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On Quasi-Projective Uniserial Modules.

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ABSTRACT - Let R be a valuation domain with maximal ideal P. We study quasi-projective uniserial modules over R. By making use of the absence of «shrinkable» uniserial modules over R we prove our main result: a characterization of quasi-projectivity of a uniserial module U over R in terms of lifting of endomorphisms of factors of U. Using this characterization allows us to describe quasi-projective ideals of R in terms of completeness of certain localizations of factor-rings of R. Archimedean ideals of R admit the best possible description from this point of view. We show that a non-principal archimedean ideal of R is quasi-projective if and only if R/K is complete in the R/K-topology for each archimedean ideal $K \nsubseteq P$. Finally, we show that taking tensor products with archimedean ideals preserves quasi-projectivity.

1. Uniserial modules.

Let us begin with necessary definitions.

DEFINITIONS. A module over a ring is called *uniserial* if its submodules form a chain under inclusion. A commutative integral domain R is a *valuation domain* if it is a uniserial module over itself.

From now on, the letter R will denote a valuation domain with maximal ideal P and quotient field Q, unless stated otherwise. We refer the reader to Fuchs and Salce [6] for a treatment of modules over valuation domains.

Submodules and factor modules of uniserial modules are likewise

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uniserial. The simplest examples of uniserial modules are besides the valuation rings themselves, their rings of quotients, their cyclic modules, and more generally, the R-modules of the form I/J where $J < I$ are R-submodules of Q. Uniserial modules of the latter kind are called *standard uniserial.* The existence of non-standard uniserial modules has been first established by Shelah in [10].

Standard uniserial modules are completely classified by the following proposition.

PROPOSITION 1.1 (Shores and Lewis [11]). *Two standard uniserial modules I/J and I'/J' over a valuation domain R are isomorphic if and only if there exists an element $0 \neq q \in Q$ such that $I = qI'$ and $J = qJ'$.*

Several properties of a uniserial module depend on its type.

DEFINITIONS. Let R be a valuation domain with the maximal ideal P. We use the following notation from [6]. If U is a torsion uniserial and $0 \neq$ $\neq u \in U$, we set

$$I = H(u) = \{r^{-1} \,|\, u \in rU\} \qquad \text{and} \qquad J = \operatorname{Ann} u = \{r \in R \,|\, ru = 0\}.$$

The fractional ideal I is called the height ideal of u, and we say that U is of type $[I/J]$ (the isomorphy class of I/J), $t(U) = [I/J]$. The type $t(U)$ does not depend on the choice of $0 \neq u \in U$. For a uniserial module U we define ideals

$$U^{\sharp} = \{r \in R \,|\, rU < U\} \qquad \text{and} \qquad U_{\sharp} = \{r \in R \,|\, ra = 0 \text{ for some } 0 \neq a \in U\}.$$

It is easy to verify that both $U^{\sharp} = I^{\sharp}$ and $U_{\sharp} = J^{\sharp}$ are prime ideals of R containing the annihilator Ann U. An ideal I of R is called *archimedean* if $I^{\sharp} = P$.

It is a good time to introduce the main objects of our study: the quasi-projective modules. In the following definition, R may be an arbitrary ring.

DEFINITION. An R-module U is called quasi-projective if it is projective relative to all exact sequences of the form $0 \to V \to U \xrightarrow{\pi} U/V \to 0$, where V is a submodule of U and π is the canonical projection. That is, for every homomorphism $f\colon U \to U/V$ there exists a map $f'\colon U \to U$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & & U \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & \\
0 \arrow[r] & V \arrow[r] & U \arrow[r, "\pi"] & U/V \arrow[r] & 0
\end{tikzcd}$$

For more on quasi-projective modules via relative projectivity see [7].

As the following proposition states, possibilities of endomorphisms of quasi-projective uniserials are limited.

