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On the Nullity Index of Isometric Immersions of Kähler Manifolds.

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1. Introduction and statement of results.

Let M be a Kähler manifold with complex dimension m and complex structure J. In the present work we analyse some obstructions to the existence of isometric immersions $\varphi : M \to N$ into certain Riemannian manifolds.

Let $\varphi^{-1} TN$ be the pull-back of the tangent bundle of N. We will use the symbol ∇ to represent either the induced connection on $\varphi^{-1} TN$ or the induced connection on $T^* M \otimes \varphi^{-1} TN$.

The covariant differential $\alpha = \nabla d\varphi$, called the second fundamental form, may be understood as a smooth section of $\odot^2 T^* M \otimes T^{\perp} N$, where $T^{\perp} N$ represents the normal bundle.

The conjunction of the second fundamental form with the complex

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structure J gives rise to two operators

$$C(X, Y) = \frac{1}{2}(\alpha(X, Y) + \alpha(JX; JY))$$

$$A(X, Y) = \frac{1}{2}(\alpha(X, Y) - \alpha(JX; JY)),$$

where $X, Y \in C(T(M))$.

We can get a different prospect of the operators working in a complex framing. The complexification of the tangent bundle of M, denoted by $T^C M$, decomposes as

$$T^C M = T^{1,0} M \oplus T^{0,1} M$$

where $T^{1,0}M$ and $T^{0,1}M$ are, respectively, the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenbundles of the complex extension of J.

The decomposition

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

of α according to type is such that

$$\alpha^{(1,1)} = C$$

$$\alpha^{(2,0)} + \alpha^{(0,2)} = A .$$

It is a well-known fact that the geometry of the second fundamental form reflects the geometry of φ. From this view point one can ask up to which extent the operators A and C influence the behaviour of the map.

The map φ is said to be pluriminimal (or $(1, 1)$-geodesic) if $\alpha^{(1,1)} \equiv 0$. Equivalently, φ is pluriminimal if, for any holomorphic immersed curve $c : \mathcal{C} \to M$, $c \circ \varphi : \mathcal{C} \to N$ is always minimal.

Trivial examples of pluriminimal immersions are the minimal immersions of a Riemannian surface. When N is flat, Dajczer and Rodriguez [D-R] showed that the pluriminimal immersions are exactly the minimal ones, i.e., the vanishing of $\alpha^{(1,1)}$ is the equivalent of the vanishing of the mean curvature $H =$ trace of α.

We define the index of relative nullity of α (resp. of $\alpha^{(1,1)}$) at x as $\nu(x) = \dim \Delta_x$ (resp. $\nu^{(1,1)}(x) = \dim \Delta_x^{(1,1)}$), where $\Delta_x = \{\varphi \in T_x M \mid \alpha_x(v, w) = 0 \ \forall \ w \in T_x M\}$ and $\Delta_x^{(1,1)} = \{v \in T_x M \mid C_x(v, w) = 0 \ \forall \ w \in T_x M\}$.

In the case $N = R^{2n+p}$, $n \geqslant 2$, Dajczer and Rodriguez classified the minimal immersions with $\nu \geqslant 2n-4$ everywhere.

In this article, when N is a conformally flat manifold we obtain

THEOREM 1. *Let N be a conformally flat Riemannian manifold with non-zero sectional curvatures. If $\nu(x_0) > 0$ at some point $x_0 \in M$, M is a Riemann surface.*

Concerning the relative nullity index of $\alpha^{(1,1)}$, we assume that N is a conformally flat Riemannian manifold with dimension n, whose scalar curvature $\mathcal{S}(N)$ never vanishes. Considering the real numbers $r = \inf\{\mathrm{Ric}^N(v, v) \mid \|v\|_x = 1, \ x \in M\}$, $R = \sup\{\mathrm{Ric}^N(v, v) \mid \|v\|_x = 1, \ x \in M\}$, $s = \inf_{x \in M} \mathcal{S}(N)_x$ and $S = \sup_{x \in M} \mathcal{S}(N)_x$, we can state:

