Examples of Birationality of Pluricanonical Maps.

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ABSTRACT - By generalizing an Enriques construction, in $P^4$ we construct a double space $V$ of degree 12, whose branch locus has a 6-ple point of the type $z^6 + \cdots + x^{12} + \cdots + y^{12} = 0$. We demonstrate that a desingularization of $V$ has birational invariants $q_1 = q_2 = 0$, $p_g = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$. Moreover, we prove that the $m$-canonical transformation has fibers that are generically finite sets if and only if $m \geq 2$ and it is birational if and only if $m \geq 6$.

Introduction.

E. Bombieri [B] proved that the $m$-canonical transformation of any nonsingular surface of general type is birational if $m \geq 5$ and $m = 5$ is the minimum for the surfaces (minimal models) with $(K^2) = 1$ and $p_g = 2$.

F. Enriques constructed a surface with $(K^2) = 1$, $p_g = 2$ (see [E] § 14, pp. 303-304); this is a desingularization of a double plane with a branch curve of degree 10, having a singular [5,5] point on it.

At a seminar, E. Stagnaro suggested generalizing the Enriques double plane to a three-dimensional double space for constructing new examples of threefolds, whose $m$-canonical transformation becomes birational if $m$ is large enough.

This paper touches first on a demonstration of the fact that the $m$-canonical transformation of the Enriques example is birational if and only if $m \geq 5$, then such a situation is generalized, constructing a double space.
space $V$. We thus have the birationality of the $m$-canonical transformation if and only if $m \geq 6$. A desingularization of $V$ has the birational invariants $q_1 = q_2 = 0$, $P_0 = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$.

We define double space of degree $2n$ the projective closure in $\mathbb{P}^4$ of the affine hypersurface given by $t^2 = f_{2n}(x, y, z)$, being $f_{2n}(x, y, z)$ a polynomial of degree $2n$; the surface of equation $f_{2n}(x, y, z) = 0$ is the branch locus of the double space.

We must bear in mind that a double plane with a branch curve of degree 10 with a singular [5,5] point on it is affinely represented by an equation of the type $z^2 + y^5 + \cdots + x^{10} = 0$. In the following paragraphs, said situation will be generalized by constructing a double space affinely given by an equation of the type $t^2 + z^6 + \cdots + x^{12} + \cdots + y^{12} = 0$.

M. Chen [C] and S. Lee [L] proved that if the canonical divisor $K$ of a threefold is «nef» and $(K^3)$ is positive, then the $m$-canonical transformation is birational for $m \geq 6$. In the proposed example the said properties are not simultaneously satisfied, but the birationality of the $m$-canonical transformation holds true for $m \geq 6$.

In this paper we consider surfaces and threefolds on the field $\mathbb{C}$ of the complex numbers and we'll write $\mathbb{P}^N$ instead of $\mathbb{P}^N_{\mathbb{C}}$.

1. Example of a double plane $S$ of degree 10 in $\mathbb{P}^3$ whose $m$-canonical transformation is birational if and only if $m \geq 5$.

1.1. Description of $S$.

Let us choose a generic curve $C$ in the linear system of curves in $\mathbb{P}^2$ defined by

$$F_{10}(X_0, X_1, X_2) = aX_0^5X_1^2 + bX_0X_2^9 + cX_1^{10} + dX_2^{10}.$$  

According to Bertini theorem, $C$ has its unique singularity at the point $A_0 = (1, 0, 0)$. To be more precise, $C$ has a [5, 5] point at $A_0$, i.e. a 5-ple point with an infinitely near 5-ple point. By using the affine coordinates

$$x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}, \quad z = \frac{X_3}{X_0}$$
we obtain the polynomial

\[ f_{10}(x, y) = ay^5 + by^9 + cx^{10} + dy^{10} \]

and hence the double plane of affine equation \( z^2 = f_{10}(x, y) \). Let \( S \) be its projective closure in \( \mathbb{P}^3 \):

\[ S : X_0^9X_2^2 - aX_0^5X_2^2 - bX_0X_2^9 - cX_1^{10} - dX_2^{10} = 0. \]

\( S \) is normal and its singularities are the points \( A_3 = (0, 0, 0, 1) \) and \( A_0 = (1, 0, 0, 0) \). To be more precise:

- \( S \) has an 8-ple point at \( A_3 \) and four double curves \( r_1, r_2, r_3, r_4 \) infinitely near in the next neighbourhoods;
- \( S \) has a double point at \( A_0 \) with a double curve \( r_5 \), a double point \( P \) and again two double curves \( r_6 \) and \( r_7 \) infinitely near, in the next neighbourhoods.

