

Algebraic Sum of Unbounded Normal Operators and the Square Root Problem of Kato.

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ABSTRACT - We prove that the algebraic sum of unbounded normal operators satisfies the square root problem of Kato under appropriate hypotheses. As application, we consider perturbed Schrödinger operators.

1. Introduction.

Let C be a normal operator (not necessarily bounded) in a (complex) Hilbert space \mathcal{H} . Using the spectral theorem for unbounded normal operators, it is well-known that C can be expressed as

$$(1.1) \quad C = C_1 - iC_2,$$

with C_1, C_2 unbounded selfadjoint operators on \mathcal{H} (see, e.g. [13, pp. 348-355]). If one supposes C_1, C_2 to be nonnegative operators, then iC is m-accretive (see, e.g. [12, Corollary 4.4, p. 15]).

Let A, B be normal operators on \mathcal{H} . Recall the algebraic sum $S = A + B$ of A and B is defined as

$$(1.2) \quad \forall u \in D(S) = D(A) \cap D(B), \quad Su = Au + Bu.$$

In this paper we are concerned with the square root problem of Kato for the sum of operators A and B .

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In section 2, we prove that the algebraic sum S satisfies the square root problem of Kato under suitable hypotheses, that is:

$$(1.3) \quad D(S^{1/2}) = D(A^{1/2}) \cap D(B^{1/2}) = D(S^{*1/2}).$$

Also, since the algebraic sum S is not always defined (see, e.g., [5, 8]). We shall define a «generalized» sum $A \oplus B$ of A and B . One can then prove that such a «generalized» sum satisfies the square root problem of Kato under suitable hypotheses. As application we shall consider perturbed Schrödinger operators.

Recall that more details about the well-known square root problem of Kato can be found in [1, 4, 7, 9, 11].

Throughout this paper we assume that A and B can be decomposed as $A = A_1 - iA_2$ and $B = B_1 - iB_2$. We denote by $\Omega = \Omega(A) \cap \Omega(B)$ where

$$\begin{aligned} \Omega(A) &= D(|A|^{1/2}) = D(A_1^{1/2}) \cap D(A_2^{1/2}), \\ \Omega(B) &= D(|B|^{1/2}) = D(B_1^{1/2}) \cap D(B_2^{1/2}). \end{aligned}$$

The following assumptions will be made

- (H₁) A_k, B_k are nonnegative ($k = 1, 2$)
- (H₂) $\overline{D(A) \cap D(B)} = \mathcal{H}$
- (H₃) there exists $c > 0$, $\langle A_2 u, u \rangle \leq c \langle A_1 u, u \rangle, \forall u \in \Omega(A)$
- (H₄) there exists $c' > 0$, $\langle B_2 u, u \rangle \leq c' \langle B_1 u, u \rangle, \forall u \in \Omega(B)$
- (H₅) $\overline{\Omega} = \mathcal{H}$
- (H₆) Ω is closed in the interpolation space $[\Omega, \mathcal{H}]_{1/2}$

Consider the sesquilinear forms associated with A and B :

$$\begin{aligned} \phi(u, v) &= \langle A_1^{1/2} u, A_1^{1/2} v \rangle - i \langle A_2^{1/2} u, A_2^{1/2} v \rangle, \quad \forall u, v \in \Omega(A) \\ \psi(u, v) &= \langle B_1^{1/2} u, B_1^{1/2} v \rangle - i \langle B_2^{1/2} u, B_2^{1/2} v \rangle \quad \forall u, v \in \Omega(B) \end{aligned}$$

Set

$$(1.4) \quad \xi(u, v) = \phi(u, v) + \psi(u, v), \quad \forall u, v \in \Omega.$$

According to Bivar-Weinholtz and Lapidus (see, e.g., [2, pp. 451]) the «Generalized» sum $A \oplus B$ of A and B is defined with the help of the

sesquilinear form ξ as follows: $u \in D(A \oplus B)$ iff $v \rightarrow \xi(u, v)$ is continuous for the \mathcal{H} -Topology, and $(A \oplus B) u$ defined to be the vector of \mathcal{H} given by the Riesz-Representation theorem

$$(1.5) \quad \langle (A \oplus B) u, v \rangle = \xi(u, v) \quad \forall v \in \Omega .$$

Applying (H_2) to (1.4), we see that ξ admits the following representation

$$(1.6) \quad \xi(u, v) = \langle (A + B) u, v \rangle, \quad \forall u \in D(A) \cap D(B), \quad \forall v \in \Omega .$$

2. Square root problem of Kato.

In this section we show the algebraic sum $S = A + B$ satisfies the square root problem of Kato under suitable conditions. We also show the same conclusion still holds if we consider the square root problem of Kato for the «generalized» sum defined above.

