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Small Almost free Modules with Prescribed Topological Endomorphism Rings (*).

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ABSTRACT - We will realize certain topological rings as endomorphism rings of $\aleph_1$-free abelian groups of cardinality $\aleph_1$ where the isomorphism is also a homeomorphism relating the topology on the given ring to the finite topology on the endomorphism ring. This way we also find $\aleph_1$-free abelian groups of cardinality $\aleph_1$ such that any non-trivial summand is a proper direct sum of an infinite number of summands. This answers a problem raised by the authors in [7, p. 447, (I)], (saying that $\aleph_1 \in \mathrm{vat}(R)$ in the cotorsion-free case). The fact that the size of the continuum could and in particular universes of set theory will be much larger then $\aleph_1$ causes difficulties in constructing pathological abelian groups G of size $\aleph_1$ in «ordinary» set theory using just ZFC: There are less possibilities to prevent potential, unwanted endomorphisms of G not to become members of $\mathrm{End}\, G$. Thus additional combinatorial arguments are needed. They come from [15], were improved for this paper, and are now ready for applications for other algebraic aspects. The results are formulated for modules over a large class of commutative rings.

1. Introduction.

First we would like to discuss the algebraic setting of this paper. Let R be a commutative ring (with 1) of cardinal $\leq \lambda$, and let S be a countable multiplicatively closed set of non-zero-divisors in R such that R is *S-re-*

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duced in the sense that $\bigcap_{s \in S} sR = 0$. List the elements of S in a sequence s_n $(n \in \omega)$, and write

$$q_0 = 1, \quad \text{and} \quad q_{n+1} = s_n q_n^2 \quad (n < \omega). \tag{1.1}$$

For a free R-module F, the submodules $q_n F$ $(n < \omega)$ constitute a basis of neighborhoods of 0 in the (Hausdorff) *S-topology* on F; the corresponding *S-completion* $\widehat{F}$ contains F as an *S-pure* submodule, in the sense that $s\widehat{F} \cap F = sF$ for all $s \in S$.

The endomorphism algebra $\operatorname{End}_R M$ of an R-module M carries the natural finite topology fin, which is generated by the annihilators $\operatorname{Ann}_{\operatorname{End} M} E$ of all finite subsets E of M. (The finite subsets may be restricted to singletons, in which case the annihilators constitute a prebasis for fin at 0.) It follows that $(\operatorname{End}_R M, \text{fin})$ is a Hausdorff, complete topological R-algebra, see [13, p. 221, Theorem 107.1] This suggest immediately to consider R-algebras A with 1 and endowed with a similar topology.

Let A be complete and Hausdorff in a topology admitting a basis of neighborhoods of 0 consisting of a set $\mathfrak{N}$ of right ideals N such that the quotients A/N are free as R-modules. We impose the cardinality restrictions that

$$|\mathfrak{N}| \leqslant \lambda \quad \text{and} \quad \sup_{N \in \mathfrak{N}} |A/N| < \lambda, \tag{1.2}$$

and define

$$\kappa = \Big(\sup_{N \in \mathfrak{N}} |A/N|\Big)^+, \tag{1.3}$$

where the plus sign denotes the cardinal successor. Then κ is a regular cardinal, and

$$\aleph_1 \leqslant \kappa \leqslant \lambda \leqslant 2^{\aleph_0}. \tag{1.4}$$

The cardinal λ can be arbitrary, if set theory permits! Our favored example however is $\lambda = \aleph_1$. The other cardinal restrictions come from our aim to produce R-modules of size close to $\aleph_1$ with the prescribed topological endomorphism ring. For larger cardinals, in particular for any cardinal λ of the form $\lambda = \lambda^{\aleph_0}$ this was established in [7, p. 465, Corollary 6.4]. Thus we can view our restriction to $\aleph_1$ as a particular way to avoid the cardinal restriction $\lambda = \lambda^{\aleph_0}$, here for $\lambda = \aleph_1$.

An obvious first attempt to find the cardinal $\aleph_1 \in \mathrm{vat}(A)$, the spectrum of cardinals λ, for which a (suitable) algebra can be expressed as $A = \mathrm{End}_R M$ with $|M| = \lambda$ (see [7, p. 447]), is replacing the arguments using the Black Box by a different combinatorial method, because the latter requires that $\lambda = \lambda^{\aleph_0}$. The Black Box prediction principle was used in [7] and relatives are discussed in Göbel [14].

However, it is important to note that there are basic difficulties for $\lambda < \lambda^{\aleph_0}$ to become a member of $\mathrm{vat}(A)$, as shown recently in Göbel, Shelah [16, pp. 240-243, Section 3, The non existence of realization theorems]: For particular classes of modules, like for abelian p-groups, or separable torsion-free abelian groups, we have $\aleph_1 \notin \mathrm{vat}(R)$ for any ring R. This motivated our aim to consider the class of $\aleph_1$-free modules of cardinality $\aleph_1$, which is still very close to separable, torsion-free abelian groups. Recall that modules are $\aleph_1$-free, if all countably generated submodules are free.

In [15] we derived a new combinatorial method, which allows to predict endomorphisms of a module (under construction) of cardinality $\aleph_1$ by a list of candidates which has only $\aleph_1$ members. We may assume (w.l.o.g) that for each unwanted member of this list a new element is added to the module under construction which prevents that member to become an endomorphism of the final module. Thus the length of this list of prediction should not pass beyond $\aleph_1$.

