbol{B}}{\times} \mathrm{T}\boldsymbol{B}$. The image

$$\mathrm{H}_{\mathfrak{C}}\boldsymbol{\mathfrak{V}} := \mathfrak{C}(\boldsymbol{\mathfrak{V}} \underset{\boldsymbol{B}}{\times} \mathrm{T}\boldsymbol{B})$$

is a vector subbundle of $\mathrm{T}\boldsymbol{\mathfrak{V}} \rightarrow \boldsymbol{\mathfrak{V}}$, with m-dimensional fibres; the restriction of $\mathfrak{C} \circ (\pi_{\boldsymbol{\mathfrak{V}}}, \mathrm{T}\,\wp)$ is the identity of $\mathrm{H}_{\mathfrak{C}}\boldsymbol{\mathfrak{V}}$. If $v : \boldsymbol{B} \rightarrow \mathrm{T}\boldsymbol{B}$ is a smooth vector field, then $\mathfrak{C}_v$: $\boldsymbol{\mathfrak{V}} \rightarrow \mathrm{T}\boldsymbol{\mathfrak{V}}$ is an F-smooth vector field, called its *horizontal lift*, with coordinate expression

$$\dot{\mathsf{x}}^a \circ \mathfrak{C}_v = v^a, \qquad \dot{\mathsf{Y}} \circ \mathfrak{C}_v = v^a\, \mathfrak{C}_a^{\mathsf{Y}}.$$

One also has the complementary map

$$\Omega := \mathbf{1} - \mathfrak{C} : \mathrm{T}\boldsymbol{\mathfrak{V}} \rightarrow \mathrm{V}\boldsymbol{\mathfrak{V}} \equiv \boldsymbol{\mathfrak{V}} \underset{\boldsymbol{B}}{\times} \boldsymbol{\mathfrak{V}},$$

so that the map $(\mathfrak{C} \circ (\pi_{\boldsymbol{\mathfrak{V}}}, \mathrm{T}\,\wp), \Omega)$ determines the decomposition

$$\mathrm{T}\boldsymbol{\mathfrak{V}} = \mathrm{H}_{\mathfrak{C}}\boldsymbol{\mathfrak{V}} \underset{\boldsymbol{\mathfrak{V}}}{\oplus} \mathrm{V}\boldsymbol{\mathfrak{V}}.$$

Let $\sigma : \boldsymbol{B} \rightarrow \boldsymbol{\mathfrak{V}}$ be an F-smooth section. The *covariant derivative* of σ is defined to be the linear morphsim over $\boldsymbol{B}$

$$\nabla\sigma \equiv \nabla[\mathfrak{C}]\,\sigma := \mathrm{pr}_2 \circ \Omega \circ \mathrm{T}\sigma : \mathrm{T}\boldsymbol{B} \rightarrow \boldsymbol{\mathfrak{V}}.$$

If $v : \boldsymbol{B} \rightarrow \mathrm{T}\boldsymbol{B}$ is a vector field, then one also writes $\nabla_v \sigma := \nabla\sigma \circ v$. The local coordinate expression of the covariant derivative is

$$(\nabla\sigma)^{\mathsf{Y}} := \mathsf{Y} \circ \nabla\sigma = \dot{\mathsf{x}}^a (\partial_a \sigma^{\mathsf{Y}} - \mathfrak{C}_a^{\mathsf{Y}} \circ \sigma).$$

The *curvature tensor* of a linear connection $\mathfrak{C}$ can be defined, as in the finite-dimensional case, as the section $\mathfrak{R} : \boldsymbol{B} \rightarrow \wedge^2 \mathrm{T}^*\boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} \mathrm{End}(\boldsymbol{\mathfrak{V}})$ given by

$$\mathfrak{R}(u, v)\, s := \nabla_u \nabla_v \sigma - \nabla_v \nabla_u \sigma - \nabla_{[u,\, v]}\sigma, \quad u, v : \boldsymbol{B} \rightarrow \mathrm{T}\boldsymbol{B},\ \sigma : \boldsymbol{B} \rightarrow \boldsymbol{\mathfrak{V}},$$

which has the local chart expression

$$\mathfrak{R}^{\mathsf{Y}}{}_{\mathsf{Y}} = \mathfrak{R}_{ab}{}^{\mathsf{Y}}{}_{\mathsf{Y}}\, \mathrm{d}\mathsf{x}^a \wedge \mathrm{d}\mathsf{x}^b = 2(\partial_b \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}} + \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}} \circ \mathfrak{C}_b{}^{\mathsf{Y}}{}_{\mathsf{Y}})\, \mathrm{d}\mathsf{x}^a \wedge \mathrm{d}\mathsf{x}^b.$$

A more general definition of curvature, valid also in the non-linear case, can be given in terms of the Frölicher-Nijenhuis bracket [FN, MK, MM, KMS]. First, one must define the Lie bracket of any two F-smooth vector fields $W, Z : \boldsymbol{\mathfrak{V}} \rightarrow \mathrm{T}\boldsymbol{\mathfrak{V}}$. Using the canonical involution $s : \mathrm{TT}\boldsymbol{\mathfrak{V}} \rightarrow$

$\to \mathrm{TT}\boldsymbol{\mathfrak{V}}$, and $\mathrm{T}Z \circ W - s(\mathrm{T}W \circ Z)\colon\ \boldsymbol{\mathfrak{V}} \to \mathrm{VT}\boldsymbol{\mathfrak{V}} \cong \mathrm{T}\boldsymbol{\mathfrak{V}} \underset{\boldsymbol{\mathfrak{V}}}{\times} \mathrm{T}\boldsymbol{\mathfrak{V}}$, one sets

$$[W, Z] := \mathrm{pr}_2(\mathrm{T}Z \circ W - s(\mathrm{T}W \circ Z))\colon \boldsymbol{\mathfrak{V}} \to \mathrm{T}\boldsymbol{\mathfrak{V}},$$

which has the local expression

$$[W, Z]^a = W^b\,\partial_b Z^a - Z^b\,\partial_b W^a + \partial_{\mathsf{Y}} Z^a \circ W^{\mathsf{Y}} - \partial_{\mathsf{Y}} W^a \circ Z^{\mathsf{Y}},$$

$$[W, Z]^{\mathsf{Y}} = W^b\,\partial_b Z^{\mathsf{Y}} - Z^b\,\partial_b W^{\mathsf{Y}} + \partial_{\mathsf{Y}} Z^{\mathsf{Y}} \circ W^{\mathsf{Y}} - \partial_{\mathsf{Y}} W^{\mathsf{Y}} \circ Z^{\mathsf{Y}}.$$

The Frölicher-Nijenhuis bracket of F-smooth tangent-valued forms $\boldsymbol{\mathfrak{V}} \to \wedge\mathrm{T}^*\boldsymbol{B} \underset{\boldsymbol{\mathfrak{V}}}{\otimes} \mathrm{T}\boldsymbol{\mathfrak{V}}$ can now be introduced by a straightforward extension of the standard definition, namely by the requirement that for decomposable forms one has

$$[\alpha \otimes W,\ \otimes Z] = \alpha \wedge \otimes[W, Z] + \alpha \wedge (W.\,\beta) \otimes Z - (Z.\,\alpha) \wedge \otimes W +$$
$$+ (-1)^r (Z|\alpha) \wedge \mathrm{d}\beta \otimes W + (-1)^r \mathrm{d}\alpha(W|\beta) \otimes Z\,,$$

where $\alpha : \boldsymbol{B} \to \wedge^r\mathrm{T}^*\boldsymbol{B}$, $\beta : \boldsymbol{B} \to \wedge^s\mathrm{T}^*\boldsymbol{B}$, and $W, Z : \boldsymbol{B} \to \mathrm{T}\boldsymbol{B}$.

If $\mathfrak{C} : \boldsymbol{\mathfrak{V}} \to \mathrm{J}\boldsymbol{\mathfrak{V}}$ is an F-smooth connection then its curvature is defined to be

$$\mathfrak{R} := -[\mathfrak{C}, \mathfrak{C}]\colon \boldsymbol{\mathfrak{V}} \to \wedge^2\mathrm{T}^*\boldsymbol{B} \underset{\boldsymbol{\mathfrak{V}}}{\otimes} \mathrm{V}\boldsymbol{\mathfrak{V}}.$$

6. Adjoint connections.

The distributional bundle $\boldsymbol{\mathfrak{V}}^* := \boldsymbol{\mathcal{D}}_{\boldsymbol{B}}(\boldsymbol{E}, \mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}^*) \to \boldsymbol{B}$ is called the *adjoint bundle* of $\boldsymbol{\mathfrak{V}} \to \boldsymbol{B}$; its fibre type is $\boldsymbol{\mathcal{Y}}^*$, the adjoint of $\boldsymbol{\mathcal{Y}}$ (§ 1).