We have

THEOREM 2.1. *Let $A = A_1 - iA_2$ and $B = B_1 - iB_2$ be unbounded normal operators on \mathcal{H} . Assume that assumptions (H_1) , (H_2) , (H_3) , and (H_4) are satisfied and that the operator $\overline{A + B}$ is maximal. Then we have*

$$D(\overline{(A + B)^{1/2}}) = D(A^{1/2}) \cap D(B^{1/2}) = D(\overline{(A + B)^{*1/2}}).$$

PROOF. Consider the sesquilinear form $\xi = \phi + \psi$ given by (1.6). Let $\Omega_\xi = (\Omega, \|\cdot\|_\xi)$ be the pre-Hilbert space Ω equipped with the inner product given as

$$\langle u, v \rangle_\xi = \langle u, v \rangle_{\mathcal{H}} + \Re e \xi(u, v), \quad \forall u, v \in \Omega .$$

Since the sum form $A_1 \oplus B_1$ is a nonnegative selfadjoint operator. It easily follows that Ω_ξ is a Hilbert space, and therefore ξ is a closed sesquilinear form. Moreover, $D(\xi) = \Omega$ is dense on \mathcal{H} ($D(A) \cap D(B) \subset \Omega$ and $\overline{D(A) \cap D(B)} = \mathcal{H}$). Thus ξ is a closed densely defined sesquilinear form. Assumptions (H_3) and (H_4) clearly imply that: there exists a constant $const. = \max(c, c') > 0$ such that

$$|\Im m \xi(u, u)| \leq const. \Re e \xi(u, u), \quad \forall u \in \Omega .$$

Therefore ξ is a sectorial sesquilinear form. In summary, ξ is a closed densely defined sectorial form. According to Kato's first representation theorem (see, e.g., [9, Theorem 2.1, pp. 322]): there exists a unique m -sectorial operator associated with ξ (m -sectorial extension of $\overline{A + B}$). Since $\overline{A + B}$ is maximal and ξ is sectorial. Then $\overline{A + B}$ is m -sectorial, and it is the m -sectorial operator associated with ξ . Since $D(A) = D(A^*)$ and $D(B) = D(B^*)$. It easily follows that $D(\overline{A + B}) \subset D((\overline{A + B})^*)$. Therefore $D((\overline{A + B})^{1/2}) \subset D((\overline{A + B})^{*1/2})$. According to [10, Theorem 5.2, p. 238], we have

$$(2.1) \quad D((\overline{A + B})^{1/2}) \subset D(\xi) \subset D((\overline{A + B})^{*1/2}).$$

In the same way, for the conjugate ξ^* of ξ we have

$$(2.2) \quad D((\overline{A + B})^{*1/2}) \subset D(\xi^*) \subset D(\overline{A + B})^{1/2}.$$

Since $D(\xi) = D(\xi^*) = \Omega$ and from (2.1), (2.2). It follows that

$$D((\overline{A + B})^{*1/2}) = \Omega = D(\overline{A + B})^{1/2}.$$

We now consider our investigation related to the square root problem of Kato for the «generalized» sum of operators defined above. We show that the same conclusion still holds under appropriate assumptions.

We have

THEOREM 2.2. *Let $A = A_1 - iA_2$ and $B = B_1 - iB_2$ be unbounded normal operators on \mathcal{H} . Assume that assumptions (H_1) , (H_3) , (H_4) , (H_5) , and (H_6) are satisfied. Then there exists a unique m -sectorial operator $A \oplus B$ satisfying the square root problem of Kato:*

$$D((A \oplus B)^{1/2}) = \Omega(A) \cap \Omega(B) = D((A \oplus B)^{*1/2}).$$

Also $A \oplus B$ and $\overline{A + B}$ coincide if $\overline{A + B}$ is maximal.

PROOF. Consider the sesquilinear form $\xi = \phi + \psi$ given by (1.4). Clearly ξ is a closed densely defined sesquilinear form. Also since $\Re \xi(u, u) = \|(A_1 \oplus B_1)^{1/2} u\|^2 \ \forall u \in \Omega$ and $\Im \xi(u, u) = -\|(A_2 \oplus B_2)^{1/2} u\|^2 \ u \in \Omega$. It easily follows ξ is a sectorial sesquilinear form. Thus there exists a $\text{const} = \max(c, c') > 0$ such that $|\Im \xi(u, u)| \leq \text{const} \cdot \Re \xi(u, u)$. According to Kato's first representation theorem: there exists a unique m -sectorial operator $A \oplus B$ associated with ξ such that

$$\xi(u, v) = \langle (A \oplus B) u, v \rangle \quad u \in D(A \oplus B), \quad v \in \Omega,$$

and in addition $D(A \oplus B), D((A \oplus B)^*) \subset D(\xi) = \Omega = \Omega_\xi$. Since Ω is closed in $[\Omega, \mathcal{D}]_{1,2}$. Then we conclude using a result of Lions (see, e.g., [11, Theorem. 6.1, p. 238]), that $A \oplus B$ satisfies the square root problem of Kato. It is not hard to see that $A \oplus B = \overline{A + B}$ if $\overline{A + B}$ is maximal. Indeed if $\overline{A + B}$ is maximal and since $A \oplus B$ is an m -sectorial extension of it. Then they coincide everywhere.