Also note that by an easy counting argument our prediction of endomorphisms must be restricted to their action on only finite subsets or singletons; otherwise the list would have length at least $2^{\aleph_0}$ which often is larger then $\aleph_1$. Therefore it is not clear at the beginning if the given predictions, which provides only very little information about the actual endomorphism, are good enough to lead to our task, prescribing A as endomorphism ring and to find out that $\aleph_1 \in \mathrm{vat}(A)$.

In [15] it was shown that any countable R-algebra with free R-module structure is the endomorphism algebra of a suitable R-module of size $\aleph_1$ which is $\aleph_1$-free. Thus the existence of various pathological modules of this kind, in particular the existence of indecomposable modules of cardinality $\aleph_1$ which are $\aleph_1$-free follows. Here we want to strengthen this result and realize any suitable topological ring as such an endomorphism ring algebraically and topologically. On the endomorphism ring $\mathrm{End}\, M$ of an R-module M we take the finite topology fin introduced above. Moreover we want to simplify arguments from [15], which will also shorten the proof in the discrete case. Thus our main result reads as follows.

THEOREM 1.1. *Let A be a topological R-algebra over a ring R as described above, then there exists an $\aleph_1$-free R-module M of cardinality λ with* $\operatorname{End} M \cong A$ *where the isomorphism is an algebraic and topological isomorphism.*

If the topology on A is discrete we obtain the main result in Göbel, Shelah [15] as a special case. Also other earlier results like the existence of an $\aleph_1$–free group M of cardinality $\aleph_1$ with $\operatorname{Hom}(M, \mathbb{Z}) = 0$ follow, see Eda [11] and Shelah [19], [10, 12].

The main difficulty here is to work exclusively in ZFC. The first example of an $\aleph_1$-free module which is not free is the Baer-Specker module R^{ω}, which is the cartesian product of countably many copies of the ring R, known for sixty years; cf. Baer [1] or [13, p. 94]. Assuming CH, this module is an example of an R-module of cardinality $\aleph_1 = 2^{\aleph_0}$. However, it is surely (assuming that R is slender) a finite but not an infinite direct sum of summands $\neq 0$. Under the same set-theoretic assumption of the continuum hypothesis it can be shown that A above can be realized as the endomorphism ring of an $\aleph_1$-free R-module M of cardinality $\aleph_1$, see Shelah [20] for the discrete case with $\operatorname{End} M = \mathbb{Z}$ and Dugas, Göbel [8] for the discrete case and $A = \operatorname{End} M$ with extensions to larger cardinals. Using Shelah's Black Box, the existence of $\aleph_1$-free modules M with $|M| = \lambda^{\aleph_0}$ and $\operatorname{End} M \cong A$ also topologically follows from Corner, Göbel [7].

Endomorphism ring results as discussed have well-known applications using the appropriate also well-known R-algebras A.

If Γ is any abelian semigroup, then we use the R-algebra A_Γ, implicitly discussed in Corner, Göbel [7], and constructed for particular $\Gamma' s$ in [5] with special idempotents, with free R-module structure and $|A_\Gamma| = \max\{|\Gamma|, \aleph_0\}$. If $|\Gamma| \leqslant \lambda$, we may apply the discrete case of the main theorem and find a family of $\aleph_1$-free R-modules $M_\alpha (\alpha \in \Gamma)$ of cardinality $\aleph_1$ such that for $\alpha, \beta \in \Gamma$,

$$M_\alpha \oplus M_\beta \cong M_{\alpha+\beta} \text{ and } M_\alpha \cong M_\beta \text{ if and only if } \alpha = \beta.$$

Observe that this induces all kinds of counterexamples to Kaplansky's test problems for suitable $\Gamma' s$. If we consider the algebra A in Corner [3] (see also [13, Vol. II, p. 145, Theorem 91.5]), then it is easy to see that A_R is free and $|A| = \aleph_0$. The particular idempotents in A and our main theorem provide the existence of an $\aleph_1$-free super-decomposable R-module of cardinality $\aleph_1$. Recall that a group is super-decomposable if any non-trivial summand decomposes into a proper direct sum. Super-decomposable abelian groups of cardinality $2^{\aleph_0}$ were also constructed (without considering endomorphism rings) in Birtz [2] and the existence of non-free but $\aleph_n$-free groups of cardinality $\aleph_n$ $(n \in \omega)$ (without any more specific algebraic properties) follows from Griffith [17].

Our main theorem allows us to strengthen these results and to establish the existence of κ-super-decomposable groups. We say that M is κ-*super-decomposable* if for any cardinal $\varrho < \kappa$ any non-trivial summand of M is a direct sum of ϱ non-trivial summands. The existence of $\aleph_1$-free κ-super-decomposable modules G of cardinality κ (e.g. for $\kappa = \aleph_1$) follows by realizing the algebra given in Corner [6] as a topological endomorphism algebra. In case $\kappa = \aleph_1$ we therefore find R-modules of size $\aleph_1$ such that every non-trivial summand is a direct sum of an **infinite** number of non-trivial summands, which is a property stronger then super-decomposable and thus sometimes called *very decomposable*.

It is fairly straight to replace the module M in the theorem by a family of modules M_X with $X \subseteq \lambda$ such that for all $X, Y \subseteq \lambda$

$$\mathrm{Hom}_R(M_X, M_Y) \cong \begin{cases} A & \text{if } X \subseteq Y \subseteq \lambda \\ 0 & \text{otherwise}. \end{cases} \tag{1.5}$$

2. The construction of modules.

Once and for all fix an uncountable cardinal $\lambda \leq 2^{\aleph_0}$ and adopt the algebraic assumptions explained in the introduction.

Write $T = {}^{\omega >}2$ for the binary tree. We will say that a subtree T' of ${}^{\omega \geq}2$ is *perfect* if for any $n \in \omega$ there is at most one finite branch $f \in T'$ of length n of the form $f = v \cap w$ for branches $v, w \in T'$.