An endomorphism $A \in \mathrm{End}(\boldsymbol{\mathcal{D}})$ of an arbitrary distributional space $\boldsymbol{\mathcal{D}}$ determines a dual endomorphism $A' \in \mathrm{End}(\boldsymbol{\mathcal{D}}_0)$ of the test space, defined by $A'u := u \circ A$, that is $\langle A'u, \phi\rangle = \langle u, A\phi\rangle$. Moreover it may happen that A' can be extended to an endomorphism A^* of the distributional completion $\boldsymbol{\mathcal{D}}^*$ of $\boldsymbol{\mathcal{D}}_0$; this possible extension is called the *adjoint* of A. This requirement is fulfilled, in particular, by the *polynomial derivation operators* [C01].

PROPOSITION 6.1. *Let the F-smooth connection* $\mathfrak{C} : \boldsymbol{\mathfrak{V}} \to \mathrm{J}\boldsymbol{\mathfrak{V}}$ *be such that, in every local F-smooth chart* $(\mathsf{x}, \mathsf{Y})\colon\ \boldsymbol{\mathfrak{V}} \to \boldsymbol{X} \times \boldsymbol{\mathcal{Y}}$, *the* *local expression* $\mathfrak{C}^{\mathsf{Y}} : \mathrm{T}\boldsymbol{B} \to \mathrm{End}(\boldsymbol{\mathcal{Y}})$ *admits an adjoint* $(\mathfrak{C}^{\mathsf{Y}})^* : \mathrm{T}\boldsymbol{B} \to$ $\to \mathrm{End}(\boldsymbol{\mathcal{Y}}^*)$.

Then, there exists a unique F-smooth connection $\mathfrak{C}^* : \boldsymbol{\mathcal{V}}^* \to \mathrm{J}\,\boldsymbol{\mathcal{V}}^*$ *such that* $\mathrm{J}c \circ (\mathfrak{C}, \mathfrak{C}^*) = 0$, *where* $c : \boldsymbol{\mathcal{V}} \underset{\boldsymbol{B}}{\times} \boldsymbol{\mathcal{V}}^* \to \boldsymbol{B} \times \mathbb{C} : (\sigma, \lambda) \mapsto (\wp(\sigma), \langle\lambda\rangle, \sigma)$. *Its chart expression is*

$$\mathfrak{C}^{*}_{a\mathsf{Y}}{}^{\mathsf{Y}} = -(\mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}})^*,$$

that is

$$(\nabla^*_v \lambda)_{\mathsf{Y}} = v^a(\partial_a \lambda_{\mathsf{Y}} - \mathfrak{C}^{*}_{a\mathsf{Y}}{}^{\mathsf{Y}} \circ \lambda_{\mathsf{Y}}) = v^a(\partial_a \lambda_{\mathsf{Y}} + \lambda_{\mathsf{Y}} \circ \mathfrak{C}_{a\mathsf{Y}}^{\mathsf{Y}}).$$

Equivalently, $\mathfrak{C}^*$ *is determined by the requirement that*

$$v.\langle\lambda, \sigma\rangle = \langle\nabla^*_v \lambda, \sigma\rangle + \langle\lambda, \nabla_v \sigma\rangle$$

hold for all smooth sections $\lambda : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}^*$ *and* $\sigma : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}$, *and for all vector field* $v : \boldsymbol{B} \to \mathrm{T}\boldsymbol{B}$, *whenever all contractions are well-defined.*

Proof. Let $\mathfrak{C}^* : \boldsymbol{\mathcal{V}}^* \to \mathrm{J}\,\boldsymbol{\mathcal{V}}^*$ be *any* linear connection; denote by $\mathsf{z} \equiv$ $\equiv \mathrm{pr}_2$ the (trivial) fibre coordinate on $\boldsymbol{B} \times \mathbb{C} \to \boldsymbol{B}$, and observe that

$$\mathrm{J}c \circ (\mathfrak{C}, \mathfrak{C}^*) : \boldsymbol{\mathcal{V}} \underset{\boldsymbol{B}}{\times} \boldsymbol{\mathcal{V}}^* \to \mathbb{C} \times \mathrm{T}^*\boldsymbol{B}$$

has the chart expression

$$\mathsf{z}_a \circ \mathrm{J}c \circ (\mathfrak{C}, \mathfrak{C}^*)(\sigma, \lambda) = \langle\lambda_{\mathsf{Y}}, \mathfrak{C}_a{}^{\mathsf{Y}}{}_{Y}(\sigma^{\mathsf{Y}})\rangle + \langle\mathfrak{C}^{*}_{a\mathsf{Y}}{}^{\mathsf{Y}}(\lambda_{\mathsf{Y}}), \sigma^{\mathsf{Y}}\rangle,$$

which holds for any $(\sigma, \lambda) \in \boldsymbol{\mathcal{V}} \underset{\boldsymbol{B}}{\times} \boldsymbol{\mathcal{V}}^*$ whenever all contractions are well-defined. This expression vanishes iff $\mathfrak{C}^{*}_{a\mathsf{Y}}{}^{\mathsf{Y}} = -(\mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}})^*$. If $s : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}^*_0 \subset \boldsymbol{\mathcal{V}}$ is a section of the subbundle of test maps in $\boldsymbol{\mathcal{V}}$, one has $\nabla_v s : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}$ in general. For every $u : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}_0$, the map

$$\nabla^*_v u : \boldsymbol{\mathcal{V}}^*_0 \to \mathbb{C} : s \mapsto v.\langle s, u\rangle - \langle\nabla_v s, u\rangle$$

is linear continuous, hence $\nabla^*_v u : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}^*$. Its chart expression is

$$\langle s, \nabla^*_v u\rangle = v^a \partial_a\langle s^{\mathsf{Y}}, u_{\mathsf{Y}}\rangle - \langle v^a \partial_a s^{\mathsf{Y}}, u_{\mathsf{Y}}\rangle + \langle v^a \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}} \circ s^{\mathsf{Y}}, u_{\mathsf{Y}}\rangle =$$

$$= \langle s^{\mathsf{Y}}, v^a(\partial_a u_{\mathsf{Y}} + \mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}})^* u_{\mathsf{Y}}\rangle.$$

By continuity, the operation ∇^*_v can be extended to all sections $\lambda : \boldsymbol{B} \to$ $\to \boldsymbol{\mathcal{V}}^*$, and is seen to define a covariant derivative. ■

REMARK. The adjoint connection $\mathfrak{C}^*$ is not reducible to the subbundle $\boldsymbol{\mathfrak{V}}_0 \to \boldsymbol{B}$.

REMARK. Similarly to the finite-dimensional case, a distributional connection $\mathfrak{C}$ determines connections on any tensor bundle over $\boldsymbol{B}$ constructed from $\boldsymbol{\mathfrak{V}} \to \boldsymbol{B}$. Together with its possible adjoint $\mathfrak{C}^*$, it determines connections on the tensor algebra of $\boldsymbol{\mathfrak{V}} \to \boldsymbol{B}$ and its subspaces.

7. Connection induced by a classical connection.

In this section, I'll show that a suitable underlying classical structure determines a connection on a distributional bundle (though not all distributional connections arise in this way).

Consider again the classical 2-fibred bundle $\boldsymbol{V} \to \boldsymbol{E} \to \boldsymbol{B}$ as before. By $\mathrm{V}\boldsymbol{V}$ and $\mathrm{J}\boldsymbol{V}$ one denotes the vertical and jet spaces of $\boldsymbol{V}$ *relatively to base* $\boldsymbol{B}$, while vertical and jet spaces relatively to base $\boldsymbol{E}$ will be denoted by $\mathrm{V}_{\boldsymbol{E}}\boldsymbol{V}$ and $\mathrm{J}_{\boldsymbol{E}}\boldsymbol{V}$.

A connection $\Gamma : \boldsymbol{V} \to \mathrm{J}\boldsymbol{V}$ is said to be *projectable* if there is a connection $\underline{\Gamma} : \boldsymbol{E} \to \mathrm{J}\boldsymbol{E}$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{V} & \xrightarrow{\Gamma} & \mathrm{J}\boldsymbol{V} \\ {\scriptstyle \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{Jq}} \\ \boldsymbol{E} & \xrightarrow[\underline{\Gamma}]{} & \mathrm{J}\boldsymbol{E} \end{array}$$

commutes; moreover, Γ is said to be *linear* if it is a linear morphism over $\underline{\Gamma}$.

Let $(\mathsf{x}^a, \mathsf{y}^i, \mathsf{y}^A) : \boldsymbol{V} \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ be a local 2-fibred coordinate chart, linear over $(\mathsf{x}^a, \mathsf{y}^i) : \boldsymbol{E} \to \mathbb{R}^m \times \mathbb{R}^n$; the coordinate expression of a linear projectable connection is then

$$\Gamma = d\mathsf{x}^a \otimes (\partial\mathsf{x}_a + \Gamma^i_a\, \partial\mathsf{y}_i + \Gamma_a{}^A{}_B\, \mathsf{y}^B\, \partial\mathsf{y}_A)\,,$$

$$\underline{\Gamma} = d\mathsf{x}^a \otimes (\partial\mathsf{x}_a + \Gamma^i_a\, \partial\mathsf{y}_i)\,,$$

with $\Gamma^i_a, \Gamma_a{}^A{}_B : \boldsymbol{E} \to \mathbb{R}$.