1.2. Birationality of the \( m \)-canonical transformation for \( m \geq 5 \).

We state the birationality of the \( m \)-canonical transformation, \( m \geq 5 \), using the theory of adjoints of Enriques. This theory has recently been revised by E. Stagnaro in [S2]. We keep the same nomenclature and notations as are used in said paper. In our examples all the singularities satisfy the hypothesis assumed in [S2].

The properties of a double plane are well known, but it may be useful to mention the ones that will be generalized to the hypersurface (double space) in \( \mathbb{P}^4 \) that we construct later on.

It is maybe less well known, however see [E], [S1], [S2] (a detailed calculation of the bicanonical adjoints is given in [S1]), that the \( m \)-canonical adjoints to a double plane of affine equation \( S : z^2 = f_{2n}(x, y) \), with a nonsingular branch curve \( f_{2n}(x, y) = 0 \), are:

\[ \phi_{m(n-3)}(x, y) + z\phi_{(m-1)n-3m}(x, y) = 0, \]

where \( \phi_i(x, y) \) denotes a polynomial of degree \( i \) in \( x, y \).

In compliance with [S2], let us call the \( m \)-canonical adjoints defined by \( \phi_{m(n-3)}(x, y) = 0 \) as **global** and the \( m \)-canonical adjoints defined by \( z\phi_{(m-1)n-3m}(x, y) = 0 \) as **non-global**.

Let us emphasize the following facts.

1. The \( m \)-canonical transformation \( \psi_{|mK|} \) coincides (on an open set), up to isomorphisms, with the rational transformation \( \psi_{|mS|} \) pro-
duced by the linear system of the \( m \)-canonical adjoints restricted to the double plane \( S \) (see \([S_2]\), section 16).

2. If we want \( \psi_{m|S} \) to be birational, it is necessary (but generally not sufficient) for at least one of the \( m \)-canonical adjoints to be of the kind \( z\phi_{(m-1)n-3m}(x, y) = 0 \). Conversely, the transformation is generically \( 2:1 \), at most.

3. It is possible to prove (but we omit the demonstration) that in every \( m \)-canonical adjoint, \( m \leq 4 \), the \( \langle z \rangle \) coefficient vanishes as soon as the branch curve has a \([5, 5]\) point on it.

4. From 2 and 3 it follows for \( m \leq 4 \) that \( \psi_{m|S} \), so \( \psi'_{|M|} \), cannot be birational. Moreover, one can prove directly that \( \psi_{5|S} \) is birational and also that \( \psi_{m|S} \) is birational for \( m \geq 5 \), because \( p_g \) is positive.

The idea for generalizing all this to double spaces is to transfer the properties 1, 2, 3 and 4 to a suitable double space. As a result, in the case of our example at least, the birationality holds true if and only if \( m \geq 6 \).

2. Example of a double space \( V \) of degree 12 in \( \mathbb{P}^4 \), whose \( m \)-canonical transformation is birational if and only if \( m \geq 6 \).

2.1. Description of \( V \).

To extend the foregoing situation to \( \mathbb{P}^4 \), let \( S \) be a generic surface in the linear system of surfaces in \( \mathbb{P}^3 \) defined by

\[
F_{12}(X_0, X_1, X_2, X_3) = aX_0^6 X_3^6 + bX_0 X_3^{11} + cX_1^{12} + dX_3^{12} + eX_4^{12}.
\]

According to Bertini theorem, \( S \) has a unique singularity at the point \( A_0 = (1, 0, 0, 0) \). To be more specific, \( S \) has a 6-ple point at \( A_0 \) with an infinitely near 6-ple curve. By using the affine coordinates

\[
x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}, \quad z = \frac{X_3}{X_0}, \quad t = \frac{X_4}{X_0}
\]

we obtain the polynomial

\[
f_{12}(x, y, z) = ax^6 + bx^{11} + cx^{12} + dy^{12} + ez^{12}
\]

and hence the hypersurface of affine equation \( t^2 = f_{12}(x, y, z) \).
Let $V$ be its projective closure in $\mathbb{P}^4$:

$$V: X_0^{10} X_1^2 - aX_0^6 X_3^6 - bX_0 X_3^4 - cX_1^{12} - dX_2^{12} - eX_3^{12} = 0.$$ 

We call $V$ a **double space**, according to our definition.

