

The Finite Free Extension of Artinian K -Algebras with the Strong Lefschetz Property.

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1. Introduction.

The main result of this paper is Theorem 28 in Section 5. This may be regarded as a generalization of the theorem which states that the Strong Lefschetz property (SLP) is preserved by tensor product. The definition of the strong Lefschetz property is the same as what the second author called the strong Stanley property in his paper [7], where it was proved that such property is preserved by tensor product. The definition forces the Hilbert functions to be symmetric.

In our previous paper [3], with two other coauthors, we used the definition of the strong Lefschetz property in the sense of [5], so that it can be applied to algebras which do not necessarily have symmetric Hilbert functions.

In the present paper we assume the symmetry of the Hilbert functions whenever we discuss the strong Lefschetz property. To explain why such a restriction is necessary we would like to point out the fact that although the symmetric unimodality of Hilbert functions is preserved by tensor product, solely the unimodality of Hilbert functions is *not*. Keeping this in mind, it is easy to construct counter examples in the general sense of SLP to our Main Theorem. (See Definition 2, Remarks 3, 4, and Example 5.)

Let A be an Artinian K -algebra, and let $y \in A$. Then y induces a linear map $\times y : A \rightarrow A$ which is represented by a nilpotent matrix if y is a non

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unit. Our method has grown out of an attempt to determine the Jordan canonical form of the linear map $\times y$ induced by a general element y . Why does the knowledge of Jordan canonical form of a general element prove or disprove the strong Lefschetz property? This is explained as follows: Suppose that two K -algebras A and B have the same Hilbert function and one of them is known to have the strong Lefschetz property. Then the other algebra has the same property if and only if a general element has the same Jordan canonical form as the other. As one notices easily the strong Lefschetz property of an Artinian algebra is an assertion about the Jordan canonical form of a general element.

The basic idea of the proof of Main Theorem is to consider two nilpotent elements $g, h \in \text{End}(V)$, where V is a vector space, commuting with each other, and deduce certain inequality for the rank of $ag + \beta h$, with some constants α, β , which assures that it is not too small in a setup of Main Theorem. This is done in Section 3 (Proposition 21). As one might expect it enables us to prove the weak Lefschetz property in certain cases. Another important tool is Proposition 18 in Section 2. It enables us to prove the strong Lefschetz property for the class of algebras in Main Theorem by showing only the weak Lefschetz property using Proposition 21. These two propositions are the keys to the proof of Main Theorem.

Considerable part of Section 2 is devoted for the proof of Proposition 18 just said above. Proposition 19 is an interesting consequence of Proposition 18. It says that the weak Lefschetz property for *all* complete intersections in certain fixed dimension, say n , implies the strong Lefschetz property of all complete intersections in dimension $(n - 1)$. Thus proving that all complete intersections in any number of variables have the weak Lefschetz property is equivalent to proving the same for the strong Lefschetz property, so in this sense the problems are the same. So far not many results are available for the strong or weak Lefschetz property of complete intersections. But at least we know that every complete intersection in codimension two has the strong Lefschetz property, and every complete intersection in codimension three has the weak Lefschetz property ([3], Theorem 2.3 and Proposition 4.4). Our result says that if there exists a complete intersection without the strong Lefschetz property in codimension three, it implies the existence of a complete intersection without the weak Lefschetz property in codimension four.

One other purpose of Section 2 is the proof of Theorem 11, which says that the strong Lefschetz property is preserved under taking tensor

product. The result itself is not new but the proof is new because it does not use the theory of \mathfrak{sl}_2 . We wanted to avoid using the theory of the Lie algebra, because it may apply to prove the strong Lefschetz property for a larger class of algebras.

This section also includes some definitions. Among others we define «CoSperner number» of Artinian algebras. This is easier to compute than the usual «Sperner number.» For example Proposition 8 is implicitly a description of the CoSperner number of certain K -algebras. Also we introduce the «Jordan second canonical form» of a nilpotent matrix. This is essentially the same as the usual Jordan decomposition, but as one will see it is easier to deal with in the theory of Artinian algebras.

As mentioned earlier, in Section 3 we are concerned with the rank of certain nilpotent matrices. Let V be a vector space over K . Let $g, g' \in \text{End}(V)$ be nilpotent matrices, both preserving a same flag of subspaces: $V_0 \subset V_1 \subset V_2 \subset \dots \subset V_r = V$ so $g(V_i) \subset V_i$ and $g'(V_i) \subset V_i$ for all i . We say that $g' \in \text{End}(V)$ is a deformation of g if they coincide in each block $\text{End}(V_i/V_{i-1})$ for $i = 1, 2, \dots, r$. Let $h \in \text{End}(V)$ be yet another nilpotent matrix preserving the same flag. In certain situations it happens that the Jordan canonical form of $g + h$ is known. In such a case we can prove that the rank of $g' + \lambda h$ is no less than the rank of $g + \lambda h$ for any deformation g' with a sufficiently general $\lambda \in K$ (Proposition 21). This plays an important role in the proof of Main Theorem.

In Section 4 we develop yet another important tool that we need for the proof of Main Theorem. We consider the commutator algebra of A in the full matrix ring $\text{End}(A)$ and also in $\text{End}(V \otimes A)$ for some V . Recall that the Double Commutant Theorem, in the simplest form, says that the commutator algebra of $\text{End}(V) \otimes 1_U$ in $\text{End}(V) \otimes \text{End}(U)$ is $1_V \otimes \text{End}(U)$. In particular if $\dim V = 1$ it reduces to the well know fact that the center of the full matrix ring is the set of scalar matrices. First we note that the commutator algebra of A embedded in $\text{End}(A)$ is A itself. Then this fact further enables us to determine the commutator algebras of A in $\text{End}(V \otimes A)$. This is given in Lemmas 25 and 26. Not only the fact but the notation is important. The vector space $\text{End}(V) \otimes \text{End}(U) \cong \text{End}(V \otimes U)$ is a set of tensors of degree four. Put $s = \dim V$ and $n = \dim U$. Then the vector space $\text{End}(V \otimes U)$ may be identified as the set of elements in K

$$(m(i_1, i_2, i_3, i_4))$$

with $1 \leq i_1, i_2 \leq s$ and $1 \leq i_3, i_4 \leq n$. With the first two indices fixed it is an $n \times n$ matrix and with the last two fixed it is an $s \times s$ matrix. In this section a «block matrix» means such a tensor, and we write $\mathcal{M} = (m_{i_3 i_4}^{(i_1 i_2)})$. The element $m_{ij}^{(kl)}$ is the (ij) entry of the (kl) block. Lemma 27 is the third key to the proof of Main Theorem. It describes the shape of these block matrices, when a finite free extension of K -algebras $A \rightarrow B$ factorizes as $A \rightarrow V \otimes A \rightarrow B$, for some $V \subset B$, as homomorphisms of vector spaces.

In Section 5 we prove Main Theorem using the results proved in earlier sections.

In Section 6 we show some classes of complete intersections which have the SLP, as an application of Main Theorem.

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2. Preliminaries.

Throughout this paper K denotes a field of characteristic 0 unless otherwise stated. A graded K -algebra is a K -algebra with grading: $A = \bigoplus_{i=0}^{\infty} A_i$. The vector space A_i is the homogeneous part of A of degree i . It will always be assumed that $A_0 = K$ and $A = K[A_1]$. When we say that (A, \mathfrak{m}) is a graded Artinian K -algebra it means that A is a finite dimensional K -vector space, so $A = \bigoplus_{i=0}^c A_i$ where $0 \leq c < \infty$ and $\mathfrak{m} = \bigoplus_{i=1}^c A_i$.

DEFINITION 1. *Let (A, \mathfrak{m}) be a graded Artinian K -algebra as above.*

- *The **Sperner number** of A is $\text{Max} \{ \dim A_i \mid i = 0, 1, 2, \dots, c \}$.*
- *The **CoSperner number** of A is $\sum_{i=0}^{c-1} \text{Min} \{ \dim A_i, \dim A_{i+1} \}$.*

Let $y \in A$ be a linear element. One sees easily that the greatest possible rank for the multiplication map $\times y : A \rightarrow A$ as an endomorphism of the vector space is the CoSperner number and similarly, the least possible value for $\dim A/yA$, or the corank of $\times y : A \rightarrow A$, is the Sperner number.