Define the map

$$\sigma : {}^{\omega}2 \to {}^{\omega}2 \quad (v \to v^{\sigma})$$

where

$$v^{\sigma}(m) = \begin{cases} v(n) & \text{if } 2^{n+1} \leq m < 2^{n+2} \text{ and } m = 2^{n+1} + \sum_{i=0}^{n} 2^{n-i} v(i) \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Clearly $v^\sigma(0) = v^\sigma(1) = 0$. Consider $v, w \in {}^\omega 2$ and suppose that $v^\sigma(m) = w^\sigma(m) = 1$. Thus $m \geqslant 2$ and $v(n) = w(n) = 1$, where $2^{n+1} \leqslant m < 2^{n+2}$, and also $m = 2^{n+1} + \sum_{i=0}^{n} 2^{n-i} v(i) = 2^{n+1} + \sum_{i=0}^{n} 2^{n-i} w(i)$. From $\sum_{i=0}^{n} 2^{n-i} v(i) = \sum_{i=0}^{n} 2^{n-i} w(i)$ and $v(i), w(i) \in \{0, 1\}$ follows that $v \restriction n = w \restriction n$ and $v \restriction n + 1 = w \restriction n + 1$ from the above. It follows that the map σ is injective. If

$$T^\sigma = \{v^\sigma \restriction m : v \in {}^\omega 2,\ m \in \omega\},$$

then we want to show the following

PROPOSITION 2.1. *If σ is the map given by* (2.1) *then*

(*a*) $\sigma : {}^\omega 2 \to {}^\omega 2$ *defines an injective tree embedding.*

(*b*) T^σ *is a perfect subtree of T with* $\mathrm{Br}(T^\sigma) = \mathrm{Im}\,\sigma$.

PROOF. Clearly T^σ is a subtree of T with $\mathrm{Br}(T^\sigma) = \mathrm{Im}\,\sigma$, so it remains to show that it is perfect. Suppose

$$m = \mathrm{br}(v^\sigma, w^\sigma) = \mathrm{br}(v'^\sigma, w'^\sigma)$$

for pairs v, w and v', w' from ${}^\omega 2$ to branch at the same level m. We may assume that $v^\sigma(m) = v'^\sigma(m) = 1$ [thus $w^\sigma(m) = w'^\sigma(m) = 0$]. From the preceding argument follows $v \restriction n + 1 = v' \restriction n + 1$ where $2^{n+1} \leqslant m < 2^{n+2}$. So also $v^\sigma \restriction 2^{n+1} = v'^\sigma \restriction 2^{n+1}$ and the two branches v^σ and v'^σ must coincide up to the branch point m, i.e.

$$v^\sigma \restriction m + 1 = v'^\sigma \restriction m + 1.$$

Similarly $w^\sigma \restriction m + 1 = w'^\sigma \restriction m + 1$ thus $v^\sigma \cap w^\sigma = v'^\sigma \cap w'^\sigma$ and T^σ must be perfect. ■

If $X \subseteq {}^{\omega \geqslant} 2$ then we call $[X] = \{\mathrm{br}(v, w) : v \neq w \in X\} \subseteq \omega$ the *support of* X. Using a sequence of almost disjoint subsets of ω and Proposition 2.1 we can fix for the remaining paper a sequence T_α $(\alpha < \lambda)$ of perfect trees with $\mathrm{Br}(T_\alpha)$ pair-wise disjoint and $[T_\alpha]$ pair-wise almost disjoint for all $\alpha < \lambda$. Moreover let $V_\alpha \subseteq \mathrm{Br}(T_\alpha)$ be a subset of infinite branches of cardinality λ for each $\alpha < \lambda$.

Identify the set

$$\Gamma = T \times \mathfrak{N} \times \lambda, \tag{2.2}$$

as a subset of a direct sum

(2.3) $$B_\infty = \bigoplus_{(\tau, N, \alpha) \in \Gamma} (\tau, N, \alpha) A$$

of cyclic A-modules with annihilators given by

(2.4) $$\mathrm{Ann}_A(\tau, N, \alpha) = N.$$

Continuity of multiplication in A implies that every element of B_∞ is annihilated by some element of $\mathfrak{N}$. It is clear that as an R-module B_∞ is free of rank λ. Each element of the S-completion $\widehat{B_\infty}$ is expressible as a countable sum $g = \sum_{(\tau, N, \alpha)} g_{(\tau, N, \alpha)}$ for suitable $g_{(\tau, N, \alpha)} \in ((\tau, N, \alpha) A)\widehat{}$; we define the *support* of g by

(2.5) $$[g] = \{(\tau, N, \alpha) \in \Gamma \mid g_{(\tau, N, \alpha)} \neq 0\}.$$

The *norm* of g is the ordinal

(2.6) $$\|g\| = \sup\{\alpha \mid (\tau, N, \alpha) \in [g] \text{ for some } \tau \in T,\ N \in \mathfrak{N}\};$$

and the norm of a subset of $\widehat{B_\infty}$ is taken to be the supremum of the norms of its elements.