$V$ is normal and only has singularities at $A_4 = (0, 0, 0, 0, 1)$ and at $A_0 = (1, 0, 0, 0, 0)$. To be more precise:

- $V$ has a 10-ple point at $A_4$ with 5 double surfaces $a_1, \ldots, a_5$ infinitely near, in the next neighbourhoods,
- $V$ has a double point at $A_0$ with 2 double surfaces $a_6, a_7$, 1 double curve $s$, and 2 double surfaces $a_8, a_9$ infinitely near, in the next neighbourhoods.

### 2.2. Computation of $p_g = P_1$ and $P_m$ of $V$.

Now we calculate the genus and plurigenera of $V$, i.e.

$$P_m = \dim \mathcal{H}^0(X, \mathcal{O}_X(mK_X)) = \dim [mK_X] + 1, \quad m \geq 1, \quad p_g = P_1,$$

where $X$ denotes a nonsingular model of $V$.

The path chosen for constructing $X$ consists in two sequences of relations owing to the singularities of $V$ at $A_4$ and $A_0$.

To solve the singularity at $A_4$ we have the following sequence of blow-ups:

(1) \[ V_6 \subset \mathbb{P}_6 \xrightarrow{\pi_6} \mathbb{P}_5 \xrightarrow{\pi_5} \mathbb{P}_4 \xrightarrow{\pi_4} \mathbb{P}_3 \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}^4 \supset V \]

where $\pi_4$ denotes the blow-up of $\mathbb{P}^4$ at $A_4$ and $\pi_i$ ($2 \leq i \leq 6$) is the blow-up of $\mathbb{P}_{i-1}$ along $a_{i-1}$. From (1) the relations follow:

\[
\begin{align*}
K_{\mathbb{P}_6} &= \pi_6^*(K_{\mathbb{P}_5}) + 3E_{A_4}, \\
K_{\mathbb{P}_5} &= \pi_5^*(K_{\mathbb{P}_4}) + E_{a_1}, \\
V_1 &= \pi_1^*(V) - 10E_{A_4}, \\
V_i &= \pi_i^*(V_{i-1}) - 2E_{a_{i-1}} (2 \leq i \leq 6),
\end{align*}
\]

where $E_{A_4}, E_{a_i}$ denote the exceptional divisors of the blow-ups at $A_4$ and $a_i$, and $V_i$ denotes the strict transformation of $V_{i-1}$.

To solve the singularity at $A_0$ we have the following sequence of blow-ups:

(2) \[ V_{12} \subset \mathbb{P}_{12} \xrightarrow{\pi_{12}} \mathbb{P}_{11} \xrightarrow{\pi_{11}} \mathbb{P}_{10} \xrightarrow{\pi_{10}} \mathbb{P}_9 \xrightarrow{\pi_9} \mathbb{P}_8 \xrightarrow{\pi_8} \mathbb{P}_7 \xrightarrow{\pi_7} \mathbb{P}_6 \supset V_6 \]

(in the following $V_{12}$ will be $X$), where $\pi_7$ is the blow-up of $\mathbb{P}_6$ at $A_0$, $\pi_s$ and $\pi_9$ are the blow-ups of $\mathbb{P}_7$ and $\mathbb{P}_8$ along $a_6$ and $a_7$, $\pi_{10}$ is the blow-up of $\mathbb{P}_9$ along $s$ and finally $\pi_{11}$ and $\pi_{12}$ are the blow-ups of $\mathbb{P}_{10}$ and $\mathbb{P}_{11}$ along
\(\alpha_8\) and \(\alpha_9\). From (2) we can say that:

\[
\begin{align*}
K_{\nu_1} &= \pi^* f(K_{\nu_3}) + 3E_{\nu_0} \\
V_7 &= \pi^* f(V_6) - 2E_{\nu_4} \\
K_{\nu_2} &= \pi^* f(K_{\nu_5}) + E_{\alpha_7} \\
V_9 &= \pi^* f(V_8) - 2E_{\alpha_7} \\
K_{\nu_3} &= \pi^* f(K_{\nu_6}) + E_{\alpha_8} \\
V_{11} &= \pi^* f(V_{10}) - 2E_{\alpha_8} \\
K_{\nu_4} &= \pi^* f(K_{\nu_7}) + E_{\alpha_9}
\end{align*}
\]

where \(E_{\nu_1}, E_{\nu_2}, E_{\nu_3}, E_{\nu_4}\) denote the exceptional divisors of the blow-ups at \(\nu_1, \nu_2, \nu_3, \nu_4\), respectively.

