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Prescribed Gauss Decompositions for Kac-Moody Groups Over Fields.

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ABSTRACT - We obtain the Gauss decomposition with prescribed torus elements for a Kac-Moody group over a field containing sufficiently many elements.

1. Introduction.

Kac-Moody groups are equipped with canonical decompositions of different types. Let us note, for instance, the decompositions of Bruhat, Birkhoff and Gauss. As in the finite dimensional case, they play an important role in calculations with these groups. However, in the Kac-Moody case each of these decompositions has its own special features (see, for example, [18] where J. Tits compares the Bruhat and the Birkhoff decompositions and presents some applications).

The aim of this paper is to establish the so-called prescribed Gauss decomposition for Kac-Moody groups.

The prescribed Gauss decomposition appeared in [16], [9] for the general linear group. It was proved for all Chevalley and twisted Chevalley groups in [3], [4], [5]. It turned out to be the main tool for proving the substantial Ore and Thompson conjectures.

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Namely, Ore [12] conjectured that every element of a finite simple group is a single commutator.

The proof of this statement for all simple groups of Lie type is given in [6]. Similar facts are also known for infinite simple groups. Let us mention the paper of Ree [14], who proved that every element of a connected semisimple algebraic group over an algebraically closed field is a commutator. A survey on the Ore problem (not including the results of Ellers-Gordeev and Lev) can be found in [20].

This paper is the continuation of [11], where the prescribed Gauss decomposition was established for Kac-Moody groups of rank 2. The general result became possible due to the recent paper [2] where the elegant idea of V. Chernousov gave rise to a uniform proof of the prescribed Gauss decomposition for all groups of Lie type. We mostly follow the method of this paper, adjusting it to the Kac-Moody case.

We do not consider in this paper Ore and Thompson type conjectures for Kac-Moody groups. These groups are perfect but, generally speaking, not necessarily simple, and their commutator structure can be very delicate.

2. – Kac-Moody groups.

Let $A = (a_{ij})$ be an $n \times n$ generalized Cartan matrix. Let \mathfrak{g} be the Kac-Moody Lie algebra over the field \mathbb{C} defined by A according to the choice of a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A with $\Pi = \{\alpha_1, \dots, \alpha_n\}$, the set of simple roots, and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, the set of simple coroots, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ (cf. [7], [10]). Let $\Delta \subset \mathfrak{h}^*$ be the root system of \mathfrak{g} with respect to the so-called Cartan subalgebra \mathfrak{h} . Let Δ_+ (resp. Δ_-) be the set of positive (resp. negative) roots defined by Π , and Δ^{re} the set of real roots. Set $\Delta_\pm^{\text{re}} = \Delta_\pm \cap \Delta^{\text{re}}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad (\text{root space decomposition})$$

and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \quad (\text{triangular decomposition}),$$

where $\mathfrak{g}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$.

Let \mathcal{G} be a Chevalley-Tits group functor from the category of commutative rings with 1 to the category of groups, corresponding to \mathfrak{g} (cf. [19]). For any commutative ring R with 1 and for any $\alpha \in \Delta^{\text{re}}$, there exists

a group homomorphism $x_\alpha(\cdot): R^+ \rightarrow \mathfrak{G}(R)$, where R^+ is the additive group of R . Using a Chevalley basis $\{e_\alpha \mid \alpha \in \Delta^{re}\}$ for Δ^{re} (cf. [19]), we can express $x_\alpha(t) = \exp(te_\alpha)$ for $t \in R$. Let $G(R)$ be the subgroup of $\mathfrak{G}(R)$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta^{re}$ and $t \in R$. We call $G(R)$ a (standard or elementary) Kac-Moody group, which depends on the choice of a root datum and here one can choose any root datum (cf. [10], [13], [15], [17], [19]). Sometimes $G(R)$ is said to be of type A . From now on, we assume that $R = K$ is a field and shorten $G(K)$ to G . We also suppose $x_\alpha(t) \neq 1$ for all $\alpha \in \Delta^{re}$ and $t \in K$ ($t \neq 0$). Let