A smooth section $\sigma : \boldsymbol{E} \to \boldsymbol{V}$ can be viewed as a section of a functional bundle, whose fibre over each $x \in \boldsymbol{M}$ is the space of all smooth sections $\boldsymbol{E}_x \to \boldsymbol{V}_x$; in the case when one considers local sections $\boldsymbol{E} \to \boldsymbol{V}$, these must be defined on a «tubelike» open subset of $\boldsymbol{E}$. Moreover, this functional bundle can be viewed as a subbundle of $\boldsymbol{\mathfrak{V}} := \boldsymbol{\mathfrak{D}}_{\boldsymbol{B}}(\boldsymbol{E}, \boldsymbol{V}) \to \boldsymbol{B}$.

Observe now that the above σ can be viewed as the vertical-valued 0-form

$$(\mathbf{1}_{\boldsymbol{V}},\,\sigma)\colon \boldsymbol{V}\to\boldsymbol{V}\underset{\boldsymbol{E}}{\times}\boldsymbol{V}\equiv \mathrm{V}_{\boldsymbol{E}}\boldsymbol{V}\subset \mathrm{T}\boldsymbol{V}\,,$$

which has the same coordinate expression $\sigma=\sigma^A\,\partial\mathsf{y}_A$. One may also view Γ as a projectable tangent-valued 1-form

$$\Gamma\colon\boldsymbol{V}\to\mathrm{T}^*\boldsymbol{M}\underset{\boldsymbol{V}}{\otimes}\mathrm{T}\boldsymbol{V}\subset\mathrm{T}^*\boldsymbol{V}\underset{\boldsymbol{V}}{\otimes}\mathrm{T}\boldsymbol{V}\,,$$

and consider the Frölicher-Nijenhuis bracket

$$[\Gamma,\,\sigma]\colon\boldsymbol{V}\to\mathrm{T}^*\boldsymbol{V}\underset{\boldsymbol{V}}{\otimes}\mathrm{T}\boldsymbol{V}\,.$$

Actually, $[\Gamma,\,\sigma]$ turns out to be a basic vertical-valued form $\boldsymbol{V}\to$ $\to\mathrm{T}^*\boldsymbol{M}\underset{\boldsymbol{V}}{\otimes}\mathrm{V}_{\boldsymbol{E}}\boldsymbol{V}$, as one immediately sees from its coordinate expression

$$[\Gamma,\,\sigma]=(\partial_a\sigma^A+\Gamma^i_a\,\partial_i\sigma^A-\Gamma_a{}^A{}_B\,\sigma^B)\,\mathsf{dx}^a\otimes\partial\mathsf{y}_A\,.$$

From this, it is clear that $[\Gamma,\,\sigma]$ can be extended to the case when σ is a section $\boldsymbol{B}\to\boldsymbol{\mathfrak{V}}$; moreover, it can be seen as the covariant derivative of a linear connection $\mathfrak{C}:\boldsymbol{\mathfrak{V}}\to\mathrm{J}\,\boldsymbol{\mathfrak{V}}$, which in the considered chart has the expression $\mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}}(\sigma^{\mathsf{Y}})=\Gamma_a{}^A{}_B\,\sigma^B-\Gamma^i_a\,\partial_i\sigma^A$, that is

$$(\mathfrak{C}_a{}^{\mathsf{Y}}{}_{\mathsf{Y}})^A{}_B=\Gamma_a{}^A{}_B-\delta^A{}_B\,\Gamma^i_a\,\partial_i\,.$$

It is not difficult (just a somewhat intricated calculation) to check that the above expression transforms in the right way under the distributional bundle chart transformation induced by a classical chart transformation.

There is a natural relation between the curvature R of Γ and the curvature $\mathfrak{R}$ of the induced distributional connection $\mathfrak{C}$. Actually one has $R=\mathsf{dx}^a\wedge\mathsf{dx}^b(R_{ab}{}^i\,\partial_i+R_{ab}{}^A{}_B\,\mathsf{y}^B\,\partial_A)$ with

$$R_{ab}{}^i=-\partial_a\Gamma^i_b+\partial_b\Gamma^i_a-\Gamma^j_a\,\partial_j\Gamma^i_b+\Gamma^j_b\,\partial_j\Gamma^i_a,$$
$$R_{ab}{}^A{}_B=-\partial_a\Gamma_b{}^A{}_B+\partial_b\Gamma_a{}^A{}_B-\Gamma^j_a\,\partial_j\Gamma_b{}^A{}_B+\Gamma^j_b\,\partial_j\Gamma_a{}^A{}_B-\Gamma_b{}^A{}_C\Gamma_a{}^C{}_B+\Gamma_a{}^A{}_C\Gamma_b{}^C{}_B.$$

A direct calculation then gives

$$\mathfrak{R}_{ab}{}^{\mathsf{Y}}{}_{\mathsf{Y}}\,\sigma^{\mathsf{Y}}=R_{ab}{}^A{}_B\,\sigma^B-R_{ab}{}^i\,\partial_i\sigma^A,$$

that is, simply, the Frölicher-Nijenhuis bracket

$$\mathfrak{R}(\sigma) = -[R, \sigma].$$

8. Induced connection and horizontal transport.

In this section it will be showed that the notion of distributional connection induced by a classical connection arises in a natural and somewhat more intuitive way in terms of the parallel (i.e. horizontal) transports related to the two connections.

Let $\mathbb{I} \subset \mathbb{R}$ be an open neighbourhood of 0, and $c : \mathbb{I} \to \boldsymbol{B}$ a smooth curve. For any $v_0 \in \boldsymbol{V}_{c(0)}$ one has, locally, a unique Γ-horizontal curve $C_{v_0}\colon \mathbb{I}_{v_0} \to \boldsymbol{V}$, with $\mathbb{I}_{v_0} \subset \mathbb{I}$, such that $C_{v_0}(0) = v_0$. Moreover C_{v_0} is linear projectable over $\underline{C}_{\underline{v}_0}\colon \mathbb{I}_{v_0} \to \boldsymbol{E}$, the horizontal $\underline{\Gamma}$-lift of c starting from $\underline{v}_0 \equiv \mathsf{q}(v_0)$.

If $t \in \mathbb{I}_{v_0}$, so that the horizontal transport of $v_0 \in \boldsymbol{V}_{c(0)}$ to $\boldsymbol{V}_{c(t)}$ is defined, then there is a neighbourhood $\boldsymbol{U} \subset \boldsymbol{V}_{c(0)}$ of v_0 such that the horizontal transport of every $u \in \boldsymbol{U}$ to $\boldsymbol{V}_{c(t)}$ is defined too (this is a consequence of the continuity of Γ). From a general result in the theory of ordinary differential equations, on the other hand, it follows that horizontal transport relatively to a *linear* connection on a vector bundle determines an isomorphism of any two fibres along any smooth curve connecting their base points. This is not the case of the presently considered setting, since $\boldsymbol{V} \to \boldsymbol{B}$ is not a vector bundle in general. But the whole fibre $\boldsymbol{V}_{\underline{v}_0}$ is *linearly* sent to the whole fibre $\boldsymbol{V}_{\underline{v}_t}$, where $\underline{C}_{\underline{v}_0}(t) \equiv \underline{v}_t \in \boldsymbol{E}_{c(t)}$; namely horizontal transport determines an isomorphism between these two fibres.

Momentarily forgetting these locality issues, assume horizontal transport along c determines, for all $t \in \mathbb{I}$, a fibred isomorphism $C_t\colon \boldsymbol{V}_{c(0)} \to \boldsymbol{V}_{c(t)}$ over a diffeomorphism $\underline{C}_t\colon \boldsymbol{E}_{c(0)} \to \boldsymbol{E}_{c(t)}$. In other terms one has a 1-parameter familiy of fibred isomorphisms over a 1-parameter familiy of diffeomorphisms, denoted by

$$C : \mathbb{I} \times \boldsymbol{V}_{c(0)} \to \boldsymbol{V}\,, \qquad \underline{C}\colon \mathbb{I} \times \boldsymbol{E}_{c(0)} \to \boldsymbol{E}\,.$$

Let now $\lambda \in \boldsymbol{\mathcal{V}}_{c(0)} \equiv \boldsymbol{\mathcal{D}}(\boldsymbol{E}_{c(0)},\ \boldsymbol{V}_{c(0)})$. Then

$$(C_t)_* \lambda \in \boldsymbol{\mathcal{V}}_{c(t)} \equiv \boldsymbol{\mathcal{D}}(\boldsymbol{E}_{c(t)},\ \boldsymbol{V}_{c(t)})\,.$$

Namely, the classical horizontal transport locally determines a lift

$$C_*: \mathbb{I} \times \boldsymbol{\mathcal{V}}_{c(0)} \to \boldsymbol{\mathcal{V}}$$

of the base curve c. It can be seen that this is exactly the horizontal lift of c relatively to the distributional connection $\mathfrak{C}$ induced by Γ, namely that

$$\mathfrak{C} : \mathrm{T}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \boldsymbol{\mathcal{V}} \to \mathrm{T}\boldsymbol{\mathcal{V}} : (\partial c(0), \lambda) \mapsto \partial(C_* \lambda)(0)\,.$$