Because \(X\) is nonsingular, we can apply the adjunction formula that states: if \(D\) is a divisor linearly equivalent to \(K_{\nu_2} + X\), i.e. \(D \equiv K_{\nu_2} + X\), and if \(D|_X\) is defined, then \(D|_X = K_X\), where \(K_X\) is a canonical divisor on \(X\).

Substituting from the above relations, we obtain

\[
K_{\nu_2} + X = K_{\nu_2} + X'
\]

We now have \(K_{\nu_4} \equiv -5H + V \equiv 12H\), where \(H\) is a hyperplane in \(\mathbb{P}^4\). If \(\Phi_7 \equiv 7H\) denotes a hypersurface of degree 7 in \(\mathbb{P}^4\), we deduce from (3)

\[
K_{\nu_2} + X = K_{\nu_2} + X'
\]

We see from the adjunction formula that, if \(D|_X\) is defined, then it is a canonical divisor \(K_X\) on \(X\), i.e. \(D|_X = K_X \equiv K_X\).

If we multiply (4) by the integer \(m \geq 1\), we obtain

\[
m(K_{\nu_2} + X) = m\Phi_7 = m\Phi_7 = D'
\]

where \(\Phi_{7m}\) is a hypersurface of degree \(7m\) in \(\mathbb{P}^4\).
As before we obtain $D_1 X = mK_X$.

Let $\sigma X: X \to V$, where $\sigma = \pi_1 \circ \cdots \circ \pi_2 \circ \pi_1$, be the desingularization of $V$ described.

Using the theory of adjoints and pluriadjoints, we can calculate $p_g = P_1$ and $P_m$; again we use the nomenclature and notations of [S2].

$\Phi_{\pi_m}$, $m \geq 1$, is an $m$-canonical adjoint to $V$ (with respect to $\sigma$) if $D_1 X$ is effective, i.e. $D_1 X \geq 0$ (see [S2], section 2).

We see first how the presence of the singular point $A_4$ characterizes the canonical and $m$-canonical adjoints.

The condition $\pi_7^* (\Phi_7) - 7E_{A_4} \geq 0$ in (4), given by $A_4$, says that if $\Phi_7$ is a global canonical adjoint, then $A_4$ must be a $7$-ple point for $\Phi_7$ itself, i.e. $\Phi_7$ is defined by a form $F_7$, in $X_0$, $X_1$, $X_2$, $X_3$. The further condition given by $A_4$

$$\pi_7^* (\pi_7^* (\pi_7^* (\pi_7^* (\Phi_7) - 7E_{A_4} - E_{a_1} - E_{a_2} - E_{a_3} \geq 0$$

(see (4)), implies that it is

$$F_7 (X_0, X_1, X_2, X_3, X_4) = X_0^5 F_2 (X_0, X_1, X_2, X_3).$$

The condition

$$[\pi_7^* (\pi_7^* (\pi_7^* (\pi_7^* (\pi_7^* (\Phi_7) - 7mE_{A_4} - mE_{a_1} - mE_{a_2} - mE_{a_3} - mE_{a_4} - mE_{a_5} \geq 0$$

imposed by $A_4$ on the $m$-canonical adjoints (see (5)) implies that

$$F_7^* (X_0, X_1, X_2, X_3, X_4) = X_0^5 F_2^* (X_0, X_1, X_2, X_3) + F_2^* (X_0, X_1, X_2, X_3).$$

So we have a situation much the same as the double plane. To be more precise, the $m$-canonical adjoints to a double space of affine equation $t^2 = f_{2m} (x, y, z)$, with a nonsingular branch locus $f_{2m} (x, y, z) = 0$, are:

$$\phi_{m(n-4)} (x, y, z) + t\phi_{(m-1)n-4} (x, y, z) = 0$$

where $\phi_i (x, y, z)$ denotes a polynomial of degree $i$ in $x, y, z$.

Here again, let us call the $m$-canonical adjoints given by $\phi_{m(n-4)} (x, y, z) = 0$ global and those given by $t\phi_{(m-1)n-4} (x, y, z) = 0$ non-global.

