Periodicity in K-groups of Certain Fields.

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Abstract - Let \( k \) be a field of characteristic different from \( p \). We study the \( p \)-torsion and the \( p \)-cotorsion in the higher algebraic K-groups of \( k \). Under a certain hypothesis we find that these groups are periodic. Some (co)-descent properties are also pointed out.

1. Introduction.

Let \( k \) be a field of characteristic different from \( p \). In the main part of this paper we will assume that the \( p \)-cohomological dimension \( cd_p(k) \) of \( k \) is less than three. Additionally, we will assume that the group \( H^2_{et}(k; \mathbb{Q}_p/\mathbb{Z}_p(i)) \) is trivial for \( i \geq 2 \). For such a \( k \) we first prove some periodicity results for its algebraic K-groups. Second we discuss some (co)-descent properties for the same groups. These results are easily deduced from the Bloch-Lichtenbaum spectral sequence, denoted by BLSS from now on, with finite coefficients. We claim no originality whatsoever for this part. The BLSS for a field such as above resembles the BLSS for a complex surface. That example was first considered by Suslin [Su2].

There are several versions of the BLSS, cf. [BL], [FS], [Le2], [RW] and [We]. Assume \( k \) has characteristic zero. The mod \( p^\infty \) BLSS for \( k \) is a third quadrant cohomological spectral sequence with input the higher Chow groups of \( k \) with mod \( p^\infty \) coefficients, and abutment the mod \( p^\infty \) algebraic K-groups of \( k \). Suslin [Su3] has proved that the higher Chow groups of \( k \) are isomorphic to the motivic cohomology groups of \( k \). We let

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subscript \( M \) indicate motivic cohomology. From the mentioned results, the mod \( p^\nu \) BLSS for \( k \) takes the form:

\[
E_2^{m,n} = H_{M}^{m-n}(k; \mathbb{Z}/p^n) \Rightarrow K_{-m-n}(k; \mathbb{Z}/p^n).
\]

The outcome of Weibel’s valuation trick from [We] is a mod \( p^\nu \) BLSS for fields of positive characteristic. The idea is to replace \( k \) by a field \( F(k) \) of characteristic zero, and whose motivic cohomology and algebraic K-theory groups are naturally isomorphic to the same groups for \( k \). Assume \( k \) has positive characteristic \( l \), where \( l \neq p \). Define \( R_0(k) \) to be the Cohen \( l \)-ring of \( k \), and define inductively \( R_n(k) \) to be \( R_{n-1} \mathbb{Z}/p^n(t)/(t^l - \pi) \) where \( \pi \) is a uniformizing parameter for \( R_{n-1}(k) \) and \( n \geq 1 \). The quotient field of the union

\[
\text{colim}(R_0(k) \subset R_1(k) \subset R_2(k) \subset \ldots)
\]

has the desired properties of \( F(k) \).

Next we explain the relation between the motivic cohomology groups and the étale cohomology groups of \( k \). The Bloch-Kato conjecture [BK] at the prime \( p \) predicts that the Galois symbol

\[
K^M_n(F)/p^\nu \to H^n_{\text{ét}}(F; \mathbb{Z}/p^\nu(n))
\]

is an isomorphism for every field \( F \) of characteristic different from \( p \). Voevodsky proved this conjecture in [Vo] for the prime \( p = 2 \). For \( p = 2 \) the Bloch-Kato conjecture was originally formulated by Milnor [Mi]. Suslin and Voevodsky proved in [SV] that if the Bloch-Kato conjecture is true at the prime \( p \), then there exists natural isomorphisms

\[
H^p_{\text{ét}}(k; \mathbb{Z}/p^\nu(i)) \cong \begin{cases} H^n_{\text{ét}}(k; \mathbb{Z}/p^\nu(i)) & \text{for } 0 \leq n \leq i, \\ 0 & \text{otherwise}. \end{cases}
\]

By specialization we get the following result (for two groups \( A \) and \( B \) we let \( A \times B \) denote an Abelian extension of \( B \) by \( A \)).

**Theorem 1.1.** Assume \( cd_p(k) \leq 2 \). If \( p \) is an odd prime, we also assume that the Bloch-Kato conjecture holds at \( p \).