$$w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t),$$

$$h_\alpha(t) = w_\alpha(t) w_\alpha(-1)$$

for $\alpha \in \Delta^{re}$, $t \in K^\times = K \setminus \{0\}$. Set

$$U = \langle x_\alpha(a) \mid \alpha \in \Delta_+^{re}, a \in K \rangle,$$

$$T = \langle h_\alpha(t) \mid \alpha \in \Delta^{re}, t \in K^\times \rangle,$$

$$V = \langle x_\alpha(a) \mid \alpha \in \Delta_-^{re}, a \in K \rangle.$$

For $i = 1, \dots, n$ we define homomorphisms $\phi_i: \text{SL}_2(K) \rightarrow G$ by

$$\phi_i: \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\alpha_i}(a),$$

$$\phi_i: \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto h_{\alpha_i}(t),$$

$$\phi_i: \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto x_{-\alpha_i}(a)$$

for all $a \in K$ and $t \in K^\times$. Let U_i, V_i be the root subgroups corresponding to the roots α_i and $-\alpha_i$, respectively. The subgroups U'_i and V'_i are defined as follows:

$$U'_i = \langle x_{\alpha_i}(s) x_\beta(t) x_{\alpha_i}(-s) \mid s, t \in K, \beta \in \Delta_+^{re} \setminus \{\alpha_i\} \rangle,$$

$$V'_i = \langle x_{-\alpha_i}(s) x_\beta(t) x_{-\alpha_i}(-s) \mid s, t \in K, \beta \in \Delta_-^{re} \setminus \{-\alpha_i\} \rangle.$$

Then $(G, U, T, V, \{\phi_1, \dots, \phi_n\})$ is a triangular system (see [11] for the definition). Hence (cf. [11]), every Kac-Moody group G over a field has a

Gauss decomposition, i.e.,

$$\begin{aligned} G &= UVTU \\ &= \bigcup_{u \in U} u(VTU) u^{-1}. \end{aligned}$$

Set

$$N = \langle w_\alpha(t) \mid \alpha \in \Delta^{re}, t \in K^\times \rangle,$$

then $T \triangleleft N$, and N/T is isomorphic to the Weyl group W (cf. [7], [10]). We sometimes identify W with N/T . For $w \in W$, we write

$$wUw^{-1} = \bar{w}U\bar{w}^{-1},$$

with $w = \bar{w}T$ and $\bar{w} \in N$, where w can also be identified with an inverse image \bar{w} of w under canonical epimorphism $N \rightarrow N/T$. Then

$$\bigcap_{w \in W} wUw^{-1} = 1$$

and

$$\bigcap_{w \in W} wVw^{-1} = 1$$

(cf.[8], [13]). Therefore, if $1 \neq u \in U$ (resp. $1 \neq v \in V$), then there exists $\bar{w} \in N$ such that

$$\begin{aligned} \bar{w}u\bar{w}^{-1} &= u_i u'_i, \quad u_i \neq 1 \\ \text{(resp. } \bar{w}v\bar{w}^{-1} &= v'_i v_i, \quad v_i \neq 1), \end{aligned}$$

for some $i = 1, 2, \dots, n$, where $u_i \in U_i, u'_i \in U'_i$ (resp. $v_i \in V_i, v'_i \in V'_i$). Otherwise $\bar{w}u\bar{w}^{-1} \in U'_i$ (resp. $\bar{w}v\bar{w}^{-1} \in V'_i$) for all $w \in W$ and all $i = 1, \dots, n$, which automatically means $u \in \bigcap_{w \in W} wUw^{-1}$ (resp. $v \in \bigcap_{w \in W} wVw^{-1}$) and $u = 1$ (resp. $v = 1$). The above property is important for us, and we will use it later.