DEFINITION 2. Suppose that (A, \mathfrak{m}) is a graded Artinian K -algebra such that $A = \bigoplus_{i=0}^c A_i$ with $A_c \neq 0$.

- The algebra A has the **weak Lefschetz property (WLP)** if there is a linear element $y \in A_1$ such that the multiplication $\times y : A_i \rightarrow A_{i+1}$ is either injective or surjective for all $i = 0, 1, \dots, c-1$.

- The algebra A has the **strong Lefschetz property (SLP)** if there is an element $y \in A_1$ such that the multiplication

$$\times y^{c-2i} : A_i \rightarrow A_{c-i}$$

is bijective for all $i = 0, 1, \dots, \lfloor c/2 \rfloor$.

- If A has the SLP, we call a linear element y with this property a **Lefschetz element** of A .

REMARK 3. It is easy to see the following:

1. The strong Lefschetz property implies the weak Lefschetz property.

2. The weak Lefschetz property implies that the Hilbert function of A is unimodal and the dimension of $\text{Ker}[\times y : A \rightarrow A]$ is equal to the Sperner number of A , where y is as in Definition 2.

3. The strong Lefschetz property implies that the Hilbert function of A is symmetric.

REMARK 4. In our paper [3] we used the more general definition of strong Lefschetz property than the one given above. Namely we said $A = \bigoplus A_i$ has the strong Lefschetz property if there is a linear form $y \in \mathfrak{m}$ such that the homomorphism $\times y^k : A_i \rightarrow A_{i+k}$ is either injective or surjective for any k and any i . (It is not difficult to see that if we assume that A has a symmetric Hilbert function, then the two definitions are equivalent.) Throughout this paper we use the definition in the specialized sense, as defined in Definition 2, which forces the Hilbert functions to be symmetric. The necessity for this will be explained in Example 5 just below.

EXAMPLE 5. **Only in this example we use SLP in the general sense.** We consider algebras with the Hilbert function like $(1, n, 1, 1, \dots, 1)$. As is easily seen such algebras exist and they have the SLP, as the homogeneous part of the highest degree is spanned by a power of a

of size n_i . Note that the dimension of $\text{Ker}[Y : V \rightarrow V]$ is the number of blocks of the Jordan canonical form of Y . It is to be understood that $P(Y)$ denotes the partition itself rather than the matrix.

REMARK 6. Suppose that $A = \bigoplus_{i=0}^c A_i$ ($A_c \neq 0$) has the SLP with a Lefschetz element $y \in A_1$. A basis for the Jordan decomposition for $\times \times y : A \rightarrow A$ is obtained as follows: First of all $1, y, y^2, \dots, y^c$ will be a part of the basis, giving us the first block for the Jordan decomposition. Now let $0 \neq \bar{a} \in \text{Ker}[A_{c-1} \rightarrow A_c]$ (if it exists). By the SLP of A there exists $a \in A_1$ such that $ay^{c-2} = \bar{a}$. Then the elements $a, ay, ay^2, \dots, ay^{c-2}$, none of these being dependent of the previously chosen basis elements, will be another part of the basis, giving us the second block. If there is $\bar{b} \in \text{Ker}[A_{c-1} \rightarrow A_c]$ independent of \bar{a} then we choose $b \in A_1$ such that $by^{c-2} = \bar{b}$ and let $b, by, by^2, \dots, by^{c-2}$ be the third part of the basis. Continue this way until elements of $\text{Ker}[A_{c-1} \rightarrow A_c]$ are exhausted. Then we move to $\text{Ker}[A_{c-2} \rightarrow A_{c-1}]$ and repeat the same process to expand basis elements. When this process is finished, it means we have decomposed $\times y$ into Jordan blocks.

Let $m_0 \leq m_1 \leq \dots \leq m_c$ be an increasing series of positive integers such that $n = m_0 + m_1 + \dots + m_c$. For $i = 1, 2, \dots, c$ let $Y_{(i-1)i}$ be the $m_{i-1} \times m_i$ matrix $\begin{bmatrix} O & E \end{bmatrix}$, where E is the identity of size m_{i-1} and O is the zero matrix of size $m_{i-1} \times (m_i - m_{i-1})$. Furthermore for (i, j) with $j \neq i + 1$ define Y_{ij} to be the zero matrix of size $m_i \times m_j$ ($0 \leq i, j \leq c$). Let $Y = (Y_{ij})$ be the $n \times n$ matrix consisting of submatrices Y_{ij} at the positions (i, j) . We call Y the **Jordan second canonical form** of a nilpotent matrix. It is determined uniquely by an increasing sequence of positive integers: $m_0 \leq m_1 \leq \dots \leq m_c$. Call the usual Jordan canonical form the first if we need to distinguish.

A basis for the second Jordan canonical form is nothing but a re-ordering of the basis elements for the Jordan (first) canonical form. It is easy to see how to reorder the basis elements, but for the purpose that we need later, we would like to make this explicit. Let B be a basis for the Jordan canonical form of a nilpotent element $Y \in \text{End}(V)$. Let c be the maximum integer such that $Y^c \neq 0$. Put $B_i = (\text{Ker } Y^{c+1-i}) \cap B \setminus (\text{Ker } Y^{c-i}) \cap B$. Furthermore put $B'_c = \{e \in B_c \mid e \notin \text{Image } Y\}$ and let $B''_c = B_c \setminus B'_c$. Then, for a second canonical form of Y , it suffices to choose any order « \ll » on the elements of B satisfying the following properties:

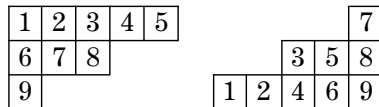
1. If $e \in B'_c$ and $f \in B'_c$, then $e < f$.
2. For $e, f \in B_i$ with $i < c$, we have $e < f$ if $eY < fY$.
3. If $e \in B_i, f \in B_j$ for $i < j$, then $e < f$.

Start with a nilpotent element $Y \in \text{End}(V)$ where $\dim V = n$. Let c be the greatest integer such that $Y^c \neq 0$. Let $m'_i = \dim \text{Ker}[Y^i: V \rightarrow V]$ for $i = c + 1, c, c - 1, \dots, 1, 0$, and put $m_i = m'_c - i + 1 - m'_{c-i}$. Then Y is represented by the Jordan second canonical form, described above with the sequence of integers, m_0, m_1, \dots, m_c . Note that this is automatically an increasing sequence. Let C be a scalar matrix. Then we note that $C + Y$ can be put in a Jordan first canonical form only by a permutation of basis elements.

Let $Y \in \text{End}(V)$ be nilpotent and let $P(Y) = n_1 \oplus \dots \oplus n_r$ be the partition associated with the Jordan decomposition of Y . Assume they are put in the decreasing order: $n_1 \geq n_2 \geq \dots \geq n_r$. Both the first and the second Jordan canonical form can be best described by using Young tableaux. Let $T = T(n_1, n_2, \dots, n_r)$ be the Young diagram (aligned left) with rows of n_i boxes. We number the boxes from left to right naturally. Using T it is easy to describe the Jordan canonical form of Y . Namely the Jordan canonical form is the matrix $J = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j \text{ is next to the right of } i \text{ in } T, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{T} = T(n_r, n_{r-1}, \dots, n_1)$ be the Young diagram *aligned right* with n_{r-i} boxes for the i th row. We number the boxes of \widehat{T} vertically from top to bottom starting with the leftmost column ending with the rightmost column. For such a Young tableau we write $\widehat{T} = \widehat{T}(m_0, m_1, \dots, m_c)$, where the integers m_0, m_1, \dots, m_c are the numbers of boxes of columns put in the increasing order. So the two partitions $n_1 \oplus n_2 \oplus \dots \oplus n_r$ and $m_0 \oplus m_1 \oplus \dots \oplus m_c$ of n are **dual** to each other. Here is an example of $T = T(5, 3, 1)$ and $\widehat{T} = \widehat{T}(1, 1, 2, 2, 3)$.



The integers in the boxes are supposed to be the indices of basis elements. So the set of elements with indices in each row form a Jordan block for Y .

