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Groups in which certain equations have many solutions

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GÉRARD ENDIMIONI (*)

1. Introduction.

Let $w(x_1, \ldots, x_n)$ be a word in the free group of rank n and let G be a group. There are several ways to mean that the equation $w(x_1, \ldots, x_n) = 1$ has «many» solutions in G. Here we adopt a combinatorial point of view and we define the class of groups $\mathfrak{V}_\infty(w)$ like this:

A group G belongs to $\mathfrak{V}_\infty(w)$ if and only if every infinite subset of G contains n (distinct) elements $x_1, \ldots, x_n$ such that $w(x_1, \ldots, x_n) = 1$.

Following a question of P. Erdös, this class appeared in a paper of B. H. Neumann [6], where he proved that if $w(x_1, x_2) = [x_1, x_2]$, then $\mathfrak{V}_\infty(w)$ coincide with the class of central-by-finite groups. Since this first paper, several authors have studied $\mathfrak{V}_\infty(w)$. For example, characterizations of finitely generated soluble groups of $\mathfrak{V}_\infty(w)$ are known when $w(x_1, x_2) = [x_1, x_2, x_2]$ [5] or when $w(x_1, x_2) = [x_1, x_2, x_2, x_2]$ [1].

In this paper, we consider the word $w(x_1, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonzero integers. Related to this question, but with a stronger condition, A. Abdollahi and B. Taeri proved that if for every n infinite subsets $X_1, \ldots, X_n$ of an infinite group G, there exist elements $x_1 \in X_1 \ldots, x_n \in X_n$ such that $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 1$, then $x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 1$ is a law in G [2]. On the other hand, by using a construction of Ol'shanskii, these authors showed that for any sufficiently large prime n, there exists

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an infinite group in $\mathfrak{V}_\infty(x_1^n \dots x_n^n)$ in which $x_1^n \dots x_n^n = 1$ is not a law (we shall see other examples in the next section). The aim of this paper is to characterize the groups of $\mathfrak{V}_\infty(x_1^{\alpha_1} \dots x_n^{\alpha_n})$.

2. – Results.

We denote by $\mathcal{F}$ the class of finite groups and by $\mathcal{B}_e$ the variety of groups satisfying the law $x^e = 1$ (for a given integer e).

Let m be a positive integer. By analogy with $\mathfrak{V}_\infty(w)$, we define the class $\mathfrak{V}_m(w)$ in the following way: a group G belongs to $\mathfrak{V}_m(w)$ if and only if every m-element subset of G contains n distinct elements $x_1, \dots, x_n$ such that $w(x_1, \dots, x_n) = 1$. Clearly, the classes $\mathfrak{V}_m(w)$ and $\mathcal{F}$ are included in $\mathfrak{V}_\infty(w)$.

From now on, we put $w(x_1, \dots, x_n) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\alpha_1, \dots, \alpha_n$ are nonzero given integers; we write α for the greatest common divisor of these integers. Observe that the variety defined by the law $w(x_1, \dots, x_n) = 1$ is equal to the variety $\mathcal{B}_\alpha$. Here we prove that as one might expect, the classes $\mathfrak{V}_\infty(w)$ and $\mathcal{B}_\alpha$ do not coincide but are relatively close:

THEOREM. *Let G be an infinite group. The following assertions are equivalent:*

(i) $G \in \mathfrak{V}_\infty(w)$;

(ii) $G \in \mathcal{B}_{(\alpha_1 + \dots + \alpha_n)} \cap (\mathcal{F}\mathcal{B}_\alpha)$;

(iii) $G \in \mathfrak{V}_m(w)$ *for some positive integer* m.

It follows immediately:

COROLLARY 1. *We have the equalities*

$$\mathfrak{V}_\infty(w) = \bigcup_{m > 0} \mathfrak{V}_m(w) = \mathcal{F} \cup (\mathcal{B}_{(\alpha_1 + \dots + \alpha_n)} \cap (\mathcal{F}\mathcal{B}_\alpha)).$$

For example, for any fixed integer $e \geq 2$, denote by H a cyclic group of order e^2 and by K the direct product of infinitely many cyclic groups of order e. Let G be the direct product of H and K. It is easy to see directly that G belongs to $\mathfrak{V}_{e+1}(x_1^e x_2^{-e})$ and so to $\mathfrak{V}_\infty(x_1^e x_2^{-e})$ (also it is a consequence of our theorem above). However, $x_1^e x_2^{-e} = 1$ is not a law in G. Notice that contrary to the example given in [2] and quoted above, G is not finitely generated. In fact, when α is «small», finitely generated groups in $\mathcal{V}_\infty(w)$ are finite. More precisely, since the Burnside problem has a positive answer when the exponent belongs to $\{1, 2, 3, 4, 6\}$ (that is, every group of $\mathcal{B}_\alpha$ is locally finite when $\alpha \in \{1, 2, 3, 4, 6\}$), we may state:

COROLLARY 2. *If* $\alpha \in \{1, 2, 3, 4, 6\}$, *every finitely generated group in* $\mathcal{V}_\infty(w)$ *is finite.*

Also, notice that $\mathcal{V}_\infty(w) = \mathcal{F}$ if $\alpha = 1$; this improves Corollary 2 of [3].