PROPOSITION 1.2. *A surjective endomorphism of a quasi-projective uniserial module U over any ring R is an automorphism.*

PROOF. Uniserial modules are indecomposable. Fuchs and Rangaswamy show in [5] that if a factor U/V of a quasi-projective module U is isomorphic to a summand of U, then V is also isomorphic to a summand of U. Thus, if $\varphi : U \to U$ is a surjective endomorphism with non-trivial kernel V, then $U/V \cong U$ and V must be isomorphic to U. Since V is a proper submodule of U, it has to be standard uniserial. Thus, U is standard uniserial either. Suppose that $U = I/J$ for some $J < I \leqslant Q$. Then $V = I_1/J$ for some I_1 such that $J < I_1 < I$. By Proposition 1, there are non-zero elements p, q in Q such that

$$I_1 = pI, \qquad J = pJ, \qquad I = qI, \qquad J = qI_1.$$

Thus, we have $pqI = J = pqI_1$, which implies $I = I_1$. This renders $U/V \cong$ $\cong U$ impossible, a contradiction. ∎

The following corollary is an immediate consequence of Proposition 1.2.

COROLLARY 1.3. *Let R be a valuation ring. If U is a quasi-projective uniserial module over R, then $U_\sharp \leqslant U^\sharp$. Moreover, if U is non-standard, then $U_\sharp = U^\sharp$.*

PROOF. Suppose that U is a quasi-projective uniserial R-module. If $r \in R$ then multiplication by r is a surjective endomorphism of U if and only if $r \notin U^\sharp$ and an injective endomorphism if and only if $r \notin U_\sharp$. The first statement follows now from Proposition 1.2.

If U is non-standard quasi-projective, then every monic endomorphism of U has to be an isomorphism because all proper submodules of U are standard uniserials. This implies $U^{\sharp} \leqslant U_{\sharp}$ and, consequently, $U^{\sharp} = U_{\sharp}$. ■

The following Lemma uses a result by Facchini and Salce [3]. They call a uniserial module U over an arbitrary ring *shrinkable* if $U \cong V/W$ for some proper submodules $0 < W < V < U$. It is proved in [3] that there are no shrinkable uniserial modules over commutative or Noetherian rings. We use the result to prove a more general statement.

LEMMA 1.4. *Let U and V be two uniserial modules over a valuation domain R. If there exist both a monomorphism f and an epimorphism g from U to V then at least one of f and g is an isomorphism. Moreover, if both U, V are standard and*

(a) $V_{\sharp} < V^{\sharp}$, *then g is an isomorphism;*

(b) $V^{\sharp} < V_{\sharp}$, *then f is an isomorphism.*

PROOF. There are four possibilities depending on whether U and V are standard or not. If both U and V are non-standard, then every monomorphism $f : U \to V$ has to be onto. If U is standard and V is not, then no epimorphism $g : U \to V$ exists. Similarly, if U is non-standard and V is standard then no monomorphism f exists. It remains to consider the case of standard U and V.

There is nothing to prove if f or g is an isomorphism. Suppose that f maps monomorphically onto a proper submodule of V and g has a non-trivial kernel. We claim that this is impossible. To prove the claim, we write $U = I/J$ and $V = I'/J'$, with submodules $J < I$, $J' < I'$ of Q. Then $\operatorname{Im} f \cong I_1'/J'$ and $\operatorname{Ker} g \cong J_1/J$ for appropriate J_1 and I_1' in Q. We have isomorphisms

$$V = I'/J' \cong I/J_1 \cong I_1'/J_1',$$

implying that V is shrinkable, which is impossible by [3]. This proves the first statement.

We conclude that whenever there is a pair f, g satisfying the hypotheses, then either $J_1' = J'$ or $I' = I_1'$. These possibilities are defined by the inclusions of the «sharps» of V and imply that either g is monic or f is epic, accordingly. ■

We are going to show that the quasi-projective property of uniserial modules is closely related to the property of lifting endomorphisms. The following definition does not require R to be a valuation domain.

DEFINITION. Let M be an R-module and N its submodule. We say that an endomorphism h of M/N *can be lifted* or *lifts* to M, if there exists an endomorphism h' of M making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, dashed, "h'"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ M/N \arrow[r, "h"] & M/N \end{tikzcd}$$

Here π denotes the canonical projection. If all the endomorphisms of each factor of a module M lift, then M will be called *weakly quasi-projective.* This term was introduced by Rangaswamy and Vanaja in [9].