Using \widehat{T} the Jordan second canonical form can be described exactly in the same manner as for the first form with T . Namely if $J = (a_{ij})$ is the Jordan second canonical form for Y then we have

$$a_{ij} = \begin{cases} 1 & \text{if } j \text{ is next to the right of } i \text{ in } \widehat{T}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular a basis element is not in the image of Y if and only if its index appears as the first integer of some row. Also a basis element is in the kernel of Y if and only if its index appears as the last integer of some row. (These apply to both T and \widehat{T} .)

Here is an example of Jordan second canonical form. Let $V = K^9$, and let $Y \in \text{End}(V)$ be nilpotent such that $P(Y) = 5 \oplus 3 \oplus 1$. Then the dual partition is $1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$. The second canonical form for Y is the matrix:

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

To summarize the above observation we have

PROPOSITION 7. *Suppose $Y \in \text{End}(V)$ is a nilpotent element with $Y^c \neq 0$ and $Y^{c+1} = 0$. Let $P(Y) = n_1 \oplus \dots \oplus n_r$, where $n_1 \geq n_2 \geq \dots \geq n_r$. Then the Jordan canonical form is described by the Young tableaux $T = T(n_1, n_2, \dots, n_r)$. The Jordan second canonical form of Y is described by $\widehat{T} = \widehat{T}(m_0, m_1, \dots, m_c)$ with $m_0 \leq m_1 \leq \dots \leq m_c$, where*

$$m_i = \dim((\text{Ker}(Y^{c-i+1})/(\text{Ker}(Y^{c-i}))).$$

Furthermore the partition $m_0 \oplus m_1 \oplus \dots \oplus m_c$ is dual to $P(Y)$.

PROPOSITION 8. *Put $V = K^n, V' = K^{n'}$. For $d = n, n'$, let $E_d \in \text{End}(K^d)$ denote the identity. Suppose that $N_d \in \text{End}(K^d)$ is a nilpotent*

element such that $P(N_d) = d$, i.e., N_d is equivalent to a single Jordan block of size d . Let

$$\tilde{N} = E_n \otimes N_{n'} + N_n \otimes E_{n'} \in \text{End}(V \otimes V').$$

Then

- (1) $P(\tilde{N}) = \underbrace{(n+n'-1) \oplus (n+n'-3) \oplus (n+n'-5) \oplus \dots \oplus (|n-n'|+1)}_{\text{Min}\{n, n'\}}$
- (2) $\dim \text{Ker}(\tilde{N}) = \text{Min}\{n, n'\}$.
- (3) $\text{rank}(\tilde{N}) = \text{rank}(N_n)^{n'} + n'(n-1) = \text{rank}(N_n)^{n'} + n(n'-1)$.

PROOF. (1) We may assume that N_n and $N_{n'}$ are already put in the Jordan canonical form. Consider the algebras $K[x]/(x^n)$ and $K[y]/(y^{n'})$. The linear map $K[x]/(x^n) \rightarrow K[x]/(x^n)$ induced by multiplication by x may be identified with N_n and similarly for y and $N_{n'}$. Thus the multiplication map $\times(x+y): A \rightarrow A$, where $A = k[x, y]/(x^n, y^{n'})$, is essentially the same as \tilde{N} in consideration. By ([3], Proposition 4.4), which uses a theorem of Galligo ([2]), A has the SLP. Assume $n \leq n'$. Then it is easy to see that the Hilbert function of A is as follows:

$$\underbrace{1, 2, 3, \dots, n-2, n-1, \overbrace{n, n, \dots, n}^{|n-n'|+1}, n-1, n-2, \dots, 3, 2, 1}_{n+n'-1}$$

(The case $n > n'$ is similar.) In Remark 6 we showed a procedure to decompose the map $\times(x+y)$ into Jordan blocks, from which (1) follows easily.

(2) Notice that the maximal integer of the Hilbert function of A is $\text{Min}\{n, n'\}$. Since A has the SLP, it has WLP (Remark 3), from which the assertion follows.

From (2) it follows that $\text{rank}(\tilde{N}) = \text{Max}\{nn' - n, nn' - n'\}$. On the other hand we have

$$\text{rank}(N_n)^{n'} = \begin{cases} n - n' & \text{if } n' \leq n, \\ 0 & \text{if } n' > n. \end{cases}$$

This proves the first equality of (3), and also the second by symmetry.

REMARK 9. The figure below illustrates the Jordan canonical form of the matrix \tilde{N} in Proposition 8, which turns out to be the same as the lin-

ear map

$$\times(x + y) : K[x, y]/(x^{n+1}, y^{n'+1}) \rightarrow K[x, y]/(x^{n+1}, y^{n'+1}).$$

It helps us see the block sizes and the number of blocks of the decomposition of \tilde{N} . This can be proved by using the representation theory of the Lie algebra \mathfrak{sl}_2 ([7]), or alternatively by the SLP of the ring $k[x, y]/(x^n, y^{n'})$ ([3]), as said above.

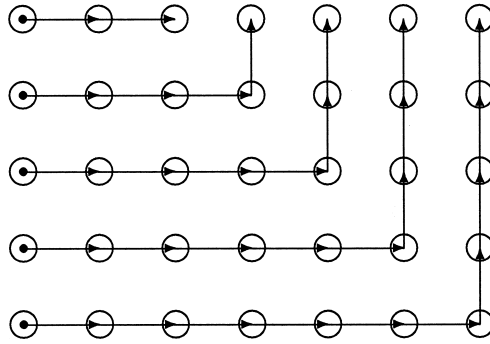


Fig. 1. – The figure illustrates the Jordan decomposition of \tilde{N} for $n=5, n'=7$.

PROPOSITION 10. Let (A, m) and (B, n) be Artinian K -algebras and let $y \in A$ and $z \in B$ be some non-unit elements (not necessarily of degree 1). Put $Y = y \otimes 1 + 1 \otimes z$, which is an element of $A \otimes_K B$. If $P(\times y) = d_1 \oplus \dots \oplus d_r$ and $P(\times z) = f_1 \oplus \dots \oplus f_s$ then

- (1) $P(\times Y) = \bigoplus_{i,j} \bigoplus_{q=1}^{\text{Min}\{d_i, f_j\}} (d_i + f_j + 1 - 2q)$
- (2) $\dim \text{Ker}(\times Y) = \sum_{i,j} \text{Min}\{d_i, f_j\}$
- (3) $\text{rank}(\times Y) = \sum_{j=1}^s (\text{rank}(y^{f_j}) + d(f_j - 1))$, where $d = \sum d_i$.

PROOF. (1) is proved using Proposition 8 block for block. Then (2) follows immediately from (1) as $\dim \text{Ker}(\times Y)$ is the number of blocks of the Jordan decomposition of $(\times Y)$. For (3) notice first that this is immediate from Proposition 8 if $s = 1$. Thus the general case also follows.

THEOREM 11. The SLP is preserved by tensor product. To be precise if A and B are graded Artinian K -algebras with the SLP then the tensor product $A \otimes_K B$ also has the SLP. In particular, $A[z]/(z^n)$ has the SLP for any $n \geq 0$ if A does.

PROOF. Probably the best proof is to use the theory of the Lie algebra \mathfrak{sl}_2 as was indicated in [7]. Alternatively one can argue as follows. Choose Lefschetz elements $g \in A_1$ and $h \in B_1$ (Definition 2). Decompose $A = \bigoplus U_i$ into subspaces so that each summand gives a Jordan block for $\times g : A \rightarrow A$, and similarly let $B = \bigoplus V_j$ be a Jordan decomposition for $\times h : B \rightarrow B$. This can be done by the procedure of Remark 2. So we may assume that all U_i and V_j have bases consisting of homogeneous elements. Now $A \otimes B$ decomposes as

$$(1) \quad A \otimes B = \bigoplus_{i,j} (U_i \otimes V_j).$$

Since this is a decomposition as graded vector spaces, we also have

$$(2) \quad (A \otimes B)_k = \bigoplus_{i,j} [(A \otimes B)_k \cap (U_i \otimes V_j)].$$

where $(A \otimes B)_k$ denotes the degree k homogeneous part of $A \otimes B$. Note that each $U_i \otimes V_j \subset A \otimes B$ is fixed under the multiplication $\times (g \otimes 1 + 1 \otimes h)$, and these subspaces further decompose into Jordan blocks as shown in Proposition 8 (1). Put $Y = g \otimes 1 + 1 \otimes h$ and let c be the maximum integer such that $(A \otimes B)_c \neq 0$. The proof of Proposition 8 (1) in fact tells us that, for each pair (i, j) , the linear map

$$(3) \quad \times Y^{c-2k} : (A \otimes B)_k \cap (U_i \otimes V_j) \rightarrow (A \otimes B)_{c-k} \cap (U_i \otimes V_j)$$

is bijective for each $k = 0, 1, \dots, [c/2]$. In view of the decomposition of (1) this proves the SLP of $A \otimes B$ with Y a Lefschetz element.