This result follows from a coordinate calculation; from the definition of a horizontal curve one has

$$\frac{\partial}{\partial t}\underline{C}^i(0, \underline{v}_0) = \dot{c}^a(0)\,\Gamma^i_a(\underline{v}_0)\,, \qquad \frac{\partial}{\partial t} C^A{}_B(0, \underline{v}_0) = \dot{c}^a(0)\,\Gamma_a{}^A{}_B(\underline{v}_0)\,,$$

while the induced horizontal curve $C_* \lambda : \mathbb{I} \to \boldsymbol{\mathcal{V}}$ can be written, by some abuse of language, as

$$(C_* \lambda)^A(t, \underline{y}) = C^A{}_B(t, \overleftarrow{\underline{C}}(t, \underline{y}))\,\lambda^B(\overleftarrow{\underline{C}}(t, \underline{y}))\,.$$

Calculating the tangent vector $\partial(C_* \lambda) : \mathbb{I} \to \mathrm{T}\boldsymbol{\mathcal{V}}$ is now a straightforward (though not immediate) task; using the relation between Γ and $\mathfrak{C}$ one gets the claimed result.

As already observed, in general this horizontal lift of c through $\mathfrak{C}$ may not exist for every $\lambda \in \boldsymbol{\mathcal{V}}_{c(0)}$, but it can defined for the restriction of λ to a suitable open subset. Furtherermore, the horizontal lift construction can be done whenever λ has compact support $\boldsymbol{K} \subset \boldsymbol{E}_{c(0)}$, by the following argument. For every $e \in \boldsymbol{E}_{c(0)}$ choose an open neighbourhood of e, $\boldsymbol{U} \subset \subset \boldsymbol{E}_{c(0)}$, such that the restriction of λ to $\boldsymbol{U}$ is horizontally transported over c up to $t = t_{\boldsymbol{U}} > 0$; from this open covering of $\boldsymbol{K}$ select a finite subcovering $\boldsymbol{\mathcal{U}}$, and define $t_{\boldsymbol{K}} := \min\{t_{\boldsymbol{U}}, \boldsymbol{U} \in \boldsymbol{\mathcal{U}}\}$. Then by a partition of unity subjected to $\boldsymbol{\mathcal{U}}$ one has horizontal transport of λ over c up to $t = t_{\boldsymbol{K}}$.

9. Induced connections and tensor products.

Consider another 2-fibred bundle $\boldsymbol{V}' \to \boldsymbol{E}' \to \boldsymbol{B}$ over the same lower base manifold $\boldsymbol{B}$. The fibred tensor product of $\boldsymbol{V}$ and $\boldsymbol{V}'$ is defined to be the 2-fibred bundle

$$\boldsymbol{W} := \boldsymbol{V} \underset{\boldsymbol{F}}{\otimes} \boldsymbol{V}' \to \boldsymbol{F} := \boldsymbol{E} \underset{\boldsymbol{B}}{\times} \boldsymbol{E}' \to \boldsymbol{B}\,.$$

Let $(\mathsf{x}^a, \mathsf{y}^i, \mathsf{y}^A)$ and $(\mathsf{x}^a, \mathsf{y}^{i'}, \mathsf{y}^{A'})$ be 2-fibred coordinate charts on $\boldsymbol{V}$ and $\boldsymbol{V}'$; then one has induced coordinates $(\mathsf{x}^a, \mathsf{y}^i, \mathsf{y}^A, \mathsf{y}^{i'}, \mathsf{y}^{A'}, \mathsf{w}^{AA'})$ on $\boldsymbol{W}$, where

$$\mathsf{w}^{AA'} \equiv \mathsf{y}^A \otimes \mathsf{y}^{A'} \quad \text{i.e.} \quad \mathsf{w}^{AA'} \circ \otimes = \mathsf{y}^A \mathsf{y}^{A'},$$

$$\otimes : \boldsymbol{V} \underset{\boldsymbol{B}}{\times} \boldsymbol{V}' \to \boldsymbol{W} : (v, v') \mapsto v \otimes v' .$$

The jet prolongation $\mathrm{J}\otimes : \mathrm{J}\boldsymbol{V} \underset{\boldsymbol{B}}{\times} \mathrm{J}\boldsymbol{V}' \to \mathrm{J}\boldsymbol{W}$ is characterized by the requirement that the diagram

$$\begin{array}{ccc} \mathrm{J}\boldsymbol{V} \underset{\boldsymbol{B}}{\times} \mathrm{J}\boldsymbol{V}' & \xrightarrow{\ \mathrm{J}\otimes\ } & \mathrm{J}\boldsymbol{W} \\ {\scriptstyle (\mathrm{j}\sigma,\, \mathrm{j}\sigma')} \nwarrow & & \nearrow {\scriptstyle \mathrm{j}(\sigma\otimes\sigma')} \\ & \boldsymbol{B} & \end{array}$$

commutes for any two sections $\sigma : \boldsymbol{B} \to \boldsymbol{V}$, $\sigma' : \boldsymbol{B} \to \boldsymbol{V}'$. Thus one finds the coordinate expression

$$\mathsf{w}^{AA'}_a \circ \mathrm{J}\otimes = \mathsf{y}^A_a \, \mathsf{y}^{A'} + \mathsf{y}^A \, \mathsf{y}^{A'}_a .$$

Let now $\Gamma : \boldsymbol{V} \to \mathrm{J}\boldsymbol{V}$ and $\Gamma' : \boldsymbol{V}' \to \mathrm{J}\boldsymbol{V}'$ be linear projectable connections over $\underline{\Gamma} : \boldsymbol{E} \to \mathrm{J}\boldsymbol{E}$ and $\underline{\Gamma}' : \boldsymbol{E}' \to \mathrm{J}\boldsymbol{E}'$, respectively; then there exists a unique connection $\Gamma \otimes \Gamma' : \boldsymbol{W} \to \mathrm{J}\boldsymbol{W}$ such that the diagram

$$\begin{array}{ccc} \mathrm{J}\boldsymbol{V} \underset{\boldsymbol{B}}{\times} \mathrm{J}\boldsymbol{V}' & \xrightarrow{\ \mathrm{J}\otimes\ } & \mathrm{J}\boldsymbol{W} \\ {\scriptstyle (\Gamma, \Gamma')} \uparrow & & \uparrow {\scriptstyle \Gamma\otimes\Gamma'} \\ \boldsymbol{V} \underset{\boldsymbol{B}}{\times} \boldsymbol{V}' & \xrightarrow[\ \otimes\ ]{} & \boldsymbol{W} \end{array}$$

commutes; moreover, $\Gamma \otimes \Gamma'$ is linear projectable over

$$(\underline{\Gamma}, \underline{\Gamma}') : \boldsymbol{E} \underset{\boldsymbol{B}}{\times} \boldsymbol{E}' \to \mathrm{J}\boldsymbol{E} \underset{\boldsymbol{B}}{\times} \mathrm{J}\boldsymbol{E}' ,$$

and its ccordinate expression is

$$(\mathsf{y}^i_a, \mathsf{y}^A_a, \mathsf{y}^{i'}_a, \mathsf{y}^{A'}_a, \mathsf{w}^{AA'}_a) \circ (\Gamma \otimes \Gamma') =$$

$$= (\Gamma^i_a, \Gamma_a{}^A{}_B \mathsf{y}^B, \Gamma^{i'}_a, \Gamma_a{}^{A'}{}_{B'} \mathsf{y}^{B'}, \Gamma_a{}^A{}_B \mathsf{y}^B \mathsf{y}^{A'} + \mathsf{y}^A \Gamma_a{}^{A'}{}_{B'} \mathsf{y}^{B'}),$$

where the components of Γ' are recognized by primed indices.