THEOREM 2. *Let N be a conformally flat Riemannian manifold with positive scalar curvature, such that $\frac{r}{S} > \frac{n}{2(n-1)}$. If $\nu^{(1,1)}(x_0) > 0$ at some point x_0, M is a Riemann surface.*

THEOREM 3. *Let N be a conformally flat Riemannian manifold with negative scalar curvature, such that $\frac{R}{s} > \frac{n}{2(n-1)}$. If $\nu^{(1,1)}(x_0) > 0$ at some point x_0, M is a Riemann surface.*

COROLLARY 1. *Let N be a conformally flat Riemannian manifold whose Ricci curvature tensor satisfies one of the following inequalities:*

(i) $\frac{nd}{2(n-1)} < \mathrm{Ric}^N < d$

(ii) $-d < \mathrm{Ric}^N < -\frac{nd}{2(n-1)}$,

for some positive real number d. Then if, at some point x_0, $\nu(x_0) > 0$, $m = 1$.

THEOREM 4. *Let N be a Riemannian manifold whose sectional curvatures satisfy one of the following conditions:*

(i) $\frac{1}{4} < K(\sigma) \leqslant 1$

(ii) $-1 \leqslant K(\sigma) < -\frac{1}{4}$.

If $\nu(x_0) > 0$, for some x_0, $m = 1$.

2. Preliminaries.

A Riemannian manifold (N, h) is said conformally flat if there exists a smooth function $f: N \to R$ such that $(N, e^{2f}h)$ is flat. Riemannian manifolds with constant sectional curvature are particular examples of conformally flat Riemannian manifolds.

For each $y \in N$ we denote by $C_y(N)$ the subspace of $S(\overset{2}{\wedge} T_y^* N)$ (the 2-symmetric forms on $\overset{2}{\wedge} T_y^* N)$ consisting of «curvature like tensors»; that means, curvature tensors satisfying the first Bianchi identity. The action of the orthogonal group $O(n)$ ($n = \dim M$) on $C_y(N)$ gives rise to the following decomposition into irreducible subspaces:

$$C_y(N) = \mathcal{U}_y(N) \oplus \mathcal{R}_y(N) \oplus \mathcal{W}_y(N),$$

where $\mathcal{U}_y(N) = RId_{\overset{2}{\wedge} T_y^* N}$ and $\mathcal{R}_y(N)$ is formed by the «traceless Ricci» tensors, that is to say, those tensors θ whose Ricci contraction $c(\theta)$ $(c(\theta)(a, b) = \operatorname{trace} \theta(a, ., b, .))$ vanishes. The orthogonal complement $\mathcal{W}_y(N)$ of $\mathcal{U}_y(N) \oplus \mathcal{R}_y(N)$ in $C_y(N)$ is called the space of Weyl tensors. The Weyl tensor of a Riemannian manifold is the Weyl part of its curvature tensor.

The Weyl curvature tensor W is the main invariant under conformal changes of the metric. The vanishing of W characterises completely the conformally flat Riemannian manifolds.

It is then easily seen that the Riemannian curvature tensor R^N of a conformally flat Riemannian manifold (N, h) with Ricci curvature Ric^N and normalised scalar curvature $s(N)$ is given by

$$R^N = \frac{1}{n-2} h \owedge \mathrm{Ric}^N - \frac{ns(N)}{(n-1)(n-2)} h \owedge h \tag{1}$$

where $\owedge$ represents the Kulkarni-Nomizu product of the symmetric 2-tensors, defined in the following way:

$$z \owedge k(u, v, w, t) =$$

$$= z(u, w)\, k(v, t) + z(v, t)\, k(u, w) - z(u, t)\, k(v, w) - z(v, w)\, k(u, t),$$

if $z, k \in \odot^2 T_y^* N$.