Among all $P(N)$ for nilpotent elements $N \in \text{End}(V)$ we put the total order $>$ as follows. (Recall that $P(N)$ denotes the partition itself rather than the Jordan matrix.)

DEFINITION 12. *Suppose that $N, N' \in \text{End}(V)$ are nilpotent elements such that $P(N) = n_1 \oplus n_2 \oplus \dots \oplus n_r$ and $P(N') = n'_1 \oplus n'_2 \oplus \dots \oplus n'_r$, where we assume that $n_1 \geq n_2 \dots \geq n_r$ and $n'_1 \geq n'_2 \dots \geq n'_r$. Then we will write $P(N) > P(N')$ if and only if (1) $r < r'$ or (2) $r = r'$ and $n_i = n'_i$ for $i = 1, 2, \dots, j-1$ and $n_j > n'_j$. $P(N)$ and $P(N')$ are regarded as the same if $\{n_i\}$ and $\{n'_i\}$ are equal as partitions of the integer $n := \dim V$. (So $>$ is a total order on the set of all partitions of n .)*

Let (A, \mathfrak{m}) be an Artinian K -algebra with $K = A/\mathfrak{m}$ (not necessarily graded). Let $\{\xi_1, \xi_2, \dots, \xi_n\}$ be a set of algebraically independent elements over A . Put $A^* = A(\xi_1, \xi_2, \dots, \xi_n)$, which is the polynomial

ring $A[\xi_1, \xi_2, \dots, \xi_n]$ localized at the minimal prime ideal $\mathfrak{m}A[\xi_1, \xi_2, \dots, \xi_n]$. Also put $\mathfrak{m}^* = \mathfrak{m}A^*$ and $K^* = A^*/\mathfrak{m}^*$.

PROPOSITION 13. *With the same notation as above, suppose that $\mathfrak{m} = (x_1, \dots, x_n)$. Put $Y = \xi_1 x_1 + \dots + \xi_n x_n$. We consider Y as an element of A^* and $\times Y \in \text{End}_{K^*}(A^*)$. Then $P(\times Y)$ is the maximum element of the set $\{P(\times y) \mid y \in \mathfrak{m}\}$ with respect to the order « $>$ » introduced in Definition 12. Also there exists an element $y \in \mathfrak{m}$ such that $P(\times y) = P(\times Y)$.*

PROOF. Let y be an element such that $P(\times y)$ is the maximum element in $\{P(\times x) \mid x \in \mathfrak{m}\}$. In ([8] Theorem A) it was proved that

$$\dim_{K^*} A^*/YA^* = \dim_K A/yA.$$

Now set $r = \dim_K A/yA$ and let

$$P(\times y) = d_1 \oplus d_2 \oplus \dots \oplus d_r \quad \text{and} \quad P(\times Y) = d'_1 \oplus d'_2 \oplus \dots \oplus d'_r,$$

where we assume that $d_1 \geq d_2 \geq \dots \geq d_r$ and $d'_1 \geq d'_2 \geq \dots \geq d'_r$. (Both $\times y$ and $\times Y$ have the same number r of blocks.) By way of contradiction assume that $P(\times y) > P(\times Y)$. This means that there exists an integer $k > 0$ such that $d_i = d'_i$ for $i < k$ and $d_k > d'_k$. Put $p = d'_k$ and let $B = K[z]/(z^p)$, where z is a new indeterminate. Compare the numbers of blocks of $P(\times(y+z)) \in \text{End}_K(A[z]/(z^p))$ and $P(\times(Y+z)) \in \text{End}_{K^*}(A^*[z]/(z^p))$. Note that $A[z]/(z^p) \cong A \otimes_K B$ and $A^*[z]/(z^p) \cong A^* \otimes_{K^*} B$. By (2) of Proposition 2, we have

$$\#\text{blocks in } P(\times(y+z)) < \#\text{blocks in } P(\times(Y+z)).$$

But the other inequality $\text{LHS} \geq \text{RHS}$ was proved in ([8] Theorem A). Thus we get a contradiction.

Any element $y \in A$ such that $P(\times y) = P(\times Y)$ will be called a **general element** of A . A general element is automatically an element of $\mathfrak{m} \setminus \mathfrak{m}^2$. It should be noted that when A is graded, a «sufficiently general» linear form will be a general element in this sense.

PROPOSITION 14. *Let A be an Artinian graded K -algebra. Then the following conditions are equivalent.*

- (1) *A has the WLP.*
- (2) *There is a linear form $g \in A$ such that $\dim A/gA$ is equal to the Sperner number of A .*

(3) For a general element $g \in A$, $\dim A/gA$ is equal to the Sperner number of A .

(4) There is a linear form $g \in A$ such that $\text{rank}[\times g : A \rightarrow A]$ is equal to the CoSperner number of A .

(5) For a general element $g \in A$, $\text{rank}[\times g : A \rightarrow A]$ is equal to the CoSperner number of A .

PROOF. Easy.

LEMMA 15. Suppose that $A = \bigoplus_{i=0}^c A_i^c$ has the SLP and $y \in A_1$ is a Lefschetz element. Put $B = A/(0 : y)$ and let \bar{y} be the natural image of y in B . Let $q = c/2$ if c is even and $q = (c + 1)/2$ if c is odd. Then

(1) B has the SLP and \bar{y} is a Lefschetz element of B .

(2) If the Hilbert function of A is h_0, h_1, \dots, h_c , then the Hilbert function of B is equal to the sequence h_0, h_1, \dots, h_c with the term h_q deleted.

PROOF. For (1) the proof of ([7], Theorem 3.8) works verbatim. (2) follows immediately from the proof of ([7], Theorem 3.8).

PROPOSITION 16. Let $A = \bigoplus_{i=0}^c A_i$ be an Artinian graded K algebra with $A_c \neq 0$. Put $n = \dim A$ and $h_i = \dim A_i$. Since $\sum h_i = n$, the positive integers h_i 's give us a partition of the integer n . Now let y be a general linear form of A , and let $P(\times y)$ be the partition of n for the Jordan decomposition of $\times y$. Then the following conditions are equivalent.

(1) A has the SLP.

(2) The sequence $\{h_0, \dots, h_c\}$ is symmetric and the dual partition of $P(\times y)$ is $h_0 \oplus h_1 \oplus \dots \oplus h_c$.

PROOF. Assume (1). As we pointed out in Remark 3, the Hilbert function of A is symmetric. Put $f_i = \dim(0 : y^{i+1})/(0 : y^i)$. First note that the dual partition of $P(\times y)$ is the set of positive integers

$$\{f_i \mid i = 0, 1, \dots, c\}$$

with multiplicity counted. We would like to show that if the sequence h_0, h_1, \dots, h_c is arranged in the decreasing order it coincides with the sequence f_0, f_1, \dots, f_c . Since A has the SLP it has the WLP. Hence f_0 is the Sperner number of A , which is the maximal integer

of $\{h_i\}$. Using Lemma 15 proof is complete by induction on c . We have proved that (1) implies (2).

Now assume (2). Put $B = A/(0 : y)$ and $\bar{y} = y \bmod (0 : y)$. First we induct on c to show that B has the SLP with a Lefschetz element \bar{y} . For this we have to show the following two things:

(i) The Hilbert function of B is equal to

$$h_0, h_1, \dots, h_{q-1}, h_{q+1}, \dots, h_c,$$

where $q = \lfloor c/2 \rfloor$.