For a subset X of Π , we denote by Δ_X the subroot system of Δ generated by X , and by Δ_X^{re} the subset of real roots generated by X . We defi-

ne the following subgroups:

$$\begin{aligned} G_X &= \langle x_{\pm\alpha}(t) \mid \alpha \in X, t \in K \rangle, \\ T_X &= \langle h_\alpha(t) \mid \alpha \in X, t \in K^\times \rangle, \\ T'_X &= \langle h_\alpha(t) \mid \alpha \in \Pi \setminus X, t \in K^\times \rangle, \\ U_X &= \langle x_\alpha(t) \mid \alpha \in \Delta_+ \cap \Delta_X^{re}, t \in K \rangle, \\ U'_X &= \langle yx_\beta(t) y^{-1} \mid \beta \in \Delta_+^{re} \setminus \Delta_X^{re}, t \in K, y \in G_X \rangle, \\ V_X &= \langle x_\alpha(t) \mid \alpha \in \Delta_- \cap \Delta_X^{re}, t \in K \rangle, \\ V'_X &= \langle yx_\beta(t) y^{-1} \mid \beta \in \Delta_-^{re} \setminus \Delta_X^{re}, t \in K, y \in G_X \rangle. \end{aligned}$$

Let $\sigma_{\alpha_i} = w_{\alpha_i}(1)T$ in $W = N/T$, and identify σ_{α_i} with $w_{\alpha_i}(1)$ if necessary.

- LEMMA. (1) $G_X \cap U = U_X$.
 (2) $\sigma_{\alpha_i}(U_X \cap U'_i) \sigma_{\alpha_i}^{-1} \subset U_X$ for every $\alpha_i \in X$.
 (3) $U = U_X U'_X \triangleright U'_X$.
 (4) $V = V_X V'_X \triangleright V'_X$.
 (5) $T = T_X T'_X$.

PROOF. (1): Let $g \in G_X \cap U$. Then, by the Bruhat decomposition of G_X , we write $g = uhv$ with $u, v \in U_X, h \in T_X$ and $w \in W_X \subset W$, where W_X is the parabolic subgroup of W corresponding to X . Using the Bruhat decomposition of G , we get $w = 1$, which implies that g can be written as $g = uh$ for some $u \in U_X$ and $h \in T_X$. Therefore $h = 1$ since $U \cap T = 1$. Hence, $g = u \in U_X$. Clearly $U_X \subset G_X \cap U$. Therefore, $G_X \cap U = U_X$.

(2): Let $z \in U_X \cap U'_i$. Then

$$\begin{aligned} \sigma_{\alpha_i} z \sigma_{\alpha_i}^{-1} &\in G_X \cap U'_i \\ &\subset G_X \cap U = U_X. \end{aligned}$$

(3): Let $U''_X = \langle yx_\beta(t) y^{-1} \mid \beta \in \Delta_+^{re} \setminus \Delta_X^{re}, t \in K, y \in U_X \rangle$. Then $U = U_X U''_X \triangleright U''_X$. To show that $U''_X = U'_X$, it is enough to verify that U''_X is σ_{α_i} -invariant since $G_X = \langle U_X, T_X, \sigma_{\alpha_i} (\alpha_i \in X) \rangle$. Let $\alpha_i \in X, \beta \in \Delta_+^{re} \setminus \Delta_X^{re}, t \in K$, and $y \in U_X$. We write $y = z_i x_{\alpha_i}(s)$ with $z_i \in U'_i$ and $s \in K$.

(a) In the case when $s = 0$,

$$\begin{aligned} & \sigma_{\alpha_i}(yx_\beta(t) y^{-1}) \sigma_{\alpha_i}^{-1} \\ &= (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1}) x_{\sigma_{\alpha_i} \beta}(\pm t) (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1})^{-1} \\ &\in U_X'', \end{aligned}$$

since

$$\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1} \in U_X.$$

(b) In the case when $s \neq 0$ and $\beta(\alpha_i^\vee) \geq 0$, where $\beta(\alpha_i^\vee)$ means the value of the linear function $\beta \in \mathfrak{h}^*$ at $\alpha_i^\vee \in \mathfrak{h}$,

$$\begin{aligned} & \sigma_{\alpha_i}(yx_\beta(t) y^{-1}) \sigma_{\alpha_i}^{-1} \\ &= (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1}) \sigma_{\alpha_i} x' \sigma_{\alpha_i}^{-1} (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1})^{-1} \\ &\in U_X'', \end{aligned}$$

where $x' = x_\beta(t)$ or $x' = x_\beta(t) x_{\beta+\alpha_i}(t')$ for some $t' \in K$.