3. - Proofs.

We start with a key result for the proof of the theorem:

LEMMA 1. *Let n be a positive integer and let $a_1, \ldots, a_n$ be elements of an infinite group G. Let $\alpha_1, \ldots, \alpha_n$ be nonzero integers. Suppose that G contains an infinite subset E satisfying the following property: each infinite subset $E' \subseteq E$ contains n (distinct) elements $x_1, \ldots, x_n$ such that $a_1 x_1^{\alpha_1} \ldots a_n x_n^{\alpha_n} = 1$. Then:*

(i) there exist an infinite subset $F \subseteq E$ and elements $c_1, \ldots, c_n$ of G such that, for each $i \in \{1, \ldots, n\}$, we have $x^{\alpha_i} = c_i$ for all $x \in F$;

(ii) *there exists an element c of G such that $x^{\alpha} = c$ for all $x \in F$, where $\alpha = \gcd(\alpha_1, \ldots, \alpha_n)$.*

PROOF. (i) We argue by induction on n. First suppose that $n = 1$. It follows from hypothesis of the lemma that the set $\{x \in E \mid a_1 x^{\alpha_1} \neq 1\}$ is finite. Thus we can conclude by taking $F = \{x \in E \mid a_1 x^{\alpha_1} = 1\}$ and $c_1 = a_1^{-1}$. Now suppose that the result is true for $n - 1$ $(n > 1)$. For any set X, we denote by $P_n(X)$ the set of subsets of X containing n elements and by S_n the set of all permutations of $\{1, \ldots, n\}$. Let E_1 be the set of subsets $\{x_1, \ldots, x_n\} \in P_n(E)$ such that $a_1 x_{\sigma(1)}^{\alpha_1} \ldots a_n x_{\sigma(n)}^{\alpha_n} = 1$ for some permutation $\sigma \in S_n$. Put $E_2 = P_n(E) \backslash E_1$. By Ramsey's Theorem, there exists an infinite subset $X \subseteq E$ such that $P_n(X) \subseteq E_1$ or $P_n(X) \subseteq E_2$. However, the second inclusion is in contradiction with the hypothesis of the lemma, so $P_n(X) \subseteq E_1$. Let $\{y_1, \ldots, y_{n-1}\}$ be a fixed element of $P_{n-1}(X)$. Then, for each $y = y_n$ in $X \backslash \{y_1, \ldots, y_{n-1}\}$, choose a permutation $f(y) = \sigma$ of $\{1, \ldots, n\}$ such that $a_1 y_{\sigma(1)}^{\alpha_1} \ldots a_n y_{\sigma(n)}^{\alpha_n} = 1$ and consider the mapping $f : X \backslash \{y_1, \ldots, y_{n-1}\} \to S_n$. By the pigeonhole principle, there exists a permutation σ of S_n such that $f^{-1}(\sigma)$ is infinite; put $k = \sigma^{-1}(n)$. Then, for all y in $f^{-1}(\sigma)$, we have $a_1 y_{\sigma(1)}^{\alpha_1} \ldots a_k y^{\alpha_k} \ldots a_n y_{\sigma(n)}^{\alpha_n} = 1$. Therefore, the elements $y_1, \ldots, y_{n-1}$ being fixed in X, y^{α_k} is constant on $f^{-1}(\sigma)$. Put $c_k = y^{\alpha_k}$ for $y \in f^{-1}(\sigma)$. Clearly, it follows from the hypothesis of the lemma that each infinite subset $E' \subseteq f^{-1}(\sigma)$ contains $n-1$ distinct elements $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ such that

$$a_1 x_1^{\alpha_1} \ldots a_{k-1} x_{k-1}^{\alpha_{k-1}} a'_{k+1} x_{k+1}^{\alpha_{k+1}} \ldots a_n x_n^{\alpha_n} = 1 \text{ (with } a'_{k+1} = a_k c_k a_{k+1})$$

if $k < n$, and such that

$$a'_1 x_1^{\alpha_1} a_2 x_2^{\alpha_2} \ldots a_{n-1} x_{n-1}^{\alpha_{n-1}} = 1 \text{ (with } a'_1 = a_n c_n a_1)$$

if $k = n$. By induction, there exist an infinite subset $F \subseteq f^{-1}(\sigma)$ and elements $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ of G such that, for each $i \in \{1, \ldots, k-1, k+1, \ldots, n\}$, we have $x^{\alpha_i} = c_i$ for all $x \in F$. Since $x^{\alpha_k} = c_k$ for all $x \in F$, the property is proved.