We shall call an element b of B_∞ *basal* if it has the two properties that bA is a direct summand of the A-module B_∞ and $\mathrm{Ann}_A(b) \in \mathfrak{N}$. Fix a sequence b_α $(\alpha < \lambda)$ running λ times through the set of all basal elements in B_∞, and set

(2.7) $$N_\alpha = \mathrm{Ann}_A(b_\alpha) \quad (\alpha < \lambda).$$

Next we construct a continuous increasing sequence G_α $(\alpha \leqslant \lambda)$ of sub-A-modules of the S-completion $\widehat{B_\infty}$. For a start, we take $G_0 = 0$. Assume inductively that for some $\alpha < \lambda$, G_α has been so constructed that

(2.8) $$\|G_\alpha\| = \alpha.$$

Set

(2.9) $$g_\alpha = \begin{cases} b_\alpha & \text{if } b_\alpha \in G_\alpha \\ 0 & \text{otherwise} \end{cases}$$

and, for $v \in V_\alpha$, $n < \omega$, write

(2.10) $$y_{v,n} = \sum_{i \geqslant n} \frac{q_i}{q_n} (v \restriction i, N_\alpha, \alpha) + g_\alpha \sum_{i \geqslant n} \frac{q_i}{q_n} v(i).$$

We now define

$$G_{\alpha+1} = G_\alpha + F_\alpha, \tag{2.11}$$

where

$$F_\alpha = \langle T \times \{N_\alpha\} \times \{\alpha\}\rangle_A + \langle \{y_{v,n} \mid v \in V_\alpha,\ n < \omega\rangle_A. \tag{2.12}$$

Our choice of $y_{v,n}$ and g_α ensures that $G_{\alpha+1}$ is an S-pure submodule of $\widehat{B_\infty}$ and that $\|G_{\alpha+1}\| = \alpha + 1$. Continuity takes care of the definition of G_α at a limit ordinal α. We define G to be equal to G_λ, so that

$$G = G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha = \sum_{\alpha<\lambda} F_\alpha. \tag{2.13}$$

We note

PROPOSITION 2.2. *$|G| = \lambda$ and G/G_α is an $\aleph_1$-free R-module for each $\alpha < \lambda$.*

PROOF. We consider any non-empty finite set $E \subseteq G/G_\beta$. Choose $\alpha<\lambda$ minimal with $E \subseteq G_\alpha/G_\beta$. First note that $\alpha > \beta$ must be a successor because E is a proper finite set, hence $\gamma = \alpha - 1 \geqslant \beta$ exists. Also note that G_α/G_β is a quotient of A-modules, hence an A-module. By induction it is enough to show that

$$E \subseteq (U + G_\beta)/G_\beta \oplus G_\gamma/G_\beta \subseteq_* G_{\gamma+1}/G_\beta \tag{2.14}$$

for a free A-module $(U + G_\beta)/G_\beta$.

First we want to find inductively an A-submodule $U \subseteq G_\alpha$. Let T^m be the set of elements in T of length $\leqslant m$ for any $m \in \omega$. From (2.11) we find a finite set $E' \subseteq G_{\gamma+1}$ of representatives of the elements in E, a finite set $F \subseteq V_\gamma$, and a number $m < \omega$ such that

$$E' \subseteq U + G_\gamma \quad \text{where} \quad U = \langle T^m \cup \{y_{vmg_\gamma} \colon v \in F\}\rangle_A.$$

Moreover we may assume that $[v_m] \cap [w_m] = \emptyset$ for all $v \neq w \in F$. A support argument shows that the defining generators of U are A-independent *modulo* G_γ, hence $U + G_\beta/G_\beta$ must be A-free and $G_\gamma/G_\beta \cap (U + G_\beta)/G_\beta = 0$.

Now it is easy to show that U is R-pure in G which also implies the purity in (2.14). If $h \in G \setminus G_{\gamma+1}$, then an easy support argument shows that $G_{\gamma+1}$ is pure in G that is to say that $dh \notin G_{\gamma+1}$ for any $0 \neq d \in R$ and in particular $dh \notin U$. We may suppose that $h \in G_{\gamma+1}$, and by the last considerations we find a finitely generated A-submodule

$$U' = \langle T_\gamma^{m'} \cup \{y_{vm'g_\gamma}: v \in F'\}\rangle_A$$

for some number $m' \geqslant m$ and finite set $F \subseteq F' \subseteq V_\gamma$ with $h \in U'$. We may assume that m' is chosen such that also

$$[v_{m'}] \cap [w_{m'}] = \emptyset \text{ for all } v \neq w \in F'.$$

One more support argument now shows that U is a summand of U', we leave it as an exercise to write down a complement of U in U'. If $dh \in U$ for some $0 \neq d \in R$, then $h \in U$ follows from $h \in U'$, which shows that U is pure in G. ■

Finally, we define

(2.15) $B_\alpha = \langle T \times \{N_\alpha\} \times \{\alpha\}\rangle_A$ for any $\alpha < \lambda$ and $B = \bigoplus_{\alpha < \lambda} B_\alpha$.

It is immediate from the construction that

(2.16) $$B \subseteq G \subseteq \widehat{B},$$

and that G is S-pure in $\widehat{B}$. It is also clear that the annihilator of every element of G is open in A. It follows that the natural map $A \to \mathrm{End}_R(G)$ is a topological embedding. To prove that it is a topological isomorphism we therefore only have to prove that it is surjective.

In fact it will be enough to prove that each endomorphism $\varphi \in \mathrm{End}_R G$ has the property that $b\varphi \in bA$ for each basal $b \in B$. For suppose that φ has this property.