Trivially, a quasi-projective module is weakly quasi-projective. The converse is not true even for uniserial modules: the abelian group $\mathbb{Z}(p^\infty)$ is an example of a weakly quasi-projective but not quasi-projective uniserial module over $\mathbb{Z}$. Interestingly, in case of valuation domains, the necessary condition of weak quasi-projectivity together with $U_\sharp \leqslant U^\sharp$ is also sufficient for U to become quasi-projective.

THEOREM 1.5. *Let R be a valuation domain and U a uniserial R-module. The following conditions are equivalent*:

(a) *U is weakly quasi-projective and $U_\sharp \leqslant U^\sharp$.*

(b) *U is quasi-projective.*

PROOF OF (a) $\Rightarrow$ (b). Suppose that every $h \in \mathrm{End}_R\, U/V$, for each $V < U$, can be lifted to an element of $\mathrm{End}_R\, U$ and $U_\sharp \leqslant U^\sharp$. Let V be a submodule of U and $f : U \to U/V$ a homomorphism. Thus, we are given the solid part of the following diagram, where π_1 is the canonical projection:

$$\begin{tikzcd} & & & U \arrow[d, "f"] \arrow[dl, dashed] & \\ 0 \arrow[r] & V \arrow[r] & U \arrow[r, "\pi_1"] & U/V \arrow[r] & 0 \end{tikzcd}$$

If we denote the kernel of f by W, then f factors through the canonical projection π_2: $U \to U/W$ and an inclusion $i : U/W \to U/V$. Since U is uniserial, either $V = W$, $V < W$, or $W < V$.

If $V = W$, then $i \in \mathrm{End}_R\, U/V$. It lifts to U by assumption.

If $V < W$, then there is canonical projection $\pi_3\colon U/V \to U/W$. Embedding it in the previous diagram results in the following:

$$\begin{tikzcd}
 & & & & U/V \arrow[d, "\pi_3"] \\
 & & & U \arrow[ur, "\pi_1"] \arrow[r, "\pi_2"] \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & U/W \arrow[dl, "i"] \\
0 \arrow[r] & V \arrow[r] & U \arrow[r, "\pi_1"] & U/V \arrow[r] & 0
\end{tikzcd}$$

The composition $i \circ \pi_3$ is an endomorphism of U/V and can be lifted to a map $f'\colon U \to U$ by the assumption.

If $W < V$, then we have the canonical epimorphism $\pi : U/W \to U/V$, and so by Lemma 1, i is an isomorphism. Fixing submodules I, J of Q such that $t(U) = [I/J]$, we can write $V \cong I_1/J$ and $W \cong I_2/J$, where $J < < I_2 < I_1 < I$ and the latter isomorphism is the restriction of the former. Since $U/V \cong U/W$, we must have $I/I_1 \cong I/I_2$. By Proposition 1, there exists a $q \in R$ such that $qI = I$ and $qI_1 = I_2$. This means $q \notin I^\natural = U^\natural \geqslant U_\natural$. Therefore, multiplication $\dot{q}$ by q is an automorphism of U such that $\dot{q}V = W$. It naturally induces an isomorphism $\overline{q}\colon U/V \to U/W$. In this case we have the following diagram:

$$\begin{tikzcd}
 & & U \arrow[rr, "\pi_2"] \arrow[drr, "f"] \arrow[dr, dashed, "f'"'] \arrow[dd, "g"'] & & U/W \arrow[d, "i"] & \\
0 \arrow[r] & V \arrow[rr] & & U \arrow[r, "\pi_1"'] \arrow[dl, "\dot{q}"'] & U/V \arrow[r] \arrow[d, "\overline{q}"] & 0 \\
 & & U \arrow[rr, "\pi_2"'] & & U/W &
\end{tikzcd}$$

Here, g is a lifting of $\overline{q} \circ i$. The map $f' \equiv \dot{q}^{-1} \circ g$ is a desired lifting of f.

(b) $\Rightarrow$ (a). This implication is trivial. ■

In particular, for ideals of R, weak quasi-projectivity always implies quasi-projectivity.