(ii) Let $\beta_1, \ldots, \beta_n$ be integers such that $\alpha = \beta_1 \alpha_1 + \ldots + \beta_n \alpha_n$. For all $x \in F$, we have $x^{\alpha} = x^{\beta_1 \alpha_1} \ldots x^{\beta_n \alpha_n} = c_1^{\beta_1} \ldots c_n^{\beta_n}$, as required. ■

Recall that in the following, we have

$$w(x_1, \ldots, x_n) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \quad \text{and} \quad \alpha = \gcd(\alpha_1, \ldots, \alpha_n).$$

Furthermore, we put $\alpha'_i = \alpha_i \alpha^{-1}$ for $i = 1, \ldots, n$.

LEMMA 2. *For each group $G \in \mathfrak{V}_\infty(w)$, the set $\{x^\alpha\}_{x \in G}$ is finite.*

PROOF. Since the result is trivial if G is finite, we can assume that G is infinite. Suppose that the set $\{x^\alpha\}_{x \in G}$ is infinite. Clearly, in this case, there exists an infinite subset $E \subseteq G$ such that $x^\alpha \neq y^\alpha$ for each pair $\{x, y\}$ of elements of E. By applying Lemma 1(ii) to G (with $a_1 = \ldots = a_n = 1$), we obtain a contradiction. ■

LEMMA 3. *Let G be a group in $\mathfrak{V}_\infty(w)$. Suppose that the set $C = \{x \in G \mid x^\alpha = c\}$ is infinite for some $c \in G$. Then $c^{\alpha'_1 + \ldots + \alpha'_n} = 1$.*

PROOF. There exist n elements $x_1,\dots,x_n \in C$ such that $w(x_1,\dots,x_n)=1$. Since

$$w(x_1, \dots, x_n) = x_1^{\alpha\alpha'_1} \dots x_n^{\alpha\alpha'_n} = c^{\alpha'_1} \dots c^{\alpha'_n} = c^{\alpha'_1 + \dots + \alpha'_n},$$

we obtain $c^{\alpha'_1 + \dots + \alpha'_n} = 1$. ■

PROOF OF THE THEOREM. (i)→(ii). Let G be an infinite group in $\mathfrak{V}_\infty(w)$. By Lemma 2, the set $\{x^\alpha\}_{x\in G}$ is finite. Clearly, this implies that G is periodic. Thus, by Dicman's Lemma, the subgroup generated by $\{x^\alpha\}_{x\in G}$ is finite and so G belongs to $\mathcal{F}\mathcal{B}_\alpha$.

Now consider an element c_i in the set $\{x^\alpha\}_{x\in G} = \{c_1, \dots, c_t\}$ and put $C_i = \{x \in G \mid x^\alpha = c_i\}$. For all $x \in C_i$, we have

$$x^{\alpha_1 + \dots + \alpha_n} = x^{\alpha(\alpha'_1 + \dots + \alpha'_n)} = c_i^{\alpha'_1 + \dots + \alpha'_n}.$$

It follows from Lemma 3 that $x^{\alpha_1 + \dots + \alpha_n} = 1$ whenever C_i is infinite. Since $C_1, \dots, C_t$ is a partition of G, the set $\{x \in G \mid x^{\alpha_1 + \dots + \alpha_n} \neq 1\}$ is finite. This implies that G belongs to the class $\mathfrak{V}_\infty(x^{\alpha_1 + \dots + \alpha_n})$. In fact, as it is observed in [4], $\mathfrak{V}_\infty(x^{\alpha_1 + \dots + \alpha_n}) = \mathcal{F} \cup \mathcal{B}_{(\alpha_1 + \dots + \alpha_n)}$ and so $G \in \mathcal{B}_{(\alpha_1 + \dots + \alpha_n)}$.

(ii)→(iii). Let H be a normal subgroup of G such that $H \in \mathcal{F}$ and $G/H \in \mathcal{B}_\alpha$. Put $m = 1 + (n-1)|H:\{1\}|$ and show that G belongs to $\mathfrak{V}_m(w)$. Let E be a subset of G containing m elements. The function $x \to x^\alpha$ maps each element of E into an element of H; thus there exists an element $c \in H$ such that the set $\{x \in E \mid x^\alpha = c\}$ contains at least n elements. Consider n distinct elements $x_1, \dots, x_n \in \{x \in E \mid x^\alpha = c\}$. We have:

$$\begin{aligned} w(x_1, \dots, x_n) &= x_1^{\alpha\alpha'_1} \dots x_n^{\alpha\alpha'_n} = c^{\alpha'_1} \dots c^{\alpha'_n} \\ &= c^{\alpha'_1 + \dots + \alpha'_n} = x_1^{\alpha(\alpha'_1 + \dots + \alpha'_n)} \\ &= x_1^{\alpha_1 + \dots + \alpha_n} = 1, \end{aligned}$$

for $G \in \mathcal{B}_{(\alpha_1 + \dots + \alpha_n)}$. Thus we have proved that G belongs to $\mathfrak{V}_m(w)$.

Since clearly (iii) implies (i), the proof is complete. ■

We finish with a question of combinatorial nature:

Suppose that G is an infinite group in $\mathfrak{V}_\infty(w)$, where w is now an arbitrary word. Does G belong to $\mathfrak{V}_m(w)$ for some integer m?

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