(ii) If the above sequence is considered as a partition of $n - h_q$, it is the dual of $P(\bar{y})$.

Note that $P(\times \bar{y})$ is obtained from $P(\times y)$ by subtracting 1 from each term. On the other hand a highest term in the dual partition of $P(\times y)$ is the number of terms of $P(\times y)$, i.e., the number of blocks in the Jordan decomposition of $\times y$. The second statement of (2) says that this is equal to the Sperner number of A . Hence by Proposition 2 A has the WLP. Thus we may identify B_j with A_j for $0 \leq j < q$ and B_j with yA_j for $j \geq q$. (Note this is possible because we are assuming that h_0, \dots, h_c is symmetric.) It follows that the Hilbert function of B is the sequence h_0, h_1, \dots, h_c with the term h_q deleted, where $q = \lfloor c/2 \rfloor$, and that it is the dual of $P(\bar{y})$. Now the induction hypothesis says that B has the SLP with a Lefschetz element \bar{y} . Let i be any integer such that $0 \leq i < c/2$ if c is even and $0 \leq i \leq (c-1)/2$ if c is odd. We would like to show that the map $\times y^{c-2i} : A_i \rightarrow A_{c-i}$ is bijective. Consider the diagram:

$$\begin{array}{ccc} [A/(0 : y)]_i & \xrightarrow{\times y^{c-1-2i}} & [A/(0 : y)]_{c-1-i} \xrightarrow{\times y} A_{c-i} \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{\times y^{c-1-2i}} & B_{c-1-i} \end{array}$$

The vertical maps are bijective by definition of B . The horizontal map indicated $\times y$ is bijective as the map $\times y : A_{c-1-i} \rightarrow A_{c-i}$ is surjective. The horizontal map downstairs is bijective by induction hypothesis. It follows that the composite $[A/(0 : y)]_i \rightarrow A_{c-i}$ upstairs is bijective. This is nothing but the desired bijection $y^{c-2i} : A_i \rightarrow A_{c-i}$ as $(0 : y)_i = 0$. Thus A has the SLP.

COROLLARY 17. *Suppose that a graded K -algebra $A = \bigoplus_{i=0}^c A_i$ has a symmetric Hilbert function. Let y be a general linear element of A .*

Suppose either (1) $A/(0 : y^i)$ has the WLP for all $i = 1, 2, \dots, c$, or (2) A has the WLP and $A/0 : y$ has the SLP. Then A has the SLP.

PROOF. It is easy to see that in either case the second condition of Proposition 16 is satisfied. Hence the SLP of A follows.

The following is very important in the proof of the main theorem.

PROPOSITION 18. *Let A be an Artinian graded ring with a symmetric Hilbert function, and let z be an algebraically independent element over A . Then the following conditions are equivalent.*

- (1) $A[z]/(z^n)$ has the weak Lefschetz property for all $n \geq 0$.
- (2) A has the strong Lefschetz property.

PROOF. Suppose that A has SLP. $A[z]/(z^n)$ is the tensor product of A and $K[z]/(z^n)$. Since the SLP is preserved by tensor product, it has SLP, hence WLP. We have proved that (2) implies (1).

We prove (1) implies (2). By way of contradiction assume (1) and assume that A does not have SLP. Let $y \in A$ be a general element and let

$$J := P(\times y) = d_1 \oplus d_2 \oplus \dots \oplus d_r$$

be the Jordan decomposition for $\times y$. Since we are assuming that A has the WLP, it follows that r is the Sperner number of A (Proposition 14). Let h_0, h_1, \dots, h_c be the Hilbert function of A and let $J' := d'_1 \oplus d'_2 \oplus \dots \oplus d'_r$ be the dual partition of $h_0 \oplus h_1 \oplus \dots \oplus h_c$. Here note that the number of terms of the partition J' is also r , as well as that of J , as it is a highest term of h_0, \dots, h_c . Proposition 16 implies that $J' \neq J$, but in fact it implies $J' > J$. Suppose that $d_i = d'_i$ for $i = 1, 2, \dots, k-1$ and $d'_k > d_k$. Put $p = d_k$. Then as in the proof of Proposition 13, one sees that the $A[z]/(z^p)$ does not have the WLP, contradicting the assumption (1).

The WLP and the SLP are different even if algebras are assumed to be Gorenstein (cf. [3], Example 4.3) but the following Proposition tells us that they are the same for complete intersections if they are considered all together in all dimensions.

PROPOSITION 19. *Let C be the family of all Artinian complete intersections with standard grading over K . Then the following conditions are equivalent.*

- (1) *All members of C have the weak Lefschetz property.*
- (2) *All members of C have the strong Lefschetz property.*

PROOF. (2) implies (1) because SLP implies WLP. Suppose that there is a member A of C which does not have the strong Lefschetz property. Then using the same idea as in the proof of Proposition 13, one can prove that there exists some integer j such that $A' = A[z]/(z^j)$ does not have the weak Lefschetz property. Since A' is a member of C , this shows that (1) implies (2).

REMARK 20. All complete intersections in codimension two have the SLP and all complete intersections in codimension three have the WLP ([3], Theorem 2.3 and Proposition 4.4). Proof of Proposition 19 tells us that if there exists a complete intersection in codimension three which does not have the SLP then there exists a complete intersection in codimension four which does not have the WLP. So far we know of no examples of complete intersections without SLP or WLP. It seems natural to conjecture that all complete intersections in all codimensions have the SLP.

3. The rank of certain nilpotent matrices.

In this section matrices may be considered over a field K of arbitrary characteristic. Let $M = (m_{ij})$ be an $s \times s$ matrix with the following properties.

1. $m_{ij} = 0$ if $j \leq i$ (so M is upper triangular and nilpotent).
2. $m_{ij} \in \{0, p, q\}$.
3. Each row of M has at most a p and a q .
4. Each column of M has at most a p and a q .
5. If $m_{ij} = p$ then $j = i + 1$.

Let X be the matrix obtained from M by setting $p = 1$ and $q = 0$ and Y by $p = 0, q = 1$. So $M = pX + qY$. Call $X' = (x'_{ij})$ a **deformation** of $X = (x_{ij})$ if it is an upper triangular nilpotent matrix and $x'_{ij} = x_{ij}$ for (i, j) such that $j = i + 1$.

PROPOSITION 21. *As described above let M be a matrix having the properties 1-5 and let X, Y be such that $M = pX + qY$. Let X' be any de-*

formation of X . Then

$$\text{rank}(X' + \lambda Y) \geq \text{rank}(X + Y) = \text{rank}(M)$$

for an algebraically independent element λ over K .

PROOF. will be preceded by two lemmas.

LEMMA 22. *By a permutation of rows and a permutation of columns of M , the matrix decomposes into blocks of the following four types of matrices and a zero block.*

$$\begin{array}{cc}
 \text{(i)} \begin{pmatrix} p & q & 0 & \cdots & \cdots & 0 \\ 0 & p & q & 0 & \cdots & 0 \\ 0 & 0 & p & q & \cdots & 0 \\ & & & \ddots & \ddots & \\ \vdots & & \cdots & p & q & \\ 0 & & \cdots & 0 & p & \end{pmatrix} & \text{(ii)} \begin{pmatrix} q & 0 & 0 & \cdots & \cdots & 0 \\ p & q & 0 & 0 & \cdots & 0 \\ 0 & p & q & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \\ \vdots & & \cdots & q & 0 & \\ 0 & & \cdots & p & q & \end{pmatrix} \\
 \\
 \text{(iii)} \begin{pmatrix} p & q & 0 & \cdots & \cdots & 0 \\ 0 & p & q & 0 & \cdots & 0 \\ 0 & 0 & p & q & \cdots & 0 \\ & & & \ddots & \ddots & \\ \vdots & & \cdots & p & q & \end{pmatrix} & \text{(iv)} \begin{pmatrix} q & 0 & 0 & \cdots & \cdots & 0 \\ p & q & 0 & 0 & \cdots & 0 \\ 0 & p & q & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & p & q & 0 & \\ \vdots & & \cdots & p & q & \\ 0 & 0 & & 0 & p & \end{pmatrix}
 \end{array}$$