(c) In the case when $s \neq 0$ and $\beta(\alpha_i^\vee) < 0$,

$$\begin{aligned} & \sigma_{\alpha_i}(yx_\beta(t) y^{-1}) \sigma_{\alpha_i}^{-1} \\ &= (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1}) x_{-\alpha_i}(s') x_{\sigma_{\alpha_i} \beta}(t') x_{-\alpha_i}(-s') (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1})^{-1} \\ &= (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1}) x_{\alpha_i}(s'') \sigma_{\alpha_i} h_{\alpha_i}(u) x_{\alpha_i}(s'') x_{\sigma_{\alpha_i} \beta}(t') x_{\alpha_i}(-s'') \\ & \quad h_{\alpha_i}(u)^{-1} \sigma_{\alpha_i}^{-1} x_{\alpha_i}(-s'') (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1})^{-1} \\ &= (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1}) x_{\alpha_i}(s'') \sigma_{\alpha_i} x'' \sigma_{\alpha_i}^{-1} x_{\alpha_i}(-s'') (\sigma_{\alpha_i} z_i \sigma_{\alpha_i}^{-1})^{-1} \\ &\in U_X'' \end{aligned}$$

for some $t', s', s'' \in K$ and $u \in K^\times$, where $x'' = x_{\sigma_{\alpha_i} \beta}(t'')$, or $x'' = x_{\alpha_i \beta}(t'')$ $x_{(\sigma_{\alpha_i} \beta) + \alpha_i}(t''')$ with $t'', t''' \in K$.

(4) can be proved in the same way as in (3), and (5) is obvious. Q.E.D.

Let $Z(G)$ be the center of the group G . Note that $Z(G) \subset T$, and we can explicitly describe it as follows (cf. [15], [19]):

$$Z(G) = \left\{ \prod_{i=1}^n h_{\alpha_i}(t_i) \mid \prod_{i=1}^n t_i^{\beta(\alpha_i^\vee)} = 1 \text{ for all } \beta \in \Pi \right\}.$$

We make a short review here. Let $g \in Z(G)$. Using the Bruhat decomposition $G = UNU$, one can write $g = uhwx$ with $u \in U$, $h \in T$, $w \in W$, $x \in U \cap w^{-1}Vw$, which has a unique expression. If $w \neq 1$, then we choose $\beta \in \Delta_+$ such that $w(\beta) \in \Delta_-$. Then $gx_\beta(1) = x_\beta(1)g$, and we obtain a contradiction by the uniqueness of expressions. In particular, $w = 1$ and $g \in UT$. Similarly we see $g \in VT$. Hence, we obtain $Z(G) \subset T = UT \cap VT$. Considering the action of T on the subgroups $U_1, \dots, U_n, V_1, \dots, V_n$, it is now easy to see that the center $Z(G)$ can be expressed as above. We notice that we need not assume «simply connectedness» here.

3. - Theorems.

We shall say that a Chevalley or Kac-Moody group G has the prescribed Gauss decomposition if given an arbitrary element $h^* \in T$, we have

$$G = Z(G) \cup \bigcup_{g \in G} g(Vh^*U)g^{-1},$$

where $Z(G)$ is the center of G ([3], [11]).

The result on the prescribed Gauss decomposition for Kac-Moody groups of rank two is as follows.

THEOREM A ([11]). Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a generalized Cartan matrix with $ab \geq 4$. Let $m = \max\{a, b\}$. Let K be a field with $|K| > m + 3$. Then every Kac-Moody group G over K of type A has a Gauss decomposition with prescribed elements in T .

A similar result holds for $0 \leq ab \leq 3$ in the rank two case without any restrictions on the cardinality of K . Moreover, in the case of Chevalley groups, there is a general result on the Gauss decomposition with prescribed semisimple elements.