(2.17) $$b\varphi = ba_b \quad (b \in B),$$

where each $a_b \in A$. For any $x \in B$ we may choose $b = (\tau, N_\alpha, \alpha) \in \Gamma \backslash ([x] \cup [x\varphi])$ with the property that $\mathrm{Ann}_A(x) \supseteq N = \mathrm{Ann}_A(b)$. Easy checks show that $\mathrm{Ann}_A(b + x) = N$ and that the direct summand bA on the right-hand side of (2.3) may be replaced by $(b + x)A$. Therefore $b + x$ is basal, and we have $(b + x)\, a_{b+x} = (b + x)\,\varphi = b\varphi + x\varphi = ba_b + x\varphi$, whence

(2.18) $$b(a_{b+x} - a_b) = x\varphi - xa_{b+x}.$$

Our choice of b implies that the supports of the two sides here are disjoint, so both sides vanish. Thus $a_{b+x} - a_b \in \mathrm{Ann}_A(b) \subseteq \mathrm{Ann}_A(x)$, and $x\varphi = xa_{b+x} = xa_b = xa_x$. It follows that if for the moment we convert B into a directed set $(B, \leqslant)$ by writing $y \leqslant z$ to mean $\mathrm{Ann}_A(y) \supseteq \mathrm{Ann}_A(z)$ for

$y, z \in B$, then the net $(a_x)_{x \in B}$ contains a cofinal Cauchy subnet, namely the subnet indexed by the basal elements. By the completeness of A, the whole net converges to an element $a \in A$, and we have proved that $x\varphi = xa$ $(x \in B)$. Thus φ agrees with scalar multiplication by a on B and, by continuity, also on G.

PIGEON-HOLE LEMMA 2.3. *Let $\alpha < \lambda$ and $\beta \leqslant \lambda$, and assume that we have a family of elements $t_v \in G_\beta$ $(v \in W)$ indexed by a subset $W \subseteq V_\alpha$ of cardinal $|W| = \kappa$ and a finitely generated sub-A-module H of G_β with the property that*

$$(2.19) \qquad v, w \in W \text{ and } \operatorname{br}(v, w) = m \Rightarrow t_v - t_w \in H + q_m G_\beta .$$

Then there exist a subset $W' \subseteq W$ with $|W'| = \kappa$ and a finite sequence of ordinals $\beta_1 < \ldots < \beta_s < \beta$ such that

$$(2.20) \qquad t_v \in \sum_{i=1}^{s} F_{\beta_i} \text{ for all } v \in W' .$$

PROOF. We induct on β. The result is vacuous for $\beta = 0$, so we suppose $\beta > 0$ and assume the analogous result for smaller ordinals.

Take first the case of a limit ordinal β. If, for some $\gamma < \beta$, G_γ contains a subfamily t_v $(v \in W')$ of cardinal κ, then since we may suppose that γ is large enough for G_γ to contain also the finitely generated H, a simple application of the modular law implies that W' satisfies the analogue of (2.19) with γ in place of β, and the result follows at once by our inductive hypothesis. Assume then, for a contradiction, that no such $\gamma < \beta$ exists. Then one easily produces a continuous increasing sequence of ordinals $\beta_\xi \leqslant \beta$ $(\xi \leqslant \kappa)$ and branches $v_\xi \in W$ $(\xi < \kappa)$ such that

$$t_{v_\xi} \in G_{\beta_{\xi+1}} \setminus G_{\beta_\xi} \quad (\xi < \kappa),$$

and it is clear that we must in fact have $\beta_\kappa = \beta$; and there will be no loss if we assume also that $H \subseteq G_{\beta_0}$. For each $\xi < \kappa$, the coset $t_{v_\xi} + G_{\beta_\xi}$ is a non-zero element of $G_{\beta_{\xi+1}}/G_{\beta_\xi}$. But by Proposition 2.2 this quotient is $\aleph_1$-free and therefore S-reduced. So for some $m_\xi < \omega$ we have $t_{v_\xi} + G_{\beta_\xi} \notin q_{m_\xi} G_{\beta_{\xi+1}}/G_{\beta_\xi}$ or, in other words,

$$(2.21) \qquad t_{v_\xi} \notin G_{\beta_\xi} + q_{m_\xi} G_{\beta_{\xi+1}} .$$

A pigeon-hole argument shows that there is a subset $C \subseteq \kappa$ of cardinal $|C| = \kappa$ on which m_ξ is constant; say $m_\xi = m$ $(\xi \in C)$. Now, if a set of branches $v \in {}^{\omega}2$ admits a bound n_0 on the levels of its branching points, since $v \mapsto v \upharpoonright n_0$ is visibly an injection into a finite set, it is clear that the set in question must be finite. Since the set $\{v_\nu \mid \nu \in C\}$ is infinite, we may certainly choose indices ξ and η such that $\mathrm{br}(v_\xi, v_\eta) = n > m$ and $\xi < \eta$. Then

$$t_{v_\eta} - t_{v_\xi} \in H + q_n G_{\beta_\eta + 1},$$

and, since t_{v_ξ} and H both lie in G_{β_η}, it follows that

$$t_{v_\eta} \in G_{\beta_\eta} + q_m G_{\beta_\eta + 1} = G_{\beta_\eta} + q_{m_\eta} G_{\beta_\eta + 1},$$

contrary to (2.21).