Now let us examine the point $A_0$, which is a singular point for the double space because there is a $6$-ple point on its branch locus.
From (4) it must be that

\[ F_7(X_0, X_1, X_2, X_3, X_4) = X_0^2 X_3(a_1 X_1 + a_2 X_2 + a_3 X_3). \]

Let \( W_7 \) be the vector space of the forms defining global canonical adjoints and \( W_7 \) be the vector space of the forms defining canonical adjoints. Since \( W_7 = \mathcal{W}_7 \) and \( p_g = \dim |K_X| + 1 \) (see \([S_2]\), section 3), it follows that

\[ p_g = 3. \]

We can move on now to consider the point \( A_0 \) for calculating the \( m \)-canonical adjoints \((m > 1)\). The conditions imposed by \( A_0 \) produce different results, depending on the value of \( m \).

For \( m < 6 \) the vector spaces of the forms defining global \( m \)-canonical adjoints, \( W_{7m} \), and those of the forms defining \( m \)-canonical adjoints, \( \mathcal{W}_{7m} \), coincide; but the equality does not hold true for \( m = 6 \). Indeed, being an \( m \)-canonical adjoint implies that

\[ \Phi_{7m} : \phi_{m(6-4)}(x, y, z) + t\phi_{(m-1)6-4m}(x, y, z) = 0 \]

must satisfy the condition (see (5)):

\begin{align*}
(6) \quad [\pi_{12}(\pi_{10}(\pi_{06}(\pi_{15}^{\phi}(\Phi_{7m}) + mE_{a_0} - mE_{a_1}) - \\
mE_{a_2} - mE_{a_3})]_{x} & \equiv 0. \nonumber
\end{align*}

Now, if \( m < 6 \), the degree of the \( m \) coefficient is too low and it satisfies the condition (6) if and only if \( \phi_{m(6-4)-4m}(x, y, z) \) vanishes. So, for \( m < 6 \), \( \Phi_{7m} \) is an \( m \)-canonical adjoint if and only if it is defined by a form

\[ F_{7m}(X_0, X_1, X_2, X_3, X_4) = X_0^{5m} X_3^m F_m(X_0, X_1, X_2, X_3), \]

i.e. if and only if \( \Phi_{7m} \) is really a global \( m \)-canonical adjoint.

To be more precise, we have

\[ \mathcal{W}_{14}' = W_{14}' = \{ X_0^{10} X_3^5(b_1 X_0 X_3 + b_2 X_2 X_3 + b_3 X_1 X_3 + \\
+ b_4 X_2^2 + b_5 X_2 X_3 + b_6 X_3^2), b_i \in \mathbb{C} \}; \]

\[ \mathcal{W}_{21}' = W_{21}' = \{ X_0^{15} X_3^4(b_1 X_0 X_1 X_3 + b_2 X_0 X_2 X_3 + \\
\cdots + b_{12} X_2 X_3^2 + b_{13} X_3^2), b_i \in \mathbb{C} \}; \]

\[ \mathcal{W}_{28}' = W_{28}' = \{ X_0^{20} X_3^3(b_1 X_0 X_2 X_3 + b_2 X_0 X_2^2 X_3 + \\
\cdots + b_{21} X_2 X_3^3 + b_{22} X_3^3), b_i \in \mathbb{C} \}; \]

\[ \mathcal{W}_{35}' = W_{35}' = \{ X_0^{25} X_3^2(b_1 X_0 X_3 X_3 + b_2 X_0 X_2 X_3^2 + \\
\cdots + b_{33} X_2 X_3^4 + b_{34} X_3^4), b_i \in \mathbb{C} \}. \]
If \( m = 6 \), the degree of the \( t \) coefficient is \((m - 1)6 - 4m = 6\). This is the minimum that can satisfy condition (6) and we have the first non-global \( m \)-canonical adjoint which is affinely given by \( tz^6 = 0 \). To be more specific, \( \Phi_{7m} \) is an \( m \)-canonical adjoint \((m = 6)\) if and only if it is defined by a form

\[
F_{42}(X_0, X_1, X_2, X_3, X_4) = X_0^{20} [X_0^5 F_6(X_0, X_1, X_2, X_3) + X_0^5 X_3^6 X_4]
\]

and, in affine coordinates, it has the equation

\[
\phi_{42}(x, y, z, t) = z^6 \phi_6(x, y, z) + tz^6 = 0.
\]

In a detailed expression we obtain

\[
\psi_{42} = \{ X_0^{20} X_3^6 (aX_0^5 X_4 + b_1 X_0^5 X_3^3 + b_2 X_0^5 X_1^2 X_3^2 + \cdots + b_{49} X_2 X_3^5 + b_{50} X_3^6) \mid a, b_i \in \mathbb{C} \}.
\]

So we have a non-global 6-canonical adjoint defined by the form \( X_0^{20} X_3^6 X_4 \).