PROOF. If we set $p = q = 1$ then M is a (01)-matrix, and we may regard it as the incidence matrix of a bipartite graph. From this view point we label the rows of M by the set $\{a_i \mid 1 \leq i \leq s\}$ and columns $\{b_j \mid 1 \leq j \leq \leq s\}$. The sets $\{a_i\}$ and $\{b_j\}$ are regarded as two different objects. Let us write $b_j \xrightarrow{p} a_i$ if p is the (ij) th entry of M and $a_j \xrightarrow{q} b_i$ if q is the (ji) th entry. A (pq) -path is a finite sequence of either one of the following:

$$\begin{cases} b_j \xrightarrow{p} a_i \xrightarrow{q} b_{j'} \xrightarrow{p} a_{i'} \xrightarrow{q} b_{j''} \xrightarrow{p} \dots \\ a_i \xrightarrow{q} b_j \xrightarrow{p} a_{i'} \xrightarrow{q} b_{j'} \xrightarrow{p} a_{i''} \xrightarrow{q} \dots \end{cases}$$

By the way p 's and q 's are arranged, it follows that $i < i' < i'' \dots$ and $j < j' < j'' < \dots$. Thus they are finite sequences. Either may end with a p

or a q . Since each row and each column of M contains at most a p and a q , it is easy to see that the bipartite graph defined by M decomposes uniquely into such maximal (pq) -paths. If it starts with a p and ends with a p it gives us a block of type (i). Similarly the other three kinds of paths correspond to matrices of types (ii)-(iv).

LEMMA 23. *Let M be the same as above. Let r the rank of M . Then the non-zero entries of the matrix $\wedge^r M$ are monomials in p and q of degree r with coefficients 1 or -1 . Let Q be the monomial in p, q that occurs in $\wedge^r M$ with the highest degree in q . Then Q occurs only at one place in the matrix $\wedge^r M$.*

PROOF. We need proof only for the last assertion. Decompose M into blocks of the four types given in Lemma 22 and a zero block. We may disregard the zero block so we assume that each row and column has at least a p or a q . The rank of the block of type (i) and (ii) is the size of the block. The rank of the block of type (iii) is the number of rows, and that of type (iv) is the number of the columns. The ranks of the blocks sum up to the rank of M . To prove the assertion we have to show that a choice of rows and columns as many as r (the rank of \mathcal{M}) is uniquely determined if the determinant of the submatrix is to be non-zero and if it is to be the highest degree in q . As in the proof of Lemma 22, regard M as the incidence matrix between the two sets of vertices $\{a_i \mid 1 \leq i \leq s\} \sqcup \{b_j \mid 1 \leq j \leq s\}$. Define the subset $R \subset \{a_i\}$ by claiming that

$$x \in R \Leftrightarrow x \text{ occurs as the last row of a block of type (iv)}$$

and similarly, the subset $C \subset \{b_i\}$ by

$$y \in C \Leftrightarrow y \text{ occurs as the first column of a block of type (iii)}.$$

Let R' and C' be the complements of R and C respectively. Then it is easy to see that the submatrix of M with rows R' and columns C' is the only possibility for the desired property.

PROOF OF PROPOSITION 21. The second equality is easy. We prove the first inequality. Recall that $M = pX + qY$ so its rank is the same if we set $p = 1$. Let R', C' be the subsets of rows and columns that were defined in the proof of Lemma 23. Let Z the submatrix of $X + \lambda Y$ with rows and columns R' and C' . Then the determinant of Z is a polynomial in λ whose head term is the same as the corresponding minor of $X + \lambda Y$. Thus the proof is complete.

4. – The commutator algebra of Artinian algebras in certain endomorphism rings.

In this section we denote by $\mathbf{M}(n)$ the $n \times n$ full matrix ring over K . For a finite dimensional vector space V we may identify $\text{End}(V)$ with $\mathbf{M}(n)$ if a basis of V is specified.

Let A be a graded K -algebra. We denote the regular representation of A by

$$\times : A \rightarrow \text{End}(A).$$

In other words, the notation $\times a$ for $a \in A$ means that it is the multiplication map $\times a : A \rightarrow A$.

LEMMA 24. *Let $A = \bigoplus_{i=0}^c A_i$ be an Artinian K -algebra with the SLP and let $y \in A_1$ be a general linear element. (We assume $A_c \neq 0$.) Let $\times : A \rightarrow \text{End}(A)$ be the regular representation of A . Then there is a homogeneous K -basis of A such that $\times y$ is put in the Jordan second canonical form as defined in Section 2, and at the same time $\times a$ is an upper triangular form for all $a \in A$. In any choice of such a basis, the first of the basis elements is $1 \in A_0$ and the last is a homogeneous element of A of the highest degree (up to a constant multiple). Furthermore $\times a$ is constant on the diagonal for all $a \in A$.*

PROOF. Let $B = \{e_i\}$ be a basis on which $\times y$ is put in the Jordan second canonical form. Let \widehat{T} be the Young tableau that is associated with B as explained in Section 2. (We may identify a box of \widehat{T} and the integer of the box and the element of B with that index.) Recall that the boxes of \widehat{T} are numbered in such a way that box i is next to box j on the left if and only if $ye_i = e_j$. Also recall that $(0 : y)$ is generated by the basis elements in the last column of \widehat{T} , and more generally $0 : y^{i+1}/0 : y^i$ is spanned by the basis elements in the $(c - i)$ th column of \widehat{T} . Since for all $x \in A$ the multiplication $\times x : A \rightarrow A$ preserves the flag

$$V = 0 : y^{c+1} \supset 0 : y^c \supset \dots \supset 0 : y^2 \supset 0 : y \supset 0 : 1 = 0,$$

the representation $\times : A \rightarrow \text{End}(A)$ is block upper-triangular. Now it suffices to prove that the induced representation on each diagonal block: $A \rightarrow \text{End}((0 : y^{t+1})/(0 : y^t))$ is upper triangular. By induction on c it suffices to prove that $A \rightarrow \text{End}(0 : y)$ is upper triangular. In Section 2 we showed a procedure to decompose $\times y$ into a Jordan canonical form

and we showed how it was converted to the Jordan second canonical form. It shows that if i and j are in the last column, then

$$i < j \Rightarrow \deg e_i \leq \deg e_j.$$

This means that if e_i and e_j are basis elements of $(0 : y)$ such that $\deg e_i < \deg e_j$ then $i < j$. Since the multiplication by element of A raises the degree unless it is a unit, we see that $\times : A \rightarrow \text{End}(0 : y)$ is upper triangular. We have proved the first part. The second part also follows immediately from the construction of \widehat{T} . Finally $\times a$ is constant on the diagonal as $(a - \alpha)$ is nilpotent for some $\alpha \in K$.

For any subset \mathfrak{A} of $\mathbf{M}(n)$ we define

$$\text{Comm}(\mathfrak{A}) = \{ \varphi \in \mathbf{M}(n) \mid \varphi\chi = \chi\varphi \text{ for all } \chi \in \mathfrak{A} \}$$

and call it the commutator algebra of \mathfrak{A} . (It is an associative algebra with identity.) The same notation $\text{Comm}(\mathfrak{A})$ will be applied to for a subset \mathfrak{A} of $\text{End}(A)$.

LEMMA 25. *Let (A, \mathfrak{m}) be an Artinian K -algebra. Then the commutator algebra of $\times A := \{ \times a \mid a \in A \}$ coincides with $\times A$ itself. More precisely we have*

$$\{ \varphi \in \text{End}(A) \mid \varphi(\times z) = (\times z)\varphi \ \forall z \in A \} = \{ \times a \mid a \in A \}.$$

PROOF. Note that if $\varphi \in \text{End}(A)$ commutes with $\times z$ for all $z \in A$ it is determined by $\varphi(1)$, from which the assertion follows.