THEOREM B ([2]). Let A be a (finite) Cartan matrix, and let K be a field. Then every Chevalley group G over K of type A has a Gauss decomposition with prescribed elements in T .

In the remaining part of this paper, we will establish the following main result on Kac-Moody groups G . For a generalized Cartan matrix $A = (a_{ij})$, we set

$$m = \max\{|a_{ij}| \mid (1 \leq i \neq j \leq n)\}.$$

MAIN THEOREM. Suppose $|K| > m + 3$. Then every Kac-Moody group G over K of type A has the Gauss decomposition with prescribed elements in T .

COROLLARY. Every noncentral element of a Kac-Moody group G can be expressed as a product of two unipotent elements in G .

4. - An inductive method.

Here we will show the following proposition. Denote $I = \{1, 2, \dots, n\}$.

PROPOSITION. Let $\Gamma = \langle \tau, G \rangle$ be a group with G normal in Γ and such that the conjugation $g \mapsto \tau g \tau^{-1}$ is a diagonal automorphism of G . Let $Z(\Gamma)$ be the center of Γ . Suppose $|K| > m + 3$. Then for every element $\tau g \in \Gamma$ with $g \in G$ and $\tau g \notin Z(\Gamma)$, and every element $h^* = \prod_{i=1}^n h_{\alpha_i}(t_{\alpha_i}^*) \in T$, there exists $z \in G$ such that

$$z(\tau g) z^{-1} = \tau(vh^*u)$$

for some $v \in V$ and $u \in U$.

This proposition can be proved in exactly the same way as in [2]. We proceed by induction on n . It is already known that the Proposition holds for $n = 1, 2$ (cf. [2], [11]), which is precisely described in Theorem A and Theorem B. Now we suppose $n \geq 3$. To make our induction complete, we have to assume that the cardinality of K is greater than $m + 3$, since we essentially use the information in case of rank two.

Take $\tau g \in \Gamma$ with $g \in G$. Since $\tau U = U\tau$ and $G = UV\tau U$, we have $\tau G = U\tau V\tau U$. Hence, for our purpose, we can assume that τg is of the form $\tau g = \tau v h u$ with $v \in V, h \in T, u \in U$. Then, we fall into one of the following three cases, using the fact $\prod_{w \in W} wUw^{-1} = \prod_{w \in W} wVw^{-1} = 1$ and the definition that τ induces a diagonal automorphism of G .

(Case 1) Let $u \neq 1$ or $v \neq 1$. Then there exists $w \in W$ satisfying

$$\gamma = \overline{w}(\tau g) \overline{w}^{-1} = \tau(v'_i v_i h' u_i u'_i)$$

with $v'_i \in V'_i, v_i \in V_i, h' \in T, u_i \in U_i, u'_i \in U'_i$ and $v_i u_i \neq 1$ for some $i \in I$.

(Case 2) Let $u = v = 1$ and the element τg be of the form

$$\gamma = \tau g = \tau h'$$

with $h' \in T$ and $\gamma \notin Z(\Gamma)$.

(Case 3) Let $u = v = 1$ and the element τg be of the form

$$\gamma = \tau g = \tau h'$$

with $h' \in T$ and $\gamma \in Z(\Gamma)$.

In (Case 2), we can find an element $x_{\alpha_j}(t)$ with $t \neq 0$ for some $j \in I$ such that

$$x_{\alpha_j}(t)(\tau h') x_{\alpha_j}(-t) = \tau(h' u_j)$$

with $1 \neq u_j \in U_j$. Thus (Case 2) can be reduced to (Case 1). Our assumption on the form of τg implies that we can skip (Case 3). So let us assume that we are in (Case 1) and fix an $i \in I$ appearing in the formula for γ .

Set $X = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$, and $Y = \{\alpha_2, \alpha_3, \dots, \alpha_n\}$. We write

$$\begin{aligned} h' &= h_{\alpha_1}(t_1) \dots h_{\alpha_n}(t_n), \\ h_X &= h_{\alpha_1}(t_1) \dots h_{\alpha_{n-1}}(t_{n-1}), \\ h_Y &= h_{\alpha_2}(t_2) \dots h_{\alpha_n}(t_n) \end{aligned}$$

with $t_1, \dots, t_n \in K^\times$.