(a) The mod \( p^\nu \) algebraic K-groups of \( k \) are given up to extensions by

\[
K_n(k; \mathbb{Z}/p^\nu) = \begin{cases} H^n_{\text{ét}}(k; \mathbb{Z}/p^\nu(i)) & \text{for } n = 2i - 1, \\ H^n_{\text{ét}}(k; \mathbb{Z}/p^\nu(i + 1)) \times H^n_{\text{ét}}(k; \mathbb{Z}/p^\nu(i)) & \text{for } n = 2i > 0. \end{cases}
\]
(b) The extension above is split by the anti-Chern classes of Kahn if \( p \) is odd, or \( p = 2 \) and \( k \) contains a primitive fourth root of unity.

**Remark 1.2.** Part (b) of Theorem 1.1 is due to Kahn, see Theorem 3.1 in [Ka2]. The results from [FS] and [Le2] make it plain that Theorem 1.1, and hence some of the results in this paper may be generalized to certain schemes with mod \( p \) étale cohomological dimension less than three.

In Section 2 we prove results which appear to be new. For this we will only consider fields with the properties stated in the beginning of the introduction. The assumptions on \( k \) can often be checked in practice. Our results reveal a periodicity phenomena for the \( p \)-torsion and the \( p \)-cotorsion in the algebraic K-groups of such a field. The proofs are very elementary and straightforward. However, the results might be useful in specific examples. The same remarks apply to the results in Section 3. Let \( k' / k \) be a Galois extension of fields as above. In Proposition 3.3 we point out the connection between the Galois (co)-invariants of the algebraic K-groups of \( k' \) and the algebraic K-groups of \( k \).

### 2. Periodicity in K-groups.

Assume \( \text{cd}_p(k) \leq 2 \). Then the long exact sequence in étale cohomology induced by the coefficient extension \( 0 \to \mathbb{Z}/p(n) \to \mathbb{Q}_p/\mathbb{Z}_p(n) \to \mathbb{Q}_p/\mathbb{Z}_p(n) \to 0 \) shows that the group \( H^1_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) \) is divisible. We impose the additional assumption that the latter group is trivial for \( n \geq 2 \).

For an Abelian group \( A \) we let \( A(4p) \) be its maximal \( p \)-torsion subgroup. Let \( \overline{k} \) be an algebraic closure of \( k \).

First we translate the additional assumption into a statement about the K-groups of \( k \). Consider the diagram

\[
\begin{array}{ccc}
K_{2n}(k; \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & K_{2n}(%(k) \mathbb{Q}_p/\mathbb{Z}_p) \\
\beta \downarrow & & \downarrow \beta \\
K_{2n-1}(k; p) & \longrightarrow & K_{2n-1}(\overline{k}; p)
\end{array}
\]

where the vertical maps are the Bockstein maps. From Theorem 1.1; the upper horizontal map is injective, since it can be identified with the natural injective map \( H^1_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^1_{\text{ét}}(\overline{k}; \mathbb{Q}_p/\mathbb{Z}_p(n)) \). We know the
Bockstein map for $\overline{k}$ is an isomorphism from [Su]. Hence the Bockstein map for $k$ is an isomorphism, and it follows that $K_{2n}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is the trivial group for all $n \geq 1$. Note also that $K_{2n-1}(k)(p)$ injects into $K_{2n-1}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

The previous remarks combined with Theorem 1.1 give an isomorphism:

\[
(2.1) \quad H^0_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) \overset{\cong}{\rightarrow} K_{2n-1}(k)(p) .
\]

Let $e_n$ denote the exponent of the multiplicative group $(\mathbb{Z}/p^n)^*$, and let $\mu_n(k)$ denote the group of $n$th roots of unity in $k$.

**Lemma 2.2.** Let $m, n \geq 1$. Then $p^rK_{2n-1}(k)$ is isomorphic to $p^rK_{2n+me_n-1}(k)$ and there is an exact sequence

\[
(2.3) \quad 0 \rightarrow H^0_{\text{ét}}(k; \mathbb{Z}/p^n(n)) \rightarrow K_{2n-1}(k) \overset{p^r}{\rightarrow} K_{2n-1}(k) .
\]

In particular, the group $K_{2n-1}(k)$ contains an element of order $p^r$.