Let \mathfrak{A} be a subset of $\mathbf{M}(n)$. After H. Weyl [9] we define the two subsets $s\mathfrak{A}$, \mathfrak{A}_s of $\mathbf{M}(sn)$ as follows:

$$s\mathfrak{A} = \left\{ \begin{pmatrix} M & & & 0 \\ & M & & \\ & & \ddots & \\ 0 & & & M \end{pmatrix} \mid M \in \mathfrak{A} \right\},$$

$$\mathfrak{A}_s = \left\{ \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1s} \\ M_{21} & M_{22} & \dots & M_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ M_{s1} & M_{s2} & \dots & M_{ss} \end{pmatrix} \mid M_{ij} \in \mathfrak{A} \right\}.$$

LEMMA 26. *Let (A, \mathfrak{m}) be an Artinian K -algebra with $\dim A = n$.*

Let V be a finite vector space over K of dimension s . Fix some bases for A , V and $V \otimes A$ and identify $\text{End}(A) = \mathbf{M}(n)$ and $\text{End}(V \otimes A) = \mathbf{M}(sn)$. Put $\mathfrak{A} = \times A$. Then the commutator algebra of $s\mathfrak{A}$ is \mathfrak{A}_s .

PROOF. If $s = 1$, then we have already proved this in Lemma 25. Let $\mathfrak{N} \in \mathbf{M}(sn)$ and let $\mathfrak{N} = (M^{(ij)})$ be the block decomposition such that $M^{(ij)} \in \mathbf{M}(n)$ and $1 \leq i, j \leq s$. An element of $s\mathfrak{A}$ may be written as $\text{diag}(\underbrace{\times a, \dots, \times a}_s)$ for some $a \in A$. Thus if M is in the commutator of $s\mathfrak{A}$ it implies that $M^{(ij)}(\times a) = (\times a)M^{(ij)}$ for all $a \in A$ and for all pairs (ij) . In other words each block $M^{(ij)}$ is in the commutator of \mathfrak{A} . By the previous lemma $M^{(ij)} \in \mathfrak{A}$. This shows that $\text{Comm}(s\mathfrak{A}) \subset \mathfrak{A}_s$. The other inclusion is obvious. Now the proof is complete.

A matrix $\mathfrak{N} \in \mathbf{M}(sn)$ may be regarded as a block matrix in two ways. Namely,

- (1) $\mathfrak{N} = (M_{ij})$, where $M_{ij} \in \mathbf{M}(s)$ and $1 \leq i, j \leq n$ and
- (2) $\mathfrak{N} = (N_{ij})$, where $N_{ij} \in \mathbf{M}(n)$ and $1 \leq i, j \leq s$.

In either case we write $\mathfrak{N} = (m_{ij}^{(lk)})$, by which we mean that the element $m_{ij}^{(lk)}$ is the (ij) entry of the block (kl) of \mathfrak{N} . In this notation, it will tacitly be assumed either

- (1) $1 \leq k, l \leq s$ and $1 \leq i, j \leq n$ or
- (2) $1 \leq k, l \leq n$ and $1 \leq i, j \leq s$.

Let $\mathcal{F}: \mathbf{M}(n) \rightarrow \mathbf{M}(sn)$ be the s -copy of M . Namely, \mathcal{F} is the homomorphism defined by $\mathcal{F}(M) = \text{diag}(\underbrace{M, \dots, M}_s)$. We are regarding $\mathcal{F}(M)$ as consisting of s^2 blocks where each block is an $n \times n$ matrix. Let $M = (m_{ij}) \in \mathbf{M}(n)$. Define $\mathcal{G}(M)$ to be the $sn \times sn$ matrix obtained from M by replacing the entry m_{ij} by the matrix $m_{ij}E$ where E is the $s \times s$ identity. Thus \mathcal{G} gives us a homomorphism $\mathcal{G}: \mathbf{M}(n) \rightarrow \mathbf{M}(sn)$. We call $\mathcal{G}(M)$ the s -inflation of M . We are regarding $\mathcal{G}(M)$ as consisting of n^2 blocks where each block is an $s \times s$ matrix.

Suppose now $\mathfrak{N} = (m_{ij}^{(kl)}) \in \mathbf{M}(sn)$. We define the «hat operation»

$$\widehat{} : \mathbf{M}(sn) \rightarrow \mathbf{M}(sn)$$

by $\widehat{\mathfrak{N}} = (m_{kl}^{(ij)})$. The hat operation is nothing but a certain conjugation.

Now we can state our key lemma.

LEMMA 27 (Key Lemma). *We use the same notation as above. Let $\mathfrak{M} \in \mathbf{M}(sn)$. Then*

(1) *If each block of \mathfrak{M} is an upper triangular matrix then $\widehat{\mathfrak{M}}$ is a block upper triangular matrix. (See Figure 2.)*

(2) *If each block of \mathfrak{M} is constant on the diagonal, then $\widehat{\mathfrak{M}}$ is constant on the diagonal blocks. (I.e., all diagonal blocks of $\widehat{\mathfrak{M}}$ are the same matrix.)*

(3) *If $M \in \mathbf{M}(n)$, then $\widehat{\mathcal{F}(M)} = \mathcal{G}(M)$.*

PROOF. (1) Write $\mathfrak{M} = (m_{ij}^{(kl)})$. Suppose that $m_{ij}^{(kl)} = 0$ for any k, l , if $j < i$. This means that (ij) -block of $\widehat{M} = (m_{kl}^{(ij)})$ is the zero matrix, if $j < i$.

(2) Write $\mathfrak{M} = (m_{ij}^{(kl)})$. Suppose that $m_{ii}^{(kl)} = m_{11}^{(kl)}$ for all i , if k and l are fixed. This implies that the i -th diagonal block, for any i , of $\widehat{M} = (m_{kl}^{(ij)})$ is the same as the first diagonal block.

(3) Write $\mathcal{F}(M) = (m_{ij}^{(kl)})$. Then by the definition of \mathcal{F} we have that

$$m_{ij}^{(kl)} = \begin{cases} 0 & \text{if } k \neq l, \\ m_{ij}^{(11)} & \text{if } k = l. \end{cases}$$

This means that each block of $\widehat{\mathcal{F}(M)} = (m_{kl}^{(ij)})$ is a diagonal matrix and moreover each diagonal entry is the same as the (11)-st entry of the block. This proves the assertion.

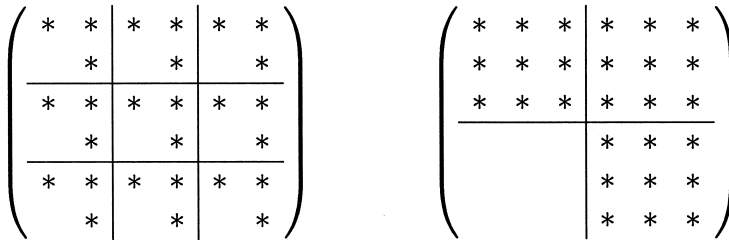


Fig. 2. – Block-wise upper triangular vs Block upper triangular.

5. – Main theorem

THEOREM 28. *Let (A, \mathfrak{m}) be a graded Artinian K -algebra and let B be a finite free algebra over A such that the algebra map $A \rightarrow B$ pre-*

serves grading. Assume that both A and $B/\mathfrak{m}B$ have the SLP. Then B has the SLP.

PROOF. Put $C = B/\mathfrak{m}B$. Let $\bar{b} \in C$ denote the image of $b \in B$ by the natural surjection. Let $z \in B$ be a linear element such that \bar{z} is a Lefschetz element of C . Let $s = \dim C$ and let

$$\langle \bar{e}_k \mid k = 1, \dots, s \rangle$$

be a basis for C such that $\times \bar{z} \in \text{End}(C)$ is represented as the Jordan (first) canonical form on this basis. Let e_k be a preimage of \bar{e}_k and let V the vector space spanned by e_k 's. Since B is free over A these elements e_k 's are a free basis of B over A (Nakayama's Lemma). Thus we may write

$$B = Ae_1 + Ae_2 + \dots + Ae_s$$

as a direct sum of A modules. This gives us a bijection of vector spaces:

$$V \otimes A \rightarrow B, \quad v \otimes a \mapsto va$$

and we may identify B with the tensor space $V \otimes A$. Let $y \in A$ be a general linear form of A and let

$$\langle f_i \mid i = 1, \dots, n \rangle$$

be a basis for A described in Lemma 24.