2. Endomorphisms of uniserial modules.

Since quasi-projectivity depends on the lifting property of endomorphisms, knowing the structure of endomorphism rings of uniserial modules will enhance our understanding of their quasi-projectivity.
If U is a uniserial module and Ann U is its annihilator ideal, then we use the following notation from Fuchs and Salce [6]: we write Ann $U = I$ if there is a $u \in U$ such that Ann $u =$ Ann U, and we write Ann $U = I^+$ otherwise. The module U is called *finitely annihilated* or *non-finitely annihilated*, respectively. It is known that a uniserial U of type $[I/J]$ is non-finitely annihilated exactly if $J^\sharp I = I$. See Bazzoni, Fuchs and Salce [2] for details.

Thus, there are two types of uniserial modules distinguished by the structure of their annihilator ideals. Each type has a specific kind of the endomorphism ring, described by the following proposition.

THEOREM 2.1 (Shores and Lewis [11]). *Let R be a valuation ring and U a uniserial R-module.*

(a) *If* Ann $U = I$, *then* $U_\sharp \leq U^\sharp$ *and U carries the natural structure of an S-module, where S is the ring R/I localized at $U^\sharp/I$, and* $\mathrm{End}_R\, U \cong S$.

(b) If Ann $U = I^+$, then $U^\sharp \leq U_\sharp$ and U carries the natural structure of a T-module, where T is the ring R/I localized at $U_\sharp/I$, and $\mathrm{End}_R\, U$ is isomorphic to the completion $\widehat{T}$ of T in the T-topology.

The endomorphisms of U can be viewed as multiplications by appropriate elements from either S or $\widehat{T}$.

COROLLARY 2.2. *A valuation domain R is maximal if and only if all uniserial R-modules U with $U_\sharp \leq U^\sharp$ are quasi-projective.*

PROOF. Theorem 3.5 of Herrmann [8] states that R is maximal if all submodules of Q are quasi-projective. This proves sufficiency of the condition.

Conversely, if R is maximal, then R and all its factors are linearly compact in the discrete topology. Thus all uniserial R-modules are standard. Hence, for every uniserial module $U = I/J$, $J < I \leq Q$, with $U_\sharp \leq U^\sharp$, the endomorphism ring $\mathrm{End}_R(U)$ is the localization $(R/\mathrm{Ann}\, U)_{(U^\sharp/\mathrm{Ann}\, U)}$, which is also maximal. Therefore, endomorphisms U and its factors are induced by multiplications by appropriate elements of $R_{U^\sharp}$. Thus, every such U is weakly quasi-projective. The statement follows from Theorem 1.5. ■

Corollary 2.2 for ideals is a particular case of a more general result by Fuchs [4]. He proves that for a cardinal κ a valuation domain R is κ-maximal if and only if every κ-generated ideal of R is quasi-projective.

We are going to show that quasi-projectivity of a uniserial module is completely determined by its type.

THEOREM 2.3. *Let R be a valuation domain and U be a uniserial R-module. Suppose that $t(U) = [I/J]$ $(J < I \leqslant Q)$. Then U is quasi-projective if and only if I/J is.*

PROOF. By Theorem 1.5, the quasi-projectivity of (non-) standard uniserial U satisfying $U_\sharp \leqslant U^\sharp$ $(U_\sharp = U^\sharp)$ is equivalent to its weak quasi-projectivity. The «sharp» ideals associated to U and I/J are necessarily the same. Thus, we have to show that the two modules are weakly quasi-projective at the same time. Suppose that V is a (proper) submodule of U, $V \leqslant U$. We can choose $I_1 \leqslant I$ such that $t(V) = [I_1/J]$ and $t(U/V) = [I/I_1]$. On the other hand, for any given ideal I_1 $(J \leqslant I_1 \leqslant I)$ there is a submodule V of U with type as above. Since uniserial modules of the same type have naturally isomorphic endomorphism rings, the endomorphisms of U/V lift to U if and only if those of I/I_1 lift to I/J. The proof is finished. ■

COROLLARY 2.4. *Let U be a uniserial R-module. If* $\mathrm{Tor}_R^1(U, K/L)$ $(L < K \leqslant Q)$ *is quasi-projective, then* $\mathrm{Tor}_R^1(U, V)$ *is quasi-projective for each V of type* $[K/L]$.