In particular, the plurigenera \( P_i = \dim [iK_X] + 1, i \geq 1 \) (see [S2]), are

\( p_0 = P_1 = 3, \quad P_2 = 7, \quad P_3 = 13, \quad P_4 = 22, \quad P_5 = 34, \quad P_6 = 51 \).

2.3. The \( m \)-canonical transformations \( \varphi_{|mK_X|} \), \( 1 \leq m \leq 5 \).

In this paragraph, we prove that \( \varphi_{|mK_X|} \) is a generically 2:1 map for \( 2 \leq m \leq 5 \).

Let us consider the following triangle

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{|mK_X|}} & p^{dim|mK_X|} - 1 \\
\sigma|X & \downarrow & \psi_{|mV|} \\
V
\end{array}
\]

where \( \sigma|X \) is the desingularization of \( V \) and \( \psi_{|mV|} \) is the rational transformation, restricted to \( V \), defined by the linear system of bicanonical adjoints to \( V \). The foregoing diagram is commutative because the divisors of \( |mK_X| \) are of the kind

\[
[x_0^5 \cdots (x_0^5 (\Phi_{7m})_m - 7mE_A_1) \cdots - mE_{\alpha_1} - mE_{\alpha_2}]|X|.
\]

To prove that \( \varphi_{|mK_X|} \) is generically 2:1, it suffices to consider such a transformation on an open set of \( X \). \( \sigma \) is a sequence of blow-ups and so it is an isomorphism outside the exceptional divisors of the single blow-ups; so, on an open set of \( X \), \( \sigma|X \) is an isomorphism. As a result, to say that \( \varphi_{|mK_X|} \) is generically 2:1 means that \( \psi_{|mV|} \) generically 2:1.
Now let us demonstrate that $\psi_{2|V}$ is generically 2:1.

Bearing in mind that

\[
\mathcal{W}_{14} = W_{14} = \{ X_0^{10} X_2^2 (b_1 X_0 X_1 + b_2 X_1^2 + b_3 X_0 X_2 + \nonumber \\
+ b_4 X_0 X_3 + b_5 X_2^2 + b_6 X_2 X_3 + b_7 X_3^2), \ b_i \in \mathbb{C} \},
\]

we shall have

\[
V \subset \mathbb{P}^4 \xrightarrow{\psi_2} \mathbb{P}^6
\]

\[
(X_0, X_1, X_2, X_3, X_4) \mapsto (Y_0, \ldots, Y_6)
\]

defined by

\[
\begin{align*}
Y_0 &= (X_0^{10} X_2^2) X_0 X_3 \\
Y_1 &= (X_0^{10} X_2^2) X_1^2 \\
Y_2 &= (X_0^{10} X_2^2) X_1 X_2 \\
Y_3 &= (X_0^{10} X_2^2) X_1 X_3 \\
Y_4 &= (X_0^{10} X_2^2) X_2^2 \\
Y_5 &= (X_0^{10} X_2^2) X_2 X_3 \\
Y_6 &= (X_0^{10} X_2^2) X_3^2.
\end{align*}
\]

Let $U = \mathbb{P}^4 - \{ X_0 = X_1 = X_4 = 0 \}$ be the affine open set chosen in $\mathbb{P}^4$, with the coordinates

\[
x = \frac{X_0}{X_1}, \quad y = \frac{X_2}{X_1}, \quad z = \frac{X_3}{X_1}, \quad t = \frac{X_4}{X_1}.
\]

Let $T = \mathbb{P}^5 - \{ Y_1 = Y_3 = 0 \}$ be the affine open set in $\mathbb{P}^5$ with the coordinates

\[
y_1 = \frac{Y_0}{Y_1}, \quad y_2 = \frac{Y_2}{Y_1}, \ldots, \quad y_6 = \frac{Y_6}{Y_1}.
\]

We shall thus have

\[
\psi_{2|U}: U \rightarrow T
\]

\[
(x, y, z, t) \mapsto (y_1, \ldots, y_6)
\]

\[
\begin{align*}
y_1 &= xz \\
y_2 &= y \\
y_3 &= z \\
y_4 &= y^2 \\
y_5 &= yz \\
y_6 &= z^2.
\end{align*}
\]
Let $\mathcal{P} = (y_1, \ldots, y_6)$ be a generic point of $Im \psi_{1|U}$; the fiber on $\mathcal{P}$ is

$$\psi_{1|U}^{-1}(\mathcal{P}) = \begin{cases} (x, y, z, t): \begin{cases} xz = y_1 \\ y = y_2 \\ z = y_3 \\ y^2 = y_4 \\ yz = y_5 \\ z^2 = y_6 \end{cases} & \begin{cases} xz = y_1 \\ y = y_2 \\ z = y_3 \end{cases} \end{cases}.$$ 