Now the set $\{e_k f_j \mid 1 \leq k \leq s, 1 \leq j \leq n\}$ is a basis for B . With these bases for A and B we make identification $\text{End}(A) = \mathbf{M}(n)$ and $\text{End}(B) = \mathbf{M}(sn)$. Consider the diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\times} & \text{End}(A) & \xrightarrow{\mathcal{F}} & \text{End}(B) \\ & & \downarrow & & \downarrow \\ & & \mathbf{M}(n) & \xrightarrow{\mathcal{F}} & \mathbf{M}(sn) \end{array}$$

The vertical maps are identifications made above. The symbol $\times : A \rightarrow \text{End}(A)$ denotes the regular representation of A . The map $\mathcal{F} : \text{End}(A) \rightarrow \text{End}(B)$ is defined by $\mathcal{F}(\phi) = I_V \otimes \phi$ for $\phi \in \text{End}(A)$, where I_V denotes the identity map of V . Note that the same map $\mathcal{F} : \mathbf{M}(n) \rightarrow \mathbf{M}(sn)$, in terms of matrices, is the s -copy of matrices of $\mathbf{M}(n)$ as defined earlier. It should be noted that for $a \in A$, the homomorphism $\mathcal{F}(\times a) \in \text{End}(B)$ is nothing but the multiplication $\times a : B \rightarrow B$ induced by $\times a : A \rightarrow A$.

Notice that $\times z \in \text{End}(B)$ commutes with all elements of $\mathcal{F}(\times A)$, which is, in the notation of Lemma 25, $s(\times A)$. By Lemma 26 we see that the endomorphism $\times z$ is represented by an $(s \text{ block})$ by $(s \text{ block})$ matrix where each block is an upper triangular matrix of size n . Consider $\widehat{\times z}$ as defined in the Section 4. Then Key Lemma 27 says that $\widehat{\times z}$ is an $(n \text{ block}) \times (n \text{ block})$ matrix with the same diagonal blocks of size s . It is easy to see that each diagonal block is the matrix for $\times \bar{z}: C \rightarrow C$ represented on the basis of C

$$\langle \bar{e}_1, \dots, \bar{e}_s \rangle$$

as we chose first. Consequently it is a Jordan first canonical form. Let $\mathcal{X} \in \mathbf{M}(sn)$ be the matrix consisting of the same diagonal blocks as those of $\widehat{\times z}$ and with zero blocks off diagonal. Then $\widehat{\times z}$ is a deformation of \mathcal{X} . Put $\mathcal{Y} = \mathcal{F}(\times y)$. Also by Key Lemma 27 we see that \mathcal{Y} is the s -inflation (as defined in a paragraph preceding Lemma 27) of $\times y \in \text{End}(A)$. It is easy to see that Lemma 21 can be applied to this situation and we get the conclusion:

$$\text{rank}[\times(z + \lambda y): B \rightarrow B] \geq \text{rank}[\mathcal{X} + \mathcal{Y} \in \mathbf{M}(sn)]$$

for some λ . Now notice that $\mathcal{X} + \mathcal{Y} \in \mathbf{M}(sn)$ is, in fact, the matrix for the multiplication $\times(\bar{z} \otimes 1 + 1 \otimes y) \in \text{End}(C \otimes A)$. Put $B' = C \otimes A$. Then B' and B have the same Hilbert function and B' has the strong Lefschetz property. This means that the rank $(z + \lambda y): B \rightarrow B$ has reached the CoSperner number of B and thus it follows that B has the weak Lefschetz property. But the same argument can be used to prove that $B[t]/(t^m)$ has the weak Lefschetz property for any $m > 0$, because we have the flat extension $A \rightarrow B[t]/(t^m)$ with the fiber $C[t]/(t^m)$ which also has the strong Lefschetz property. Now the strong Lefschetz property of B follows from Proposition 18.

COROLLARY 29. (1) *Let (A, \mathfrak{m}) be a graded K -algebra with the strong Lefschetz property. Let $B = A[z]/(h)$, where h is a monic homogeneous polynomial in the variable z with coefficients in A . Then B has the strong Lefschetz property.*

(2) *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring and let $I \subset R$ be a height n homogeneous complete intersection ideal. Suppose that a power of a linear element z of R can be a member of a minimal generating set for I . Put $B = R/I$. If $R/I + (z)$ has the strong Lefschetz property, so does B .*

PROOF. (1) Main theorem applies since the ring B is a flat extension of A and $B/\mathfrak{m}B \cong K[z]/(z^r)$ for some r .

(2) Suppose that z^r is one of the members of a minimal generating set for I . Put $A = K[z]/(z^r)$. One notices that $I : z^s = I + (z^{r-s})$, which shows that B is a flat extension of A , because the injection $0 \rightarrow (z^s) \rightarrow A$ remains exact by making the tensor with B . Now one may apply Main Theorem.

6. – Some applications.

PROPOSITION 30. *Let R be the polynomial ring in three variables over K . Let I be a homogeneous complete intersection ideal of R . If a power of a linear form can be an element of a minimal generating set of I , then R/I has the SLP.*

PROOF. Let z be a linear form such that $I = (f, g, z^n)$ for some n . Then $R/I + (z)$ has the SLP by [3] Proposition 4.4. Hence the SLP of R/I follows from Corollary 29 (2).

PROPOSITION 31. *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring and let $s_i(\underline{x})$ be the basic symmetric polynomial of degree i , for $i = 1, 2, \dots, n$. Let r be a positive integer. Put*

$$f_i = s_i(x_1^r, x_2^r, \dots, x_n^r), \quad \text{for } i = 1, 2, \dots, r,$$

and $I = (f_1, \dots, f_r)$. Then R/I has the strong Lefschetz property.

PROOF. Let X be an indeterminate. Put $z = x_n$. z^r satisfies the equation

$$X^n - f_1 X^{n-1} + \dots + \pm f_n = 0$$

which shows that $I = (f_1, \dots, f_{n-1}, z^r)$. We induct on n . The induction hypothesis means that $R/(z)$ has the SLP. So the SLP of R/I follows from Corollary 29 (2).

REMARK 32. In the above corollary it is well known that the elements

$$f_1, f_2, \dots, f_n$$

are an integrity basis of the ring of invariants by a certain pseudo-reflection group acting on the polynomial ring $K[x_1, \dots, x_n]$ by linear transformation of the variables. Moreover if $r = 1$, the ring R/I appears as the co-

homology ring of the algebraic variety G/B where $G = GL(n)$ is the general linear group and B a Borel subgroup. So G/B is the complete flag variety. In this case the ring $K[f_1, \dots, f_n]$ is the ring of invariants by the Weyl group of type A_{n-1} . If $r = 2$, it is the cohomology ring of G/B where $G = SO(2n + 1)$ is the special orthogonal group with B a Borel subgroup. The ring $K[f_1, \dots, f_n]$ is the ring of invariants by the Weyl group of type B_n . Since G/B is non-singular the Hard Lefschetz theorem states the same thing as the SLP on these rings. For detail we refer the reader to [4] and [6].

PROPOSITION 33. *Let R be the polynomial ring over K in n variables and let $I \subset R$ be a height n complete intersection ideal. Let $\text{In}(I)$ denote the ideal generated by the head terms of elements in I with respect to the graded reverse lexicographic order. If $\text{In}(I)$ is a complete intersection, then R/I has the strong Lefschetz property.*

PROOF. That $\text{In}(I)$ is a complete intersection means that it is generated by powers of the variables. Let z be the last variable of R and let r be the least integer such that $z^r \in \text{In}(I)$. Then it follows that $z^r \in I$ and it can be an element of a minimal generating set of I . Moreover, since $\text{In}(I + (z)) = \text{In}(I) + (z)$ ([1] Proposition 15.12), we may use Corollary 29 (2) to prove the SLP of R/I by induction on the number of variables.

REMARK 34. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring and let I be the ideal generated by the set of basic symmetric polynomials in

$$x_1^r, x_2^r, \dots, x_n^r.$$

Then $\text{In}(I)$ is a complete intersection. In fact we have

$$\text{In}(I) = (x_1^r, x_2^{2r}, \dots, x_n^{nr}).$$

To prove this, first note that

$$I = (f_1, f_2, \dots, f_{n-1}, x_n^{nr}),$$

as was shown in the proof of Proposition 31. Then induct on n noticing the equality

$$\text{In}(I) = \text{In}(f_1, f_2, \dots, f_{n-1}) + (x_n^{nr}).$$

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