PROOF. It is known (see [2]) that, for uniserial R-modules U and V with types $[I/J]$ and $[K/L]$ respectively, the Tor_R^1 product of U and V is uniserial and

$$t(\mathrm{Tor}_R^1(U, V)) = \left[\frac{IL \cap JK}{JL}\right].$$

Thus, $\mathrm{Tor}_R^1(U, K/L)$ and $\mathrm{Tor}_R^1(U, V)$ have the same type. A reference to Theorem 2.3 finishes the proof. ■

Bazzoni, Fuchs, and Salce prove in [2] that the isomorphy classes of torsion uniserial R-modules form a commutative semigroup under operation Tor_R^1. Corollary 2 implies that the orbit of a module U under the Tor_R^1 operation with uniserials of the same type consists entirely of quasi-projective modules if it contains at least one quasi-projective.

3. Ideals of valuation domains.

In this section we use Theorems 1.5 and 2.1 to study quasi-projectivity of ideals of a valuation domain R. We will show that quasi-projectivity of an archimedean ideal depends on the completeness of certain localizations of factor-rings of R.

THEOREM 3.1. *Let R be a valuation domain. An ideal I of R is quasi-projective if and only if for each ideal $J < I$ with* $\mathrm{Ann}\, I/J = (J : I)^+$ *the ring*

$$S = (R/(J : I))_{(J^\sharp/J : I)}$$

is complete in its S-topology.

PROOF. This is a direct consequence of the Theorem 1 and Theorem 2. Since for an ideal I, $0 = I_\sharp \leq I^\sharp$ always, the quasi-projectivity of I is equivalent to lifting of each endomorphism of every factor I/J to I. When $\mathrm{Ann}\, I/J = (J : I)^+$ and the ring S is not complete, the ring $\mathrm{End}_R\, I/J = \widehat{S}$ contains elements which are not induced by elements of $\mathrm{End}_R\, I = R_{I^\sharp}$. The absence of such elements is equivalent to the completeness of S (for every such J), and the quasi-projectivity of I. ■

It is possible to give a more explicit characterization of quasi-projectivity of archimedean ideals. There are two cases, according as P is principal or not. If P is principal, then only the principal ideals are archimedean. Thus, they all are projective and, therefore, quasi-projective. To consider the second alternative, we need a special case of Lemma 1.4 from [2]. The proof is provided for the sake of completeness.

LEMMA 3.2. *Let $J < I$ be archimedean ideals of a valuation domain R. Then* $\mathrm{Ann}\, I/J = (J : I)^+$ *if and only if I is not principal.*

PROOF. The «only if» part is trivial. To prove the converse, let I be an infinitely generated archimedean ideal. Assume that there is an element $i \in I$ such that $\mathrm{Ann}\, iR/J = \mathrm{Ann}\, I/J$. Since I is not principal, we can choose $i' \in I$ such that $iR < i'R < I$. That is, $i = si'$ for a non-unit s of R.

Then Ann $iR/J = \text{Ann}\, i'R/J$, hence $i^{-1}J = i'^{-1}J$, and i, i' are associates. A contradiction. Hence, no such element $i \in I$ exists and Ann $I/J = (J : I)^+$. ■

THEOREM 3.3. *Let R be a valuation domain with infinitely generated maximal ideal P. A non-principal archimedean ideal I is quasi-projective if and only if R/K is complete in the R/K-topology for each archimedean ideal $K \ncong P$.*

PROOF. By Theorem 3.1, I is quasi-projective if and only if R/K is complete whenever $\text{Ann}(I/J) = K^+$ for some ideal J. The ideal J has to be archimedean by Theorem 2.1. It remains to prove that for different J, $\text{Ann}(I/J)$ can be any archimedean ideal $\ncong P$, but nothing else.

If $K \ncong P$ then Ann $I/IK = K^+$.

If $K \cong P$ then $K = sP$ for some $s \in R$. Assume that Ann $I/J = K^+$ for some (archimedean) ideal J. We have $IK = IsP = sI \leqslant J$. But then $(sR)I \leqslant J$, which means that I/J has annihilator larger than K. Contradiction. Hence, ideals isomorphic to P cannot be annihilators of I/J. This completes the proof. ■

The following corollary is an immediate consequence of Theorem 3.3.