The distributional bundle $\boldsymbol{\mathfrak{W}} := \boldsymbol{\mathfrak{D}}_{\boldsymbol{B}}(\boldsymbol{F}, \boldsymbol{W}) \to \boldsymbol{B}$ is easily seen to coincide with the fibred tensor product of $\mathfrak{V}$ and $\mathfrak{V}'$, namely

$$\mathfrak{W} := \boldsymbol{\mathcal{D}}_M(\boldsymbol{F},\, \boldsymbol{W}) = \boldsymbol{\mathcal{D}}_M(\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}',\, \boldsymbol{V} \underset{\boldsymbol{E} \times_{\boldsymbol{M}} \boldsymbol{E}'}{\otimes} \boldsymbol{V}') =$$

$$= \boldsymbol{\mathcal{D}}_M(\boldsymbol{E},\, \boldsymbol{V}) \underset{\boldsymbol{M}}{\otimes} \boldsymbol{\mathcal{D}}_M(\boldsymbol{E}',\, \boldsymbol{V}') \equiv \mathfrak{V} \underset{\boldsymbol{M}}{\otimes} \mathfrak{V}'.$$

Let $\mathfrak{C} : \mathfrak{V} \to \mathrm{J}\,\mathfrak{V}$ and $\mathfrak{C}' : \mathfrak{V}' \to \mathrm{J}\,\mathfrak{V}'$ be the distributional connections induced by Γ and Γ'. These yield, exactly by the same argument which is valid in the finite-dimensional case, a linear connection $\mathfrak{C} \otimes \mathfrak{C}' : \mathfrak{W} \to$ $\to \mathrm{J}\,\mathfrak{W}$; it is not difficult to proof:

Proposition 9.1. *The tensor product connection $\mathfrak{C} \otimes \mathfrak{C}'$ is exactly the distributional connection associated with the classical connection $\Gamma \otimes \Gamma'$. For $\omega \in \mathfrak{W}$ one has*

$$(\mathfrak{C} \otimes \mathfrak{C}')_a{}^{\mathsf{YY}'}{}_{\mathsf{YY}'}\, \omega^{\mathsf{YY}'} = \Gamma_a{}^A{}_B\, \omega^{BA'} - \Gamma^i_a\, \partial_i\, \omega^{AA'} + \Gamma_a{}^{A'}{}_{B'}\, \omega^{AB'} - \Gamma^{i'}_a\, \partial_{i'}\, \omega^{AA'}.$$

If $\boldsymbol{E} = \boldsymbol{E}'$ then one also has the 2-fibred bundle $\boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}' \to \boldsymbol{E} \to \boldsymbol{B}$. The distributional bundle $\boldsymbol{\mathcal{D}}_M(\boldsymbol{E},\, \boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}')$ is *different* from $\mathfrak{V} \underset{\boldsymbol{M}}{\otimes} \mathfrak{V}'$. If $\Gamma : \boldsymbol{V} \to \mathrm{J}\boldsymbol{V}$ and $\Gamma' : \boldsymbol{V}' \to \mathrm{J}\boldsymbol{V}'$ are now linear projectable connections over the *same* connection $\underline{\Gamma} : \boldsymbol{E} \to \mathrm{J}\boldsymbol{E}$, then, besides $\Gamma \otimes \Gamma'$, they also determine a different kind of tensor connection, that is

$$\Gamma \underline{\otimes} \Gamma' : \boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}' \to \mathrm{J}(\boldsymbol{V} \otimes \boldsymbol{E}\boldsymbol{V}'),$$

which is characterized by the commuting diagram

$$\begin{array}{ccc}
\mathrm{J}(\boldsymbol{V} \underset{\boldsymbol{B}}{\times} \boldsymbol{V}') \equiv \mathrm{J}\boldsymbol{V} \underset{\mathrm{J}\boldsymbol{E}}{\times} \mathrm{J}\boldsymbol{V}' & \xrightarrow{\mathrm{J}\otimes} & \mathrm{J}(\boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}') \\
{\scriptstyle (\Gamma,\Gamma')}\uparrow & & \uparrow{\scriptstyle \Gamma \underline{\otimes} \Gamma'} \\
\boldsymbol{V} \underset{\boldsymbol{E}}{\times} \boldsymbol{V}' & \xrightarrow[\otimes]{} & \boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}'
\end{array}$$

and has the coordinate expression

$$(\mathsf{y}^i_a,\, \mathsf{y}^A_a,\, \mathsf{y}^{A'}_a,\, \mathsf{w}^{AA'}_a) \circ (\Gamma \underline{\otimes} \Gamma') =$$

$$= (\Gamma^i_a,\, \Gamma_a{}^A{}_B\, \mathsf{y}^B,\, \Gamma_a{}^{A'}{}_{B'}\, \mathsf{y}^{B'},\, \Gamma_a{}^A{}_B\, \mathsf{y}^B \mathsf{y}^{A'} + \mathsf{y}^A\, \Gamma_a{}^{A'}{}_{B'}\, \mathsf{y}^{B'}).$$

The induced distributional connection

$$\mathfrak{C} \underline{\otimes} \mathfrak{C}' : \boldsymbol{\mathcal{D}}_M(\boldsymbol{E},\, \boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}') \to \mathrm{J}\,\boldsymbol{\mathcal{D}}_M(\boldsymbol{E},\, \boldsymbol{V} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}')$$

has the cordinate chart expression

$$(\mathfrak{C}\underline{\otimes}\mathfrak{C}')_a{}^{\mathsf{YY}'}{}_{\mathsf{YY}'}\,\omega^{\mathsf{YY}'} = \Gamma_a{}^A{}_B\,\omega^{BA'} + \Gamma_a{}^{A'}{}_{B'}\,\omega^{AB'} - \Gamma^i_a\,\partial_i\,\omega^{AA'}.$$

10. Induced connection: vertical bundle and adjoint case.

A linear projectable connection $\Gamma : \boldsymbol{V} \to \mathrm{J}\boldsymbol{V}$, as considered in the previous sections, determines a unique «dual» connection $\Gamma^* : \boldsymbol{V}^* \to \mathrm{J}\boldsymbol{V}^*$; this is again linear projectable over the same $\underline{\Gamma}$, and is characterized by

$$\mathrm{J}c \circ (\Gamma, \Gamma^*) = 0\,,$$

where $c : \boldsymbol{V} \underset{\boldsymbol{E}}{\times} \boldsymbol{V}^* \to \boldsymbol{E} \times \mathbb{C}$ denotes the duality contraction; it has the coordinate expression

$$\Gamma^*_{aA}{}^B = -\Gamma_a{}^A{}_B\,.$$

On turn, Γ^* determines a connection on the distributional bundle $\boldsymbol{\mathcal{D}}_{\boldsymbol{B}}(\boldsymbol{E}, \boldsymbol{V}^*)$. In general, this is *not* the adjoint connection $\mathfrak{C}^*$ of $\mathfrak{C}$, which is actually a connection on a different distributional bundle. In order to study the relation between $\mathfrak{C}^*$ and the classical connection Γ one has to perform some further constructions.

The first step consists in the vertical extension of $\underline{\Gamma}: \boldsymbol{E} \to \mathrm{J}\boldsymbol{E}$. Recalling the natural isomorphism $\mathrm{JV}\boldsymbol{E} \cong \mathrm{VJ}\boldsymbol{E}$, one gets the morphism

$$\check{\Gamma} := \mathrm{V}\underline{\Gamma} : \mathrm{V}\boldsymbol{E} \to \mathrm{JV}\boldsymbol{E}\,,$$

which turns out to be a linear projectable connection over $\underline{\Gamma}$. Its coordinate expression is

$$\check{\Gamma}_a{}^i{}_j = \partial_j \Gamma^i_a\,.$$

Its dual connection $\check{\Gamma}^* : \mathrm{V}^*\boldsymbol{E} \to \mathrm{JV}^*\boldsymbol{E}$ has the coordinate expression

$$(\check{\Gamma}^*)_{aj}{}^i = -\check{\Gamma}_a{}^i{}_j = -\partial_j \Gamma^i_a\,.$$

Now one finds induced linear projectable connections over $\underline{\Gamma}$ in all tensor product bundles over $\boldsymbol{E} \to \boldsymbol{B}$ constructed from $\mathrm{V}\boldsymbol{E}$ and $\mathrm{V}^*\boldsymbol{E}$. Most noticeably, one has projectable linear connections over $\underline{\Gamma}$ on the 2-fibre bundles

$$\wedge^r \mathrm{V}^*\boldsymbol{E} \to \boldsymbol{E} \to \boldsymbol{B}\,, \qquad r \in \mathbb{N}\,,$$

$$\mathbb{V}^*\boldsymbol{E} \to \boldsymbol{E} \to \boldsymbol{B}\,,$$

and, using Γ, in their tensor products with $\boldsymbol{V}$ and $\boldsymbol{V}^*$ over $\boldsymbol{E}$. In particular, the connection $\widehat{\Gamma}\colon \mathbb{V}^*\boldsymbol{E} \to \mathrm{J}\mathbb{V}^*\boldsymbol{E}$ has the coordinate expression

$$\widehat{\Gamma}_a = (\check{\Gamma}^*)_{ai}{}^i = -\partial_i \Gamma^i_a .$$

All these classical connections determine linear connections on the corresponding distributional bundles, and, in particular, in the distributional bundle

$$\boldsymbol{\mathfrak{V}}^* := \boldsymbol{\mathfrak{D}}_{\boldsymbol{B}}(\boldsymbol{E}, \mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}^*) .$$

The classical connection

$$\Gamma' \equiv (\widehat{\Gamma} \otimes \Gamma^*) \colon \mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}^* \to \mathrm{J}(\mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}^*) ,$$

which is again linear projectable over $\underline{\Gamma}$, has the coordinate expression

$$\mathsf{z}_{Ba} \circ \Gamma' = (-\delta_B{}^A \partial_i \Gamma^i_a + \Gamma^*_{aB}{}^A)\, \mathsf{y}_A = -(\delta_B{}^A \partial_i \Gamma^i_a + \Gamma_a{}^A{}_B)\, \mathsf{y}_A ,$$

where (z_B) and (y_A) are the induced coordinates in the fibres of $\mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{V}^* \to \boldsymbol{E}$ and $\boldsymbol{V}^* \to \boldsymbol{E}$, respectively.