3. Applications.

In this section we give an application related to the algebraic sum case. Consider the algebraic sum given by a perturbation of the Schrödinger operator $S_Z = -Z\Delta + V$ with Z a complex number and V is a singular complex potential. Let $X \subset \mathbf{R}^d$ be an open set and assume that our Hilbert space $\mathcal{H} = L^2(X)$. Let Φ be the sesquilinear form given by

$$(3.1) \quad \Phi_Z(u, v) = \int_X Z \nabla u \overline{\nabla v} dx, \quad \forall u, v \in D(\Phi) = \mathbf{H}_0^1(X),$$

where $Z = \alpha - i\beta$ is a complex number satisfying the following conditions: $\alpha > 0, \beta > 0$, and $\beta \leq \alpha$. The previous conditions on Z clearly that the sesquilinear form Φ is sectorial.

Let V be a measurable complex function and let Ψ be the sesquilinear form given as

$$(3.2) \quad \Psi(u, v) = \int_X V u \overline{v} dx, \quad \forall u, v \in D(\Psi),$$

with $D(\Psi) = \{u \in L^2(X) : V|u|^2 \in L^1(X)\}$. Throughout this section we assume that the potential $V \in L^1_{loc}(X)$ and that there exists $\theta \in \left(0, \frac{\pi}{2}\right)$ such that

$$(3.3) \quad |\arg(V(x))| \leq \theta, \quad \text{almost everywhere}$$

From (3.3) we have

$$(3.4) \quad |\Im m \Psi(u, u)| \leq \tan \theta \Re e \Psi(u, u), \quad \forall u \in D(\Psi).$$

In other words, the sesquilinear form Ψ is sectorial. Under the previous assumptions Φ and Ψ are closed densely defined sectorial forms. The op-

erators associated with both Φ and Ψ are given as

$$D(A_Z) = \{u \in \mathbf{H}_0^1(X) : Z\Delta u \in L^2(X)\}, \quad A_Z u = -Z\Delta u, \quad \forall u \in D(A_Z)$$

$$D(B) = \{u \in L^2(X) : Vu \in L^2(X)\}, \quad Bu = Vu, \quad \forall u \in D(B).$$

It is not hard to see that A_Z, B are unbounded normal operators in $L^2(X)$. One can decompose them as $A_Z = A_Z^1 - iA_Z^2$ where $A_Z^1 = -\alpha\Delta$ and $A_Z^2 = -\beta\Delta$ are nonnegative selfadjoint operators, and $B = B_V^1 - iB_V^2$ where B_V^1, B_V^2 are nonnegative selfadjoint operators (that is justified by (3.3)). Now one assumes that $X = \mathbf{R}^d$. Thus the operator associated with $\Xi = \Phi + \Psi$ (by the first representation theorem of Kato) is the closure of the algebraic sum S_Z (i.e., $\overline{-Z\Delta + V}$) and it is m -sectorial (see, e.g., [3]). In fact such an operator is defined as

$$D(\overline{-Z\Delta + V}) = \{u \in \mathbf{H}^1(\mathbf{R}^d) : V|u|^2 \in L^1(\mathbf{R}^d) \text{ and } -Z\Delta u + Vu \in L^2(\mathbf{R}^d)\}$$

$$\overline{-Z\Delta + V}u = -Z\Delta u + Vu, \quad \forall u \in D(\overline{-Z\Delta + V}).$$

Let us notice that $D(A_Z) = \mathbf{H}^2(\mathbf{R}^d), D(B) = \{u \in L^2(\mathbf{R}^d) : Vu \in L^2(\mathbf{R}^d)\}$, and their intersection is dense in $L^2(\mathbf{R}^d)$. Therefore applying Theorem 2.1 to the operators A_Z and B . It easily follows that

$$D(\overline{(-Z\Delta + V)^{1/2}}) = \mathbf{H}^1(\mathbf{R}^d) \cap D(B^{1/2}) = D(\overline{(-Z\Delta + V)^{*1/2}}).$$

For instance considering the case $d = 1$. Then we have

$$D(\overline{(-Z\Delta + V)^{1/2}}) = \mathbf{H}^1(\mathbf{R}) = D(\overline{(-Z\Delta + V)^{*1/2}}).$$

REMARK 3.1. We may illustrate theorem 2.2 by considering both A_Z and B defined above and by assuming that the potential $V \neq 0$ satisfies: (3.3) and the following

$$(3.5) \quad V \in L^1(\mathbf{R}^d), V/\in L_{loc}^2(\mathbf{R}^d).$$

In such a case, it is not hard to show that $D(A_Z) \cap D(B) = \{0\}$ (see, e.g., [5, 8]). Therefore the algebraic sum $S_z = -Z\Delta + V$ is not defined. Nevertheless, consider the sum $\Xi = \Phi + \Psi$. Clearly Ξ is a closed densely defined sectorial form ($C_0^\infty(\mathbf{R}^d) \subset D(\Phi) \cap D(\Psi)$). According to the first representation theorem of Kato: there exists a unique m -sectorial operator $(-Z\Delta \oplus V)$ associated with Ξ . Thus theorem 2.2 can applied to A_Z and B . Therefore it easily follows the operator $(-Z\Delta \oplus V)$ satisfies the square root problem of Kato.

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