COROLLARY 3.4. *Let R be a valuation domain with infinitely generated maximal ideal P. If there exists a non-principal quasi-projective archimedean ideal, then all archimedean ideals of R are quasi-projective.* ■

It is difficult to give a more explicit characterization of the quasi-projectivity of non-archimedean ideals beyond Theorem 3.1. However, we show that one has a way of «producing» new quasi-projective ideals starting with a quasi-projective ideal I and taking tensor products of I with archimedean ideals. This result is in a way analogous to Theorem 3.3 from [1].

First, we need the following lemma.

LEMMA 3.5. *Let R be a valuation domain with maximal ideal P. Suppose that I is a non-principal ideal of R. If $0 \neq r \in R$, then the endomorphism rings of I/rR and I/rP are (naturally) isomorphic.*

PROOF. This is a consequence of Theorem 2.1 and the following two observations. Firstly, I/rR and I/rP have the same annihilator A^+, where $A = rR : I = rP : I$. For, if $sI \subseteq rR$ $(0 \neq s \in A)$, then $sI \subseteq rP$ since I is not principal and rP is the maximal submodule of rR. Secondly, both modules have the same «sharp» ideals because rR and rP are archimedean. By Theorem 2.1, the endomorphism rings of I/rR and I/rP are isomorphic to the completion of R/A in the R/A-topology. This isomorphism is natural. ■

We have the following theorem.

THEOREM 3.6. *Let R be a valuation domain with non-principal maximal ideal P. Suppose that I is a non-principal ideal of R and J is an archimedean ideal. Then I is quasi-projective if and only if $I \otimes_R J$ is.*

PROOF. Over valuation domains $I \otimes_R J$ is naturally isomorphic to IJ. This allows us to consider submodules of $I \otimes_R J$ as ideals contained in IJ and vice-versa. For ideals of valuation domains the quasi-projectivity is equivalent to the weak quasi-projectivity. Thus we need to prove that I is weakly quasi-projective if and only if $I \otimes_R J$ is. Since the arguments in each way are similar, we give the proof of the «if» part only. The case of principal J is trivial.

Suppose that $I \otimes_R J$ is weakly quasi-projective, $K < I$ and $\varphi \in \mathrm{End}_R(I/K)$. If K is not principal, tensoring with J is «reversible», that is, $K \otimes_R J \otimes_R (R : J) \cong K \otimes_R P \cong K$. The isomorphisms are natural. Consider the following diagrams.

$$\begin{tikzcd}
I \arrow[r, dashed, "f'"] \arrow[d, "\pi"'] & I \arrow[d, "\pi"] & \xleftarrow{\otimes_R R.J} & I \otimes_R J \arrow[r, "f"] \arrow[d, "\pi\otimes_R 1"'] & I \otimes_R J \arrow[d, "\pi\otimes_R 1"] \\
\frac{I}{K} \arrow[r, "\varphi"] & \frac{I}{K} & \xrightarrow[\otimes_R J]{} & \frac{I \otimes_R J}{K \otimes_R J} \arrow[r, "\varphi\otimes_R 1"] & \frac{I \otimes_R J}{K \otimes_R J}
\end{tikzcd}$$

Taking the tensor product of the solid part of the diagram on the left with J, one obtains the diagram on the right. Here, map f is a lifting of $\varphi \otimes_R 1$, which exists by assumption. Maps π and $\pi \otimes_R 1$ are canonical projections. Taking the tensor product with $R : J$, one returns to the left diagram. Map $f' \cong f \otimes_R 1_{R:J}$ is a lifting of φ, which makes the left square commute. It remains to consider the special case of principal K.

If $K = rR$ $(0 \neq r \in R)$, then $K \otimes_R J \otimes_R (R : J) \cong rP$. By Lemma 3.5, the endomorphism $\varphi : I/rR \to I/rR$ is an element of the $R/\mathrm{Ann}\,(I/rR)$-completion of the ring $R/\mathrm{Ann}\,(I/rR)$. Since this ring is also the endomorphism ring of I/rP, φ can be considered as an endomorphism of I/rP. As such, it can be lifted to I using technique of the previous paragraph. Clearly, this lifting is also a lifting of the original $\varphi : I/rR \to I/rR$. The proof is finished. ■

We conclude with the following observation. Since isomorphy classes of archimedean ideals form a group under the tensor product operation, application of Theorem 3.6 delivers an alternative proof of Corollary 3.4.

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