Now, Γ' induces a linear distributional connection $\mathfrak{C}'\colon \boldsymbol{\mathfrak{V}}^* \to \mathrm{J}\,\boldsymbol{\mathfrak{V}}^*$; if $\tau : \boldsymbol{B} \to \boldsymbol{\mathfrak{V}}^*$ is an F-smooth section, with coordinate expression $\tau = \tau_A\, \mathrm{d}^n \underline{\mathsf{y}} \otimes \mathsf{y}^A$, then one finds

$$\mathfrak{C}'_{a\mathsf{Y}}{}^{\mathsf{Y}} \tau_{\mathsf{Y}_B} = \Gamma'_{aB}{}^A \tau_A - \Gamma^i_a \partial_i \tau_B = -\Gamma_a{}^A{}_B \tau_A - \partial_i \Gamma^i_a \tau_B - \Gamma^i_a \partial_i \tau_B .$$

Now it is a straightforward matter to proof:

PROPOSITION 10.1. *The distributional connection* $\mathfrak{C}'\colon \boldsymbol{\mathfrak{V}}^* \to \mathrm{J}\,\boldsymbol{\mathfrak{V}}^*$ *coincides with the adjoint connection* $\mathfrak{C}^*$ *of* $\mathfrak{C} : \boldsymbol{\mathfrak{V}} \to \mathrm{J}\,\boldsymbol{\mathfrak{V}}$ (*proposition* 6.1).

11. Quantum Dirac connection.

Let $(\boldsymbol{M}, g)$ be an Einstein spacetime. A *time map* is a bundle $\mathsf{t} : \boldsymbol{M} \to \boldsymbol{T}$, where $\boldsymbol{T}$ is an oriented 1-dimensional real manifold whose fibres $\boldsymbol{M}_t \equiv \mathsf{t}^{-1}(t)$, $t \in \boldsymbol{T}$, are spacelike (this is one possible extension of the notion of *observer* to the curved spacetime case). The assignment of t determines a splitting of the spacetime's tangent bundle as $\mathrm{T}\boldsymbol{M} = \mathrm{T}^{\|}\boldsymbol{M} \underset{\boldsymbol{M}}{\oplus} \mathrm{T}^{\perp}\boldsymbol{M}$, where, for each $x \in \boldsymbol{M}$, $\mathrm{T}^{\|}_x\boldsymbol{M}$ is defined to be the timelike subspace of $\mathrm{T}_x\boldsymbol{M}$

which is orthogonal to the spacelike fibre through x, and $\mathrm{T}^{\perp}_{x}\boldsymbol{M}$ is the subspace orthogonal to $\mathrm{T}^{\|}_{x}\boldsymbol{M}$; namely $\mathrm{T}^{\perp}\boldsymbol{M} \equiv \mathrm{V}\boldsymbol{M}$ is constituted by all vectors tangent to the spacelike fibres.

The bundle $\boldsymbol{M} \to \boldsymbol{T}$ has a natural trivialization $(\mathsf{t}, \mathsf{x}) : \boldsymbol{M} \to \boldsymbol{T} \times \boldsymbol{X}$, determined by the integral lines of any vector field $\boldsymbol{M} \to \mathrm{T}^{\|}\boldsymbol{M}$: the family of these lines can be identified with the fibre type $\boldsymbol{X}$ of t. It should be noted that, in general (differently from the flat case), the manifolds $\boldsymbol{T}$ and $\boldsymbol{X}$ do not inherit distinguished metric structures. One may choose *adapted* coordinate charts $(\mathsf{x}^a) = (\mathsf{x}^i, \mathsf{x}^4)$ on $\boldsymbol{M}$, determined by a chart (x^4) on $\boldsymbol{T}$ and a chart (x^i) on $\boldsymbol{X}$. Obviously, one has $g_{4i} = 0$, $i = 1, 2, 3$.

Besides adapted charts, it is also convenient to work with a *tetrad*, which is defined to be an ortonormal frame $(\Theta_\lambda) \equiv (\Theta_0, \Theta_j)$ such that $\Theta_0 : \boldsymbol{M} \to \mathrm{T}^{\|}\boldsymbol{M}$ and $\Theta_j : \boldsymbol{M} \to \mathrm{T}^{\perp}\boldsymbol{M}$, $j = 1, 2, 3$. One also sets $\partial \mathsf{x}_a = \Theta^{\lambda}_{a}\Theta_\lambda$, with $\Theta^{\lambda}_{a} : \boldsymbol{M} \to \mathbb{R}$.

The given time and spacetime orientations of $\boldsymbol{M}$ yield a space orientation, namely an orientation of each $\boldsymbol{M}_t$; one has the positive semi-vector bundle

$$\mathbb{V}^{\perp} := (\wedge^3 \mathrm{T}^{\perp}\boldsymbol{M})^{+} \subset \wedge^3 \mathrm{T}\boldsymbol{M} \to \boldsymbol{M} ,$$

and the spacetime volume form can be decomposed as $\eta = \Theta^0 \wedge \eta_0$, $\eta_0 : \boldsymbol{M} \to \mathbb{V}^{\perp *}$. It is not difficult to see that the spacetime connection determines connections on $\mathrm{T}^{\|}\boldsymbol{M} \to \boldsymbol{M}$ and $\mathrm{T}^{\perp}\boldsymbol{M} \to \boldsymbol{M}$ by the rules

$$\nabla^{\|}_{a} u := (\nabla_a u)^{\|}, \qquad u : \boldsymbol{M} \to \mathrm{T}^{\|}\boldsymbol{M} ,$$

$$\nabla^{\perp}_{a} v := (\nabla_a u)^{\perp}, \qquad v : \boldsymbol{M} \to \mathrm{T}^{\perp}\boldsymbol{M} ,$$

and that $\nabla^{\|}\Theta_0 = 0$, $\nabla^{\perp}\eta_0 = 0$.

Next, consider a *4-spinor bundle* (see also [CJ, C00b] for details); this is defined to be a complex vector bundle $\boldsymbol{W} \to \boldsymbol{M}$ with 4-dimensional fibres, endowed with a fibred Hermitian metric k with signature $(++--)$, a Clifford map $\gamma : \mathrm{T}\boldsymbol{M} \to \mathrm{End}(\boldsymbol{W})$ over $\boldsymbol{M}$ fulfilling $k(\gamma(v)\,\psi', \psi) = k(\psi', \gamma(v)\,\psi)$ $\forall (v, \psi', \psi) \in \mathrm{T}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \boldsymbol{W} \underset{\boldsymbol{M}}{\times} \boldsymbol{W}$, and a k-preserving linear connection $\digamma : \boldsymbol{W} \to \mathrm{J}\boldsymbol{W}$ such that $\nabla[\Gamma \otimes \digamma]\,\gamma = 0$. Then, in suitable linear fibre coordinates, $\digamma$ is related to the spacetime connection Γ by the expression

$$\digamma_{a}{}^{\alpha}{}_{\beta} = \mathrm{i}A_a\,\delta^{\alpha}{}_{\beta} + \frac{1}{4}\Gamma_{a}{}^{\lambda\mu}(\gamma_\lambda\gamma_\mu)^{\alpha}{}_{\beta}, \qquad \gamma_\lambda \equiv \gamma(\Theta_\lambda), \qquad \alpha, \beta = 1, 2, 3, 4 ,$$

where the functions $A_a : \boldsymbol{M} \to \mathbb{R}$ can be seen as the components of the connection induced on $\wedge^2 \boldsymbol{S} \to \boldsymbol{M}$, $\boldsymbol{S} \subset \boldsymbol{W}$ being a maximal k-isotropic subbundle (2-dimensional fibres). The time fibration yields a further Hermitian structure h in the fibres of $\boldsymbol{W}$, given by

$$h(\psi', \psi) := k(\gamma^0 \psi', \psi) = k(\psi', \gamma^0 \psi),$$

which turns out to have positive signature.