The fiber on $\mathcal{P}$ intersects $V_U = V \cap U$ at two points; indeed,

$$V_U \cap \psi_{1|U}^{-1}(\mathcal{P}) = \begin{cases} x^2 = y_1 \\ y = y_2 \\ z = y_3 \\ \left( \frac{y_4}{y_3} \right)^{10} t^2 = a y_4^6 + b y_4 y_2^4 + c + d y_2^2 + e y_4^2 \\ y = y_2 \\ z = y_3 \\ x = \frac{y_1}{y_3} \end{cases} = \begin{cases} x^2 = y_1 \\ y = y_2 \\ z = y_3 \end{cases}.$$ 

This means that $\psi_{1|U}: V \to \mathbb{P}^6$, so $\phi_{1|K_X}: X \to \mathbb{P}^6$, is generically $2:1$. In particular, we find that $V$ is of general type (Kodaira dimension 3). It follows that $\phi_{1|mK_X}$, $m > 2$, is also generically $n:1$, with $n \leq 2$.

Let us consider an effective canonical divisor $K$, which exists because $p_g$ is positive; putting $nK + |2K_X| = \{ nK + D, D \in |2K_X| \}$ for $n = 1, 2, \ldots$ (assuming $K$ fixed part of the linear system), we consider the linear systems

$$K + |2K_X| \subset |3K_X|, 2K + |2K_X| \subset |4K_X|, \ldots, (m - 2)K + |2K_X| \subset |mK_X|, \ldots.$$ 

All these linear systems $K + |2K_X|$, $2K + |2K_X|$, $\ldots$ give rise to rational transformations which are generically $n:1$, $n \leq 2$, and so are the transformations $\phi_{1|mK_X}$, $m \geq 2$.

If $2 \leq m \leq 5$, the absence of any non-global $m$-canonical adjoint implies that $n = 2$, which is the statement.

REMARK 1. We said previously that the canonical transformation $\phi_{1|K_X}$ coincides, up to isomorphisms, with $\psi_{1|U}$ on an open set. We
can now note that \( \psi_1|V \) is generically the projection map of \( V \) from the straight line \( X_1 = X_2 = X_3 = 0 \) on a plane.

2.4. The 6-canonical transformation \( \varphi_{|6K_X|} \).

Our aim is to prove that \( \varphi_{|6K_X|} \) is birational. Unlike the foregoing cases, this will be based on the existence of the non-global 6-canonical adjoint defined by the form \( G_7 = X_0^3 X_4 X_5 X_6 \).

As we did previously, we choose a canonical effective divisor \( K \) (e.g. let \( K \) be given by \( L = X_0^5 X_3 X_4 \)) and we construct the linear system \( 4K + |2K_X| \sigma |6K_X| \). The linear system \( 4K + |2K_X| \sigma |6K_X| \) defines a rational transformation which coincides with \( \varphi_{|2K_X|} \) on an open set, so it defines a generically 2:1 transformation. Now let’s consider the non-global 6-canonical adjoint given by \( G_7 \) and let \( \mathcal{D} \) be the divisor on \( X \) defined by it. Note that \( \mathcal{D} = 6K_X \). Let \( \Sigma \) be the linear system

\[
\{ L^4(\lambda_0 F_0 + \cdots + \lambda_6 F_6) + \lambda_7 G_7 = 0, \ \lambda_j \in \mathbb{C} \},
\]

with \( F_0 = (X_0^{10} X_3^2) X_0 X_2, \quad F_1 = (X_0^{10} X_3^2) X_1^2, \quad F_2 = (X_0^{10} X_3^2) X_1 X_2, \quad F_3 = (X_0^{10} X_3^2) X_1^3, \quad F_4 = (X_0^{10} X_3^2) X_2 X_3, \quad F_5 = (X_0^{10} X_3^2) X_2 X_3, \quad F_6 = (X_0^{10} X_3^2) X_0^3.
\]