We now assume $\beta = \gamma + 1$ and that the lemma holds for all ordinals $\leqslant \gamma$. Using support arguments, (1.1) and (2.9)-(2.12) we will show first that

(2.22) $$G_\gamma \cap F_\gamma = g_\gamma A.$$

Choose any $v \in V_\gamma$ and apply (2.10) for some $n \in \omega$ with $v(n) = 1$. Thus $s_n q_n = q_{n+1}/q_n$ and

$$y_{v,n} - s_n q_n y_{v,n+1} = y_{v,n} - (q_{n+1}/q_n)\, y_{v,n+1} =$$

$$= (v \upharpoonright n, N_\gamma, \gamma) + \sum_{i \geqslant n+1} (q_i/q_n)(v \upharpoonright i, N_\gamma, \gamma) + g_\gamma + \sum_{i \geqslant n+1} (q_i/q_n)\, v(i)$$

$$- \sum_{i \geqslant n+1} (q_i/q_n)(v \upharpoonright i, N_\gamma, \gamma) - \sum_{i \geqslant n+1} (q_i/q_n)\, v(i) = (v \upharpoonright n, N_\gamma, \gamma) + g_\gamma \in F_\gamma.$$

However $(v \upharpoonright n, N_\gamma, \gamma) \in F_\gamma$ and $g_\gamma \in G_\gamma$, thus

$$g_\gamma A \subseteq F_\gamma \cap G_\gamma.$$

On the other hand $h \in F_\gamma$ can be expressed as a sum

$$h = \sum_{v \in E} a_v y_{v,m} + a_\gamma g_\gamma + t, \quad (a_v, a_\gamma \in A)$$

for large enough $m \in \omega$, a finite subset $E \subseteq V_\gamma$ and $t \in B_\gamma$. We may assume for $v \neq w \in E$ that

$$[g_\gamma] \cap [y_{v,m}] = [y_{v,m}] \cap [y_{w,m}] = \emptyset\,.$$

If also $h \in G_\gamma$, consider any $\tau \in [y_{v,m}]$ to see that $a_v = 0$ and similarly $t = 0$. Hence $h = a_\gamma g_\gamma$ and $G_\gamma \cap F_\gamma \subseteq g_\gamma A$, so (2.22) follows.

Now the proof of the lemma when $\beta = \gamma + 1$ will be easy.

Let $H \subseteq G_{\gamma+1}$, $t_v \in G_{\gamma+1} (v \in W)$ satisfying the hypothesis of the lemma. Using (2.22), we can write

$$t_v = t_v^0 + t_v^1 \quad (t_v^0 \in G_\gamma,\ t_v^1 \in F_\gamma) \ \text{ for all } \ v \in W,$$

so that

$$t_v - t_w \in H + q_m G_{\gamma+1} \ \text{ whenever } \ \mathrm{br}(v, w) = m.$$

We can also find finitely generated sub-A-modules H^0, H^1 of G_γ and F_γ respectively such that $H \subseteq H^0 + H^1$. Thus

$$(t_v^0 - t_w^0) + (t_v^1 - t_w^1) + h^0 + h^1 + q_m g + q_m f = 0$$

for suitable elements $h^i \in H^i$ and $g \in G_\gamma, f \in F_\gamma$. From (2.22) follows

$$(t_v^0 - t_w^0) + h^0 + q_m g = (t_w^1 - t_v^1) - h^1 - q_m f \in g_\gamma A.$$

Hence $(t_v^0 - t_w^0) \in \langle H^0, g_\gamma A \rangle + q_m G_\gamma$, and by induction $t_v^0 \in \sum_{i=1}^{s} F_{\gamma_i}$ for all $v \in W'$, a suitable $W' \subseteq W$ with $|W'| = |W|$ and $\gamma_1 < \ldots < \gamma_s < \gamma + 1$. If we add $\gamma_{s+1} = \gamma$, then

$$t_v^0 \in \sum_{i=1}^{s+1} F_{\gamma_i} \ \text{ for all } \ v \in W'$$

completes the induction. ■

3. Comparing branch point.

Proposition 3.1. *Let $m_0 \in \omega$, $\beta_1 < \ldots < \beta_s$ be a sequence of ordinals and $\beta_s, \alpha < \lambda$. Assume that we have a family $t_v \in \sum_{i=1}^{s} F_{\beta_i}$ for all $v \in W$ of elements indexed by a subset $W \subseteq V_\alpha$ of cardinal $|W| = \kappa$, then we can find natural numbers $m, j^* \geq m_0$, a subsequence of ordinals, renamed $\beta_1 < \ldots < \beta_s$ and another family*

$$t_v \in \sum_{i=1}^{s} F_{\beta_i} \ \textit{indexed by } W, \textit{ which is of the form } \ t_v = \sum_{i=1}^{n} a_i y_{v_i m}$$

in which the a_is, n and m do not depend on v. The new family is obtained by passing to an equipotent subset of W which we rename W and adding to members of the old family a fixed element in $\sum_{i=1}^{s} F_{\beta_i}$. The new *family has additional properties for* $i, j \leqslant s$, $v \in W$

(a) $\bigcup_{j \leqslant s} [g_{\beta_j}] \cap [y_{v_i m}] = \emptyset$

(b) $[v_{im}] \cap [v_{jm}] = \emptyset$ *for all* $i \neq j$ *and the branches* v_i $(v \in W)$ *are pair-wise distinct.*

(c) $v_i \in V_{\beta_j}$ *for some* $j = j(i) \leqslant s$ *which is independent of* v.

(d) $\operatorname{Ann} y_{v_i} = N_i \in \mathfrak{N}$ *for all* $v \in W$.

(e) *The* $v_i \restriction j^*$ *are pair-wise distinct and independent of* v.

(f) $a_i + N_i \in A/N_i \setminus q_{j^*-1}(A/N_i)$

(g) $[T_{\beta_i}] \cap [T_{\beta_j}] \subseteq j^*$ for all $i \neq j$ and $[T_{\beta_i}] \cap [T_\alpha] \subseteq j^*$ for all $\beta_i \neq \alpha$.