The Dirac equation for a (generalized) section $\psi : \boldsymbol{M} \to \boldsymbol{W}$,

$$\mathrm{i}\gamma^\lambda \nabla_\lambda \psi - \mu\psi + \frac{\mathrm{i}}{2} T_\lambda \gamma^\lambda \psi = 0, \quad \mu \in \mathbb{R}^+$$

(here $\gamma^\lambda := g^{\lambda\nu}\gamma_\nu$ and $T_\lambda := T_\lambda{}^\nu{}_\nu$, T being the torsion of the spacetime connection), can be rewritten, after composition by γ^0 on the left, as [5]

$$\partial_4 \psi - \Gamma_4 \psi + \Theta^0_4 (\Theta^{-1})^h_j \gamma^0 \gamma^j (\partial_h \psi - \Gamma_h \psi) + \Theta^0_4 \left(\mathrm{i}\mu\gamma^0 \psi + \frac{1}{2} T_\lambda \gamma^0 \gamma^\lambda \psi \right) = 0 .$$

Let now $\mathcal{W} := \mathcal{D}_{\boldsymbol{T}}(\boldsymbol{M}, \boldsymbol{W}) \to \boldsymbol{T}$ be the distributional bundle whose fibre over any $t \in \boldsymbol{T}$ is the space of all generalized sections of the classical bundle $\boldsymbol{W}_{\boldsymbol{M}_t} \to \boldsymbol{M}_t$. This is called the bundle of *1-electron states*, and a section $\psi : \boldsymbol{T} \to \mathcal{W}$ is called a *1-electron quantum history*. It is clear, from the latter way of writing it, that the Dirac equation can be seen as an equation for quantum histories of the form $\nabla[\mathfrak{C}]\,\psi = 0$, relatively to a linear connection $\mathfrak{C} : \mathcal{W} \to \mathrm{J}\,\mathcal{W}$ which I call the *quantum Dirac connection*. It should be noted that $\mathfrak{C}$ does *not* derive from a connection on the underlying classical bundle (§ 7).

The adjoint bundle of $\mathcal{W} \to \boldsymbol{T}$ is

$$\mathcal{W}^* = \mathcal{D}_{\boldsymbol{T}}(\boldsymbol{M}, \mathbb{V}^{\perp *}\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W}^*) \to \boldsymbol{T},$$

its fibres being constituted by $\boldsymbol{W}^*$-valued generalized densities on the spacelike fibres of t. Because the Hermitian metric k determines an anti-isomorphism $\boldsymbol{W} \leftrightarrow \boldsymbol{W}^*$, the conjugate Dirac equation is a field equation for (generalized) sections $\phi : \boldsymbol{M} \to \boldsymbol{W}^*$, namely

$$\mathrm{i}\nabla_\lambda \phi\gamma^\lambda + \mu\phi + \frac{\mathrm{i}}{2} T_\lambda \phi\gamma^\lambda = 0 .$$

As one has a connection on $\mathbb{V}^{\perp *} \to \boldsymbol{M}$, determined by the spacetime con-

[5] As customary, here spinor indices are not explicitely shown.

nection, and since $\nabla\eta_0 = 0$, one can equivalently write the above equation for ϕ as a formally identical equation for $\check{\phi} \equiv \eta_0 \otimes \phi : \boldsymbol{M} \to \mathbb{V}^{\perp *} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{W}^*$ (coordinates expressions, however, are not exactly the same). One can rewrite the equation for $\check{\phi}$ using the same procedure used for ψ above, getting

$$0 = \partial_4 \check{\phi} - (\partial_4 \log\det \Theta^{\perp})\, \check{\phi} + \check{\phi}\Gamma_4 + \\ + \Theta^0_4 (\Theta^{-1})^h_j [\partial_h \check{\phi} - (\partial_h \log\det \Theta^{\perp})\, \check{\phi} + \check{\phi}\Gamma_j]\, \gamma^j \gamma^0 + \\ + \Theta^0_4 (-\mathrm{i}\mu\check{\phi}\gamma^0 + \frac{1}{2} T_\lambda \check{\phi}\gamma^\lambda\gamma^0),$$

where $(\Theta^{\perp})$ denotes the «spacelike» matrix (Θ^k_i), $k, i = 1, 2, 3$. Then, one sees that the equation for $\check{\phi}$ can be also written in the form $\nabla[\mathfrak{C}^\flat]\, \check{\phi} = 0$, relatively to a connection $\mathfrak{C}^\flat \colon \boldsymbol{\mathfrak{W}}^* \to \mathrm{J}\,\boldsymbol{\mathfrak{W}}^*$. Naturally, one wishes to compare this connection with the distributional adjoint of $\mathfrak{C}$. It turns out that $\mathfrak{C}^\flat$ is not $\mathfrak{C}^*$, but rather it is the adjoint of $\mathfrak{C}$ relatively to a contraction mediated by the observer through γ_0 (thus related to the positive Hermitian metric h). In fact:

PROPOSITION 11.1. *Whenever all contractions are defined, one has*

$$\partial_4 \langle \check{\phi}, \gamma_0 \psi \rangle = \langle \nabla_4 [\mathfrak{C}^\flat]\, \check{\phi}, \gamma_0 \psi \rangle + \langle \check{\phi}, \gamma_0 \nabla_4 [\mathfrak{C}]\, \psi \rangle .$$

PROOF. By an argument similar to the proof of proposition 6.1 there is a connection $\mathfrak{C}' \colon \boldsymbol{\mathfrak{W}}^* \to \mathrm{J}\,\boldsymbol{\mathfrak{W}}^*$ determined by the requirement $\partial_4 \langle \check{\phi}, \gamma_0 \psi \rangle = \langle \nabla_4 [\mathfrak{C}']\, \check{\phi}, \gamma_0 \psi \rangle + \langle \check{\phi}, \gamma_0 \nabla_4 [\mathfrak{C}]\, \psi \rangle$. The operator $\nabla_4[\mathfrak{C}']$ can be calculated by assuming that $\check{\phi}$ and ψ are represented in each fibre by ordinary sections, and $\check{\phi}$ in particular by a test section. Then contractions can be written as integrals, and integration by parts gives

$$\nabla_4 [\mathfrak{C}']\, \check{\phi} = \partial_4 \check{\phi} + \check{\phi}\Gamma_4 + \tilde{\Gamma}_4{}^0{}_j \check{\phi}\gamma^j\gamma^0 + \Theta^0_4 (\Theta^{-1})^h_j (\partial_h \check{\phi} + \check{\phi}\Gamma_h)\, \gamma^j\gamma^0 + \\ + \partial_h [\Theta^0_4 (\Theta^{-1})^h_j]\, \check{\phi}\gamma^j\gamma^0 + \Theta^0_4 (\Theta^{-1})^h_j \tilde{\Gamma}_h{}^j{}_\lambda \check{\phi}\gamma^\lambda\gamma^0 + \\ - \mathrm{i}\Theta^0_4 \mu\check{\phi}\gamma^0 - \frac{1}{2} \Theta^0_4 T_\lambda \check{\phi}\gamma^\lambda\gamma^0 .$$

The comparison between $\mathfrak{C}^b$ and $\mathfrak{C}'$ now involves some coordinate calculations by which one relates the derivatives of the tetrad components to the torsion; eventually, these two distributional connections are seen to coincide. ■

By similar arguments, one can show that $\mathfrak{C}^*$ is related to the field equation obeyed by $\psi^\dagger$, the adjoint of ψ through the positive Hermitian metric h.

12. Connections in phase-distributional bundles.

A convenient way of describing quantum states consists in viewing them as distributions on the phase bundle of the particle under consideration. Let $\mu \in \{0\} \cup \mathbb{R}^+$ be the particle's mass (6) and consider the subbundle $\boldsymbol{K}^+_\mu \subset \mathrm{T}\boldsymbol{M}$ over $\boldsymbol{M}$ constituted by all future-pointing vectors $v \in \mathrm{T}\boldsymbol{M}$ such that $g(v, v) = \mu^2$ (using spacetime metric signature $(+\,-\,-\,-)$); the fibres are 3-hyperboloids for $\mu > 0$, null half-cones for $\mu = 0$.

Let $(\mathsf{y}^0, \mathsf{y}^i)$ be (not necessarily orthonormal) coordinates in the fibres of $\mathrm{T}\boldsymbol{M} \to \boldsymbol{M}$ such that $g_{00} > 0$ (namely y^0 is timelike) and $g_{0i} = 0$, $i = 1, 2, 3$. Then the restrictions of (y^i) are coordinates in the fibres of $\boldsymbol{K}^+_\mu \to \boldsymbol{M}$.

The following is a generalization of a result by Janyška and Modugno [JM96].

Proposition 12.1. *The spacetime connection Γ is reducible to a (non-linear) connection Γ_μ in $\boldsymbol{K}^+_\mu \to \boldsymbol{M}$; in orthonormal fibred coordinates $(\mathsf{y}^0, \mathsf{y}^i)$, its expression is*

$$(\Gamma_\mu)_a{}^i = \Gamma_a{}^i{}_0 (\mu^2 + \delta_{hk}\, \mathsf{y}^h \mathsf{y}^k)^{1/2} + \Gamma_a{}^i{}_j\, \mathsf{y}^j .$$

Proof. The subbundle $\boldsymbol{K}_\mu \subset \mathrm{T}\boldsymbol{M}$ over $\boldsymbol{M}$, constituted by all $v \in \mathrm{T}\boldsymbol{M}$ (of any time orientation) such that $g(v, v) = \mu^2$, is characterized in coordi-

(6) For a precise physical setting, physical constants should be described as elements of certain «unit spaces», namely 1-dimensional vector spaces or semi-vector spaces [3, 5, 7, 12]. Accordingly, some geometric structures and fields, such as the spacetime metric, the Dirac map γ and a quantum history ψ have unit spaces attached to them as tensor products. The metric, in particular, is valued into $\mathbb{L}^2 \equiv \mathbb{L} \otimes \mathbb{L}$ where $\mathbb{L}$ is the unit space of lengths. For the purpose of this paper, however, one can simply work with (arbitrarily) chosen units.

nates by the condition $g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\mathsf{y}^{\nu}=\mu^2$; hence, $\mathrm{T}\boldsymbol{K}_{\mu}$ is the submanifold of $\mathrm{TT}\boldsymbol{M}$ characterized by

$$g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\mathsf{y}^{\nu}=\mu^2, \quad \dot{\mathsf{x}}^{a}\,\partial_a g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\mathsf{y}^{\nu}+2g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\dot{\mathsf{y}}^{\nu}=0\,,$$

and $\mathrm{V}\boldsymbol{K}_{\mu}$ is the submanifold of $\boldsymbol{K}_{\mu}\underset{\boldsymbol{M}}{\times}\mathrm{T}\boldsymbol{M}$ characterized by $g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\dot{\mathsf{y}}^{\nu}=0$.