**Proof.** From (2.1) we find an isomorphism $H^0_{\text{ét}}(k; \mathbb{Z}/p^n(n)) \overset{\cong}{\rightarrow} H^0_{\text{ét}}(k; \mathbb{Z}/p^n(n)) \rightarrow K_{2n-1}(k)$. Now employ the Gal($k^*/k$)-module isomorphism $\mathbb{Z}/p^n(n) \cong \mathbb{Z}/p^n(n + e_n)$ where $k^*$ is a separable closure of $k$. The last claim follows from $p^rK_{2n-1}(k) \equiv p^rK_{2n-1}(k) \equiv H^0_{\text{ét}}(k; \mathbb{Z}/p^n(0))$ and the fact that the absolute Galois group of $k$ acts trivially on $\mathbb{Z}/p^n(0)$ by definition of the Tate twist.

**Remark 2.4.** If $k$ contains a primitive $p^r$th root of unity, then:

\[
\mu_{p^r}(k) \equiv p^rK_3(k) \equiv p^rK_5(k) \equiv \ldots
\]

This follows since $\mathbb{Z}/p^n(i)$ is independent of the twist $i$ under the given assumption.

We claim the Bockstein exact sequence in K-theory and Theorem 1.1 combine to make a commutative diagram:

\[
0 \rightarrow K_{2n}(k)/p^r \rightarrow K_{2n}(k; \mathbb{Z}/p^n) \rightarrow p^rK_{2n-1}(k) \rightarrow 0
\]

\[
0 \rightarrow H^0_{\text{ét}}(k; \mathbb{Z}/p^n(n + 1)) \rightarrow K_{2n}(k; \mathbb{Z}/p^n) \rightarrow H^0_{\text{ét}}(k; \mathbb{Z}/p^n(n)) \rightarrow 0
\]

For $\overline{k}$ there is a unique choice of isomorphism on the right hand side that
makes the diagram commutative. For \( k \) we choose the isomorphism that is compatible with the inclusion into \( \bar{k} \). This gives a natural isomorphism:

\[
K_2/k \rightarrow K_2/\mathbb{Z}/p^n(\mathbb{Z}/p^n(n + 1))
\]

**Lemma 2.6.** Let \( m, n \geq 1 \). Then \( K_2/k \rightarrow K_2/\mathbb{Z}/p^n(\mathbb{Z}/p^n(n + 1)) \) and there is an exact sequence

\[
K_2(k)/p^r \rightarrow K_{2n-2}(k) \rightarrow H^2_{\text{et}}(k; \mathbb{Z}/p^n(n)) \rightarrow 0.
\]

**Proof.** Given (2.5), the proof is a verbatim copy of the argument for Lemma 2.2. The periodicity can be decreased according to Remark 2.4.

The mod \( p^r \) Bockstein exact sequence in K-theory and Theorem 1.1 give the short exact sequence

\[
0 \rightarrow K_{2n-1}(k)/p^r \rightarrow H^1_{\text{et}}(k; \mathbb{Z}/p^n(n)) \rightarrow p^r K_{2n-2}(k) \rightarrow 0.
\]

The sequence (2.8) splits if \( n \) is a multiple of \( e \) and \( k \) is a number field which satisfies the assumptions in Theorem 1.1. These assumptions are satisfied unless \( k \) is real and \( p = 2 \), cf. Theorem 4.5 [RW]. Indeed, Lemma 2.2 shows that the mod \( p^r \) reduction of \( K_2(k) \) is a full subgroup of \( H^1_{\text{et}}(k; \mathbb{Z}/p^n(n)) \), hence a direct summand. These remarks motivate the following observation.

**Lemma 2.9.** If (2.8) splits for \( n \) and \( n + me \), then:

\[
K_{2n-1}(k)/p^r \oplus p^r K_{2n-2}(k) \cong K_{2(n + me) - 1}(k)/p^r \oplus p^r K_{2(n + me) - 2}(k).
\]

In particular, if \( K_{2n-1}(k)/p^r \) is finite and isomorphic to \( K_{2(n + me) - 1}(k)/p^r \), then \( p^r K_{2n-2}(k) \cong p^r K_{2(n + me) - 2}(k) \). Likewise, if \( p^r K_{2n-2}(k) \) is finite and isomorphic to \( p^r K_{2(n + me) - 2}(k) \), then \( K_{2n-1}(k)/p^r \cong K_{2(n + me) - 1}(k)/p^r \).