Proof. We will apply several pigeon-hole arguments and after passing to a subset of W, we will silently name it W. As a first application we may assume that W in the hypothesis of the proposition satisfies

$$\operatorname{br}(v, w) > m_0 \text{ for all distinct pairs } v, w \in W. \tag{3.1}$$

By (2.10) any sum $\sum_n a_n y_{v,n}$ can be expressed as a summand $a_m y_{v,m}$ for a large enough m, and any t_v $(v \in W)$ by (2.12) can be written as a sum of elements in F_{β_i}, which are therefore of the form

$$\sum_{l=1}^{t} a_{li} y_{v_{li} m_i} + b_i + a_i g_{\beta_i} \tag{3.2}$$

with $b_i \in B_{\beta_i}$, $a_i, a_{li} \in A$, $t, m_i \in \omega$ depending on v. However $|T_{\beta_i}|$, $|A|$, $|\omega| < \kappa = |W|$ and another pigeon-hole argument shows that these elements no longer depend on v. Similarly, using (2.10), we may assume that $m_i = m$ in (3.2) does not depend on $i \leqslant s$ and that the supports satisfy (a) and the first part of (b) of the proposition. We may assume that $a_{li} y_{v_{li}} \neq 0$ for all pairs li. Since $\operatorname{Ann} v_{li} = N_{\beta_i} \in \mathfrak{N}$ as in (2.10) we have $a_{li} \notin N_{\beta_i}$ and can choose $m_1 > m_0$ such that

$$a_{li} + N_{\beta_i} \in A/N_{\beta_i} \setminus q_{m_1}(A/N_{\beta_i}) \text{ for all pairs } li.$$

Thus (f) will follow if we choose $j^* > m_1$.

We have

$$t_v = \sum_{i=1}^{s} \left(\sum_{l=1}^{t} a_{li} y_{v_{li} m} + b_i + a_i g_{\beta_i} \right) \quad (v \in W). \tag{3.3}$$

For any fixed v we can choose $j_v > m_1$ such that all the $v_{li} \restriction j_v$ are distinct and by a pigeon-hole argument $j_v = j^*$ does not depend on v and surely $j^* > m_1$, hence (f) holds. Also (e) of the proposition is shown. Enlarging j^* further if needed, condition (g) is obvious. Now we identify all pairs li with natural numbers $\leqslant n = st$. Subtracting the constant element $\sum_{i=1}^{s}(b_i + a_i g_{\beta_i})$ we get a family of new elements $t_v = \sum_{j=1}^{n} a_j y_{v_j m}$. Apply the Δ-Lemma (see Jech [18, p. 225]) to the finite sets $\{v_j : j \leqslant n\}$ $(v \in W)$. We can pass to a subset W again such that $\{v_j : j \leqslant n\} \cap \{w_j : j \leqslant n\} = \Delta$ for all distinct $v, w \in W$. Again subtract the constant element $d = \sum_{x \in \Delta} a_x y_{v_x m}$ from each t_v. Hence all new v_is are distinct and (b) of the proposition follows, which provides finally the new family in the proposition. ■

PROPOSITION 3.2. *Let* $t_v = \sum_{i=1}^{n} a_i y_{v_i m} \in \sum_{i=1}^{s} F_{\beta_i}$ $(v \in W \subseteq V_\alpha)$ *be the family of elements given by Proposition* 3.1 *and let* $z \in G \setminus s_z G$ *for* $s_z \in S$, $s_z \mid q_{m_0}$ *having the property*

(3.4) $t_v - t_w \pm q_b z \in q_{b+1} G$ *for all* $v, w \in W$ *with* $b = \mathrm{br}(v, w)$.

Then there is a map $\pi : \{1, \ldots, n\} \to \{1, \ldots, s\}$ *such that for all* $i \leqslant n$

(a) $v_i, w_i \in V_{\pi(i)}$

(b) $\mathrm{br}(v_i, w_i) \geqslant \mathrm{br}(v, w) \geqslant j^*$, *in particular all* $\mathrm{br}(v_i, w_i)$ *are distinct.*

(c) *There is some* $j \leqslant s$ *with* $\mathrm{br}(v_j, w_j) = \mathrm{br}(v, w)$ *and* $\pi(j) = \alpha$, $g_\alpha \neq 0$.

PROOF. By Proposition 3.1 (c) for each $i \leqslant n$ there is a unique $\pi(i) \leqslant s$ such that $v_i, w_i \in V_{\beta_{\pi(i)}}$ come from the free $T_{\beta_{\pi(i)}}$, hence π and (a) are obvious.

First we claim that the $k_i = \mathrm{br}(v_i, w_i)$ $(i \leqslant n)$ are pair-wise distinct, which is part of (b). If there are $i \neq j$ with $k_i = k_j$ then $\pi(i) = \pi(j)$ follows from $k_i = k_j \geqslant j^*$ and Proposition 3.1 (e) and (g). Hence v_i, w_i, v_j, w_j come from the same tree $T_{\beta_{\pi(i)}}$. However $v_i \cap w_i \neq v_j \cap w_j$ are distinct by Proposition 3.1 (e) and of the same length $k_i = k_j$ which is impossible for the perfect tree $T_{\beta_{\pi(i)}}$. Hence it remains to show for (b) that

$$b \leqslant k_i \text{ for all } i \leqslant n,$$

which we postpone until (c) is shown. For this we need the hypothesis (3.4) in the form

$$t_v - t_w \pm q_b z \in q_{b+1} G \subseteq \frac{q_{b+1}}{q_m} G.$$

From (2.10) follows

$$(3.5) \qquad y_{v_i m} - y_{w_i m} \pm \frac{q_{k_i}}{q_m} g_{\beta_{\pi(i)}} = \frac{q_{k_i+1}}{q_m} (y_{v_i m+1} - y_{w_i m+1})$$

and if $b = \mathrm{br}(v, w) < k = \min\{k_i, i \leqslant n\}$ then $b+1 \leqslant k$ for all $i \leqslant n$. We derive $y_{v_i m} - y_{w_i m} \in \frac{q_{b+1}}{q_m} G$, hence

$$t_v - t_w = \sum_{i \leqslant n} a_i (y_{v_i m} - y_{w_i m}) \in \frac{q_{b+1}}{q_m} G$$

and $q_b z \in \frac{q_{b+1}}{q_m} G$ follows. So $z \in \frac{q_{b+1}}{q_b q_m} G = \frac{s_b q_b}{s_m} G \subseteq s_z G$ from $s_z \mid q_{m_0}$ and $m_0 < < m < b$, which contradicts the choice of z.