The vertical-valued form $\Omega:\mathrm{TT}\boldsymbol{M}\to\mathrm{VT}\boldsymbol{M}\cong\mathrm{T}\boldsymbol{M}\underset{\boldsymbol{M}}{\times}\mathrm{T}\boldsymbol{M}$, associated with the spacetime connection restricts to a form $\Omega_{\mu}:\mathrm{T}\boldsymbol{K}_{\mu}\to\boldsymbol{K}_{\mu}\underset{\boldsymbol{M}}{\times}\mathrm{T}\boldsymbol{M}$; using the above coordinates identities, and $\Omega=(\dot{\mathsf{y}}^{\lambda}-\dot{\mathsf{x}}^{a}\,\Gamma_{a}{}^{\lambda}{}_{\nu}\,\mathsf{y}^{\nu})\,\partial_{\lambda}$, it is immediate to check that Ω_{μ} is actually valued onto $\mathrm{V}\boldsymbol{K}_{\mu}$, namely it is the vertical-valued form associated with a connection on $\boldsymbol{K}_{\mu}\to\boldsymbol{M}$. On turn, this is obviously reducible to the subbundle $\boldsymbol{K}^{+}_{\mu}\subset\boldsymbol{K}_{\mu}$ of future-pointing vectors. In orthonormal fibre coordinates, on $\mathrm{T}\boldsymbol{K}^{+}_{\mu}$ one has

$$\mathsf{y}^{0}=\sqrt{\mu^2+\delta_{hk}\,\mathsf{y}^{h}\,\mathsf{y}^{k}}, \quad g_{\lambda\nu}\,\mathsf{y}^{\lambda}\,\dot{\mathsf{y}}^{\nu}=0\,, \quad \Gamma_{a\lambda\nu}\,\mathsf{y}^{\lambda}\,\mathsf{y}^{\nu}=0\,,$$

$$\Rightarrow\ \dot{\mathsf{y}}^{i}\circ\Omega_{\mu}=\dot{\mathsf{y}}^{i}-\dot{\mathsf{x}}^{a}(\Gamma_{a}{}^{i}{}_{0}\,\mathsf{y}^{0}+\Gamma_{a}{}^{i}{}_{j}\,\mathsf{y}^{j})\,, \quad \mathsf{y}^{0}=\sqrt{\mu^2+\delta_{hk}\,\mathsf{y}^{h}\,\mathsf{y}^{k}}. \quad \blacksquare$$

Let $\boldsymbol{W}\to\boldsymbol{M}$ be the spinor bundle introduced in § 11 and $\boldsymbol{V}\equiv\boldsymbol{K}^{+}_{\mu}\underset{\boldsymbol{M}}{\times}\boldsymbol{W}$. The couple $(\Gamma_{\mu},\tilde{\Gamma})$ is a classical connection on the 2-fibred bundle $\boldsymbol{V}\to\to\boldsymbol{K}^{+}_{\mu}\to\boldsymbol{M}$, linear projectable over Γ_{μ}; thus one gets (§ 7) a linear connection $\mathfrak{C}$ on the distributional bundle $\boldsymbol{\mathcal{V}}:=\boldsymbol{\mathcal{D}}_{\boldsymbol{M}}(\boldsymbol{K}^{+}_{\mu},\boldsymbol{V})\to\boldsymbol{M}$ (which is related to the quantum description of electrons and other massive $\frac{1}{2}$-spin particles: here $\boldsymbol{K}^{+}_{\mu}$ is the particle's phase bundle). Its coordinate expression is

$$(\mathfrak{C}_{a}{}^{\mathsf{Y}}{}_{\mathsf{Y}})^{\alpha}{}_{\beta}=\tilde{\Gamma}_{a}{}^{\alpha}{}_{\beta}-\delta^{\alpha}{}_{\beta}[\Gamma_{a}{}^{i}{}_{0}(\mu^2+\delta_{hk}\,\mathsf{y}^{h}\,\mathsf{y}^{k})^{1/2}+\Gamma_{a}{}^{i}{}_{j}\,\mathsf{y}^{j}]\,\partial_{i}.$$

For massless particles, the phase bundle is not $\boldsymbol{K}^{+}\equiv\boldsymbol{K}^{+}_{0}$ but rather its projective bundle over $\boldsymbol{M}$

$$\boldsymbol{P}\equiv\mathrm{P}\boldsymbol{K}^{+}:=\boldsymbol{K}^{+}/\mathbb{R}^{+}.$$

That is, $\boldsymbol{P}$ is the quotient of $\boldsymbol{K}^{+}$ by the action of the multiplicative group $\mathbb{R}^{+}$: its fibres are the sets of generatrices of the future null cone, namely 2-spheres (the so-called *celestial spheres*).

PROPOSITION 12.2. *There exists a unique connection* $\Gamma_{\boldsymbol{P}}\colon \boldsymbol{P} \to \mathrm{J}\boldsymbol{P}$ *such that the diagram*

$$\begin{array}{ccc} \boldsymbol{K}^{+} & \xrightarrow{\Gamma_0} & \mathrm{J}\boldsymbol{K}^{+} \\ {\scriptstyle \mathrm{P}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{JP}} \\ \boldsymbol{P} & \xrightarrow[\Gamma_{\underline{\boldsymbol{P}}}]{} & \mathrm{J}\boldsymbol{P} \end{array}$$

commutes, where $\mathrm{P}:\boldsymbol{K}^{+}\to\boldsymbol{P}$ *is the natural projection.*

PROOF. Let $k \in \boldsymbol{K}^{+}$, $r \in \mathbb{R}^{+}$; then, by means of coordinate expressions, it is not difficult to see that $\Gamma_0(k)$, $\Gamma_0(rk) \in \mathrm{J}\boldsymbol{K}^{+}$ are in the same orbit of the prolonged $\mathbb{R}^{+}$-action. ■

In order to write down a coordinate expression for $\Gamma_{\boldsymbol{P}}$, one may take spherical fibre coordinates $(\mathsf{r}, \theta, \phi)$ associated with orthonormal fibre coordinates (y^i). Then (θ, ϕ) are fibre coordinates for $\boldsymbol{P}$, and after some calculations one finds

$$(\Gamma_{\boldsymbol{P}})^{\theta}_{a} = \cos\theta\cos\phi\,\Gamma_a{}^1{}_0 + \cos\theta\sin\phi\,\Gamma_a{}^2{}_0 - \sin\theta\,\Gamma_a{}^3{}_0 + {} \\ + \cos\phi\,\Gamma_a{}^1{}_3 + \sin\phi\,\Gamma_a{}^2{}_3,$$

$$(\Gamma_{\boldsymbol{P}})^{\phi}_{a} = -\Gamma_a{}^1{}_2 + {} \\ + \frac{1}{\sin\theta}\left(-\sin\phi\,\Gamma_a{}^1{}_0 + \cos\phi\,\Gamma_a{}^2{}_0 - \cos\theta\sin\phi\,\Gamma_a{}^1{}_3 + \cos\theta\cos\phi\,\Gamma_a{}^2{}_3\right).$$

A classical photon field can be described as a section $\Phi : \boldsymbol{M} \to \mathrm{V}\boldsymbol{P}$ (see [C00b] for details). Accordingly, in view of its quantum description one is lead to consider the distributional bundle $\boldsymbol{\mathcal{P}} := \boldsymbol{\mathcal{D}}_{\boldsymbol{M}}(\boldsymbol{P}, \mathrm{V}\boldsymbol{P})$. The vertical prolongation of $\Gamma_{\boldsymbol{P}}$ is a connection (§ 10) $\mathrm{V}\boldsymbol{P} \to \mathrm{JV}\boldsymbol{P}$ which is linear projectable over $\Gamma_{\boldsymbol{P}}$, thus one obtains a linear connection $\boldsymbol{\mathcal{P}} \to \to \mathrm{J}\,\boldsymbol{\mathcal{P}}$.

Applications of these constructions to quantum field theory will be expounded in a forthcoming paper.

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