For $i > 1$, we put $\tau_Y = \tau h_{\alpha_1}(t_1)$ and $\Gamma_Y = \langle \tau_Y, G_Y \rangle$. Then

$$\gamma = v'_Y (\tau_Y v_Y h_Y u_Y) u'_Y$$

for some $v'_Y \in V'_Y$, $v_Y \in V_Y$, $u_Y \in U_Y$, $u'_Y \in U'_Y$, and $v_Y u_Y \neq 1$ since $i \in Y$ and $u_i v_i \neq 1$. Let us choose $t'_2 \in K^\times$ such that $\tau h^* h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1}) h_{\alpha_2}(t'_2)$ is non-central in $\Gamma_X = \langle \tau_X, G_X \rangle$, where $\tau_X = \tau h_{\alpha_n}(t_{\alpha_n}^*)$. This is actually possible since there are enough elements in K . In fact, we can choose $t'_2 \neq \pm 1$ if $\tau h^* h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1})$ centralizes both of U_2 and V_2 , otherwise we just put $t'_2 = 1$. In any case, $\tau h^* h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1}) h_{\alpha_2}(t'_2)$ does not centralize $G_{\alpha_2} = \langle U_2, V_2 \rangle$, which implies that $\tau h^* h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1}) h_{\alpha_2}(t'_2)$ is not central in Γ_X . Applying induction to non-central element

$$\gamma_Y = \tau_Y (v_Y h_Y u_Y) \in \Gamma_Y$$

with $v_Y u_Y \neq 1$, we can find an element $z_Y \in G_Y$ such that $z_Y \gamma_Y z_Y^{-1} = \tau_Y v''_Y h''_Y u''_Y$ with $v''_Y \in V_Y$, $u''_Y \in U_Y$ and $\tau h^* h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1}) h_{\alpha_2}(t'_2) = \tau_Y h''_Y$,

where $h_Y'' \in T_Y$. Then we have

$$\gamma' = z_Y \gamma z_Y^{-1} = \tau v'' h'' u''$$

with $v'' \in V$, $u'' \in U$ and $h'' = h * h_{\alpha_1}(t_1 t_{\alpha_1}^{*-1}) h_{\alpha_2}(t_2')$. Next, we write

$$\gamma' = v_X' (\tau_X v_X h_X' u_X) u_X'$$

with $v_X' \in V_X'$, $v_X \in V_X$, $u_X \in U_X$, $u_X' \in U_X'$ and

$$h_X' = h_{\alpha_1}(t_1) h_{\alpha_2}(t_{\alpha_2}^* t_2') h_{\alpha_3}(t_{\alpha_3}^*) \dots h_{\alpha_{n-1}}(t_{\alpha_{n-1}}^*).$$

Then we apply induction to

$$\gamma_X = \tau_X (v_X h_X' u_X) \in \Gamma_X.$$

Since γ_X is noncentral in Γ_X , we can find an element $z_X \in G_X$ such that $z_X \gamma_X z_X^{-1} = \tau_X v_X'' h_X'' u_X''$ with $v_X'' \in V_X$, $u_X'' \in U_X$ and $\tau h^* = \tau_X h_X''$. Therefore, we have

$$\gamma'' = z_X \gamma' z_X^{-1} = \tau v''' h^* u'''$$

with $v''' \in V$, $u''' \in U$, and we are done.

If $i = 1$, we take first X and then Y , and repeat the same process as above. Then by choosing α_n, α_{n-1} and α_1 instead of α_1, α_2 and α_n respectively, we complete the proof of Proposition. We can also explain the matter in the following way. The order on Π is irrelevant for the proof. Therefore, if in our initial order i turns out to be one, we can reorder Π in such a way, that taking the corresponding X and Y we come to the already considered case $i > 1$.

The main theorem follows from the proposition in an obvious way.

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