**Proof.** The first claim is clear from periodicity of \( H^1_{\text{et}}(k; \mathbb{Z}/p^n(n)) \). The remaining claims follow from the cancellation property of finite groups, see [Hi].

The exact sequences (2.3), (2.7) and (2.8) imply the next result.
THEOREM 2.10. Let \( n \geq 2 \). Then we have the exact sequence

\[
0 \rightarrow H^0_{/\mathbb{A}}(k; \mathbb{Z}/p^{n}((n))) \rightarrow K_{2n-1}(k) \rightarrow K_{2n-1}(k) \rightarrow H^1_{/\mathbb{A}}(k; \mathbb{Z}/p^{n}(n)) \rightarrow \]
\[
\rightarrow K_{2n-2}(k) \rightarrow K_{2n-2}(k) \rightarrow H^2_{/\mathbb{A}}(k; \mathbb{Z}/p^{n}(n)) \rightarrow 0 .
\]

REMARK 2.11. Sequence (2.11) inserted \( n = 2 \) and with \( K_2(k) \) replaced with its indecomposable part is known from [Le1] and [MS].

3. (Co)-descent.

Let \( k'/k \) be a Galois extension of fields with group \( \Gamma \). We keep the assumptions that \( \text{cd}_p(k) \leq 2 \) and \( H^2_{/\mathbb{A}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0 \) for all \( n \geq 2 \), and likewise for \( k' \). Consider the Hochschild-Serre spectral sequence

\[(3.1) \quad E_2^{s,t} = H^s(\Gamma, H^t_{/\mathbb{A}}(k', \mathbb{Q}_p/\mathbb{Z}_p(n))) \Rightarrow H^{s+t}_{/\mathbb{A}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) \]

and the Tate spectral sequence:

\[(3.2) \quad E_2^{-s,t} = H_s(\Gamma, H^t_{/\mathbb{A}}(k', \mathbb{Q}_p/\mathbb{Z}_p(n))) \Rightarrow H^{s+t}_{/\mathbb{A}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)).\]

Here (3.1) is a first quadrant cohomological spectral sequence. Moreover, (3.2) is discussed in Chapter I Appendix 1 [Se] and in Proposition 3.1.1 [Ka1]. This is a second quadrant cohomological spectral sequence. The following result is now trivial to prove.

PROPOSITION 3.3. Let \( M^q \) denote \( H^q_{/\mathbb{A}}(k'; \mathbb{Q}_p/\mathbb{Z}_p(n)) \), \( n \geq 2 \). We have the exact sequences

\[
0 \rightarrow H^1(\Gamma, M^0) \rightarrow K_{2n-1}(k; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^2(\Gamma, M^0) \rightarrow 0
\]

and:

\[
0 \rightarrow H^2(\Gamma, M^1) \rightarrow K_{2n-2}(k; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(\Gamma, M^1) \rightarrow 0 .
\]

In addition we have the naturally induced isomorphisms

\[
K_{2n-2}(k; \mathbb{Q}_p/\mathbb{Z}_p) \stackrel{\sim}{\rightarrow} K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p)
\]

and:

\[
K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p) \stackrel{\sim}{\rightarrow} K_{2n-1}(k; \mathbb{Q}_p/\mathbb{Z}_p) .
\]
The $d^2$-differentials in (3.1) and (3.2) give isomorphisms

$$H^q(\Gamma, K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)) \xrightarrow{\cong} H^{q+2}(\Gamma, K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p))$$

and

$$H^{q+2}(\Gamma, K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)) \xrightarrow{\cong} H_q(\Gamma, K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p))$$

for all $q \geq 1$.

**Remark 3.4.** It follows that $K_{2n-1}(k)\{p\} \xrightarrow{\cong} K_{2n-1}(k')\{p\}$, and the transfer map induces a surjection $K_{2n-2}(k')\{p\}_T \rightarrow K_{2n-2}(k)\{p\}$. That surjection is an isomorphism if $K_{2n-2}(k')\{p\}$ is reduced. The first claim follows from the diagram displayed in the beginning of Section 2, and the second claim follows from an obvious Bockstein sequence argument.

**References**