Hence $k \leqslant b$ and suppose $k < b$. We note that there is a **unique** $j \leqslant n$ with $k = k_j$ and $k_j < k_i$ for all $i \neq j$ by our first claim in the proof. From $k+1 \leqslant b$ and (3.4) follows $t_v - t_w \in q_{k+1} G \subseteq \frac{q_{k+1}}{q_m} G$, moreover

$$\sum_{j \neq i \leqslant n} a_i (y_{v_i m} - y_{w_i m}) \in \frac{q_{k+1}}{q_m} G,$$

hence

$$a_j (y_{v_j m} - y_{w_j m}) \in \frac{q_{k+1}}{q_m} G$$

and $a_j \frac{q_k}{q_m} g_{\beta_{\pi(i)}} \in \frac{q_{k+1}}{q_m} G$ from (3.5). We get $a_j g_{\beta_{\pi(j)}} \in \frac{q_{k+1}}{q_k} G = s_k q_k G$. Recall $k \geqslant j^* - 1$ and $\mathrm{Ann}\, g_{\beta_{\pi(j)}} = N_{\pi(j)}$, hence $a_j + N_{\pi(j)} \in q_{j^*-1}(A/N_{\pi(j)})$ contradicting Proposition 3.1 (f). Thus the first part of (c) follows, which immediately implies $\pi(j) = \alpha$ by Proposition 3.1 (g) (the second part) and (b), (c) are shown. ■

4. Proof of the Theorem.

The main theorem will follow from a corollary of the results in section 3.

COROLLARY 4.1. *Let* $t_v = \sum_{i=1}^{n} a_i y_{v_i m} \in \sum_{i=1}^{s} F_{\beta_i}$ $(v \in W \leqslant F_\alpha)$ *be the family of elements given by Proposition* 3.1 *and let* $z \in G \setminus s_z G$ *be as in Proposition* 3.2 *with the property* (3.4) [*such that* $\operatorname{Ann} z \supseteq \operatorname{Ann} g_\alpha$], *then* $z \in A g_\alpha$.

PROOF. If $t_v - t_w = \sum_{i=1}^{n} a_i (y_{v_i m} - y_{w_i m})$ and $b = \operatorname{br}(v, w)$, then by Proposition 3.1 and Proposition 3.2 we have

$$t_v - t_w - a_j (y_{n_j m} - y_{w_j m}) \in q_{b+1} G$$

with $\pi(j) = \alpha$ as in Proposition 3.2 (c), where $\operatorname{br}(v_{jm}, w_{jm}) = \operatorname{br}(v, w) = b$. From (3.4) follows

$$q_b z - a_j (y_{v_j m} - y_{w_j m}) \in q_{b+1} G \subseteq \frac{q_{b+1}}{q_m} G$$

and similarly, using (3.5) we have

$$q_b z - a_j \frac{q_b}{q_m} g_\alpha \in \frac{q_{b+1}}{q_m} G$$

thus $q_m z - a_j q_\alpha \in \frac{q_{b+1}}{q_b} G \subseteq s_b q_b G$.

Choose a sequence (v, w) of pairs $v, w \in W$ with $\operatorname{br}(v, w) = b \in w$ converging to infinity. Thus $q_m z - a_j g_\alpha \in \bigcap_{b \in w} s_b q_b G = 0$ and $q_m z = a_j g_\alpha$. Note that g_α in B_∞ is basal with $\operatorname{Ann}_A b_\alpha = N_\alpha$, hence $a_j = a' q_m$ and $z \in A g_\alpha$. ■

We finally prove the main theorem.

PROOF. By (2.17) it remains to show that $b\varphi \in bA$ for all basal $0 \neq b \in B$. We fix a basal element $b \in B$ and suppose that $z = b\varphi \notin bA$. Choose any $\alpha < \lambda$ such that $b = g_\alpha \in G_\alpha$ and $\operatorname{Ann} g_\alpha = N_\alpha \in \mathfrak{N}$. In particular $z \neq 0$ and there is $m_0 \in w$ with $z \notin s_z G$ for some $s_z \in S$ and $s_z \mid q_{m_0}$. Consider F_α as in (2.12) and take its family

$$t_v = y_v \varphi \quad (v \in V_\alpha).$$

From (2.10) follows $y_v - y_w \pm q_b g_\alpha \in q_{b+1} G$ for $b = \mathrm{br}(v, w)$, hence

$$t_v - t_w \pm q_b z \in q_{b+1} G \text{ for any } v, w, \in V_\alpha \text{ with } b = \mathrm{br}(v, w).$$

The $t_v' s$ constitute a family satisfying the hypothesis of Proposition 3.1 and Proposition 3.2. Passing to the subfamily in Proposition 3.1 we can apply Proposition 3.2 and get $z \in g_\alpha A$ from Corollary 4.1, which is a contradiction. It follows $b\varphi \in bA$ as desired. ∎

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