Note that \( F_0, \ldots, F_6 \) span \( W_{14} = \mathbb{W}_{14} \) and \( L^4 F_0, \ldots, L^4 F_6, G_7 \) span a vector subspace of \( \mathbb{W}_{14} \). We obtain \( 4K + |2K_X| \sigma |6K_X| \). The linear system \( \Sigma \) defines a rational transformation

\[
V \subset \mathbb{P}^4 \quad \rightarrow \quad \mathbb{P}^7
\]

\[
(X_0, X_1, X_2, X_3, X_4) \quad \mapsto \quad (Y_0, \ldots, Y_7)
\]

given by:

\[
\begin{align*}
Y_0 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_0 X_3 \\
Y_1 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_1^2 \\
Y_2 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_1 X_2 \\
Y_3 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_1 X_3 \\
Y_4 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_2^2 \\
Y_5 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_2 X_3 \\
Y_6 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_3^2 \\
Y_7 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2) X_4.
\end{align*}
\]

Let us now consider the open affine set \( U = \mathbb{P}^4 - \{ X_0 = X_1 = X_3 = 0 \} \) in \( \mathbb{P}^4 \) with the coordinates

\[
x = \frac{X_0}{X_1}, \quad y = \frac{X_2}{X_1}, \quad z = \frac{X_3}{X_1}, \quad t = \frac{X_4}{X_1}.
\]
and the open affine set $T = \mathbb{P}^7 - \{ Y_1 = Y_3 = 0 \}$ in $\mathbb{P}^7$ with the coordinates

$$ y_1 = \frac{Y_0}{Y_1}, \ldots, y_7 = \frac{Y_7}{Y_1}. $$

We obtain:

$$ \psi |_U : U \rightarrow T $$

$$(x, y, z, t) \mapsto (y_1, \ldots, y_7) : \begin{cases} y_1 = xz \\ y_2 = y \\ y_3 = z \\ y_4 = y^2 \\ y_5 = yz \\ y_6 = z^2 \\ y_7 = x^5 t. \end{cases} $$

$\psi |_U$ is 1:1. Indeed let $P_1(x_1, y_1, z_1, t_1)$ and $P_2(x_2, y_2, z_2, t_2)$ be two points on $U$ such that $\psi |_U(P_1) = \psi |_U(P_2)$, i.e.

$$ x_1 z_1 = x_2 z_2, \quad y_1 = y_2, \quad z_1 = z_2, \ldots, \quad x_1^5 t_1 = x_2^5 t_2. $$

From $y_1 = y_2$ and $z_1 = z_2$, it follows that $x_1 = x_2$ and finally that $t_1 = t_2$. This proves that $\psi$, so $\psi |_{6K_X}$ is birational.

The birationality of $\psi |_{6K_X}, m > 6$, follows from this last fact. Indeed, let us consider an effective canonical divisor $K$, and let us construct the linear systems $\mathcal{K} + | 6K_X | \subset | 7K_X |, \ 2 \mathcal{K} + | 6K_X | \subset | 8K_X |, \ldots$ All these linear systems give rise to rational transformations which are generically 1:1. So all the transformations $\psi |_{mK_X}, m \geq 6$, are birational.

**Remark 2.** Note that if we «delete» $y_7 = x^5 t$ in the expression of $\psi |_U : U \rightarrow T$, we obtain the $\psi_{2|U}$ of section 2.3. So we have obtained all the informations we need on the pluricanonical transformations only considering the linear system of bicanonical adjoints to $V$ and the non-global 6-canonical adjoint given by $X_0^3 X_1^6 X_4$.

**2.5. Irregularities of $V$.**

We have to show that the following two relations hold true:

$$ q_1(X) = \dim_{\mathbb{C}} H^1(X, \Omega_X) = 0, \quad q_2(X) = \dim_{\mathbb{C}} H^2(X, \Omega_X) = 0. $$

To do this, we use the arguments of [S2], section 4. We consider the surface of degree 12 $S = \sigma^{-1}(H \cap V)$, where $H$ is the generic hyperplane in $\mathbb{P}^4$. Since $A_6$ and $A_4$ are isolated singular points on $V$, then $H \cap V$, and so $S$, is nonsingular. Thus it is well known (and easy to see, cf. for instance
formula (36)), that $q(S) = 0$. We deduce from remark 8 that
\[ q_1(X) = q(S) = 0. \]

In addition from formula (36), we have
\[ q_2(X) = p_g(X) + p_g(S) - \dim C W_8, \]
where $W_8$ is the vector space of the forms defining global adjoints to $V$ in $\mathbb{P}^4$ of degree 8. Thus
\[ q_2(X) = 3 + 165 - 168 = 0. \]

This proves the statement.

REFERENCES


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