Let d be a positive real number. The Riemannian manifold (N, h) is said to be positively (resp. negatively) d-pinched at a point $y \in N$ if there exists a positive real number τ such that

$$\tau d < K_y(\sigma) \leqslant \tau \ (\text{resp.} \quad -\tau \leqslant \mathrm{K_y}(\sigma) < \tau \mathrm{d})$$

for any 2-dimensional subspace σ of $T_y N$. N is said to be positively (resp. negatively) d-pinched if it is positively (resp. negatively) d-pinched at each point $y \in N$.

LEMMA 1. [B] *Let N be a Riemannian manifold whose sectional curvatures satisfy one of the following inequalities:*

(i) $-1 \leqslant K(\sigma) < -\frac{1}{4}$,

(ii) $\frac{1}{4} < K(\sigma) \leqslant 1$.

Then if X, Y, Z, W is a local orthonormal frame field, the following inequality holds:

$$|\langle R(X, Y)Z, W\rangle| \leqslant \frac{1}{2}. \tag{2.11}$$

Let M be a Kähler manifold with complex structure J. Denoting respectively by π' and π'' the projections of the complexified tangent bundle $T^C M$ into its holomorphic and anti-holomorphic parts, $T^{1,0}M$ and $T^{0,1}M$, we use the following notation:

$$\alpha^{(1,1)}(X, Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

where $X' = \pi'(X)$ and $X'' = \pi''(X)$.

3. Proof of the statements.

PROOF OF THEOREM. 1. Choose a point $x_0 \in M$ such that $\Delta_{x_0} \neq \emptyset$ and consider $X, Y \in C(TM)$ such that $X(x_0) \in \Delta_{x_0}$, $\langle X, Y\rangle = \langle X, JY\rangle = 0$ and $|X| = |Y| = 0$.

It is clear from Gauss equation that for all $W \in C(TM)$

$$\begin{aligned}\langle R^N(X, Y)X, W\rangle &= \langle R^M(X, Y)X, W\rangle = \\ &= \langle R^M(JX, JY)X, W\rangle = \\ &= \langle R^N(JX, JY)X, W\rangle.\end{aligned}$$

From (1) we obtain

(2) $\langle R^N(X, Y)\, JX, JY\rangle = 0$, so that
(3) $\langle R^N(X, Y)\, X, Y\rangle = 0$, which cannot happen. Thus, $\dim M = 1$. ■

PROOF OF THEOREMS 2, 3 AND 4 As above we consider $x_0 \in M$ with $\Delta_{x_0}^{(1,1)} \neq \emptyset$ and $X, Y \in C(TM)$ such that $X(x_0) \in \Delta_{x_0}^{(1,1)}$, $\langle X, Y\rangle = \langle X, JY\rangle = 0$ and $|X| = |Y| = 1$.

Using the complex multilinear extension of Gauss equation, we can write

$$\langle R^N(X', Y')\, X'', Y''\rangle = \frac{1}{4}\langle \alpha^{(1,1)}(X, X), \alpha^{(1,1)}(Y, Y)\rangle - \frac{1}{4}\langle \alpha^{(1,1)}(X, Y), \alpha^{(1,1)}(X, Y)\rangle .$$

In the case of theorems 2 and 3 we now follow section 3 of [F-R-T] and use equation (1) to get

$$\langle R^N(X', Y')\, X'', Y''\rangle = \frac{1}{4(n-2)}\left\{ \mathrm{Ric}^N(X, X) + \mathrm{Ric}^N(Y, Y) + \mathrm{Ric}^N(JX, JX) + \mathrm{Ric}^N(JY, JY) - \frac{2ns(N)}{n-1}\right\}.$$

Under the assumptions of theorem 2 we have

$$\langle R^N(X', Y')\, X'', Y''\rangle \geq \frac{1}{2(n-2)}\left(2r - \frac{Sn}{n-1}\right) > 0 ,$$

a contradiction.

The conditions of theorem 3 imply that

$$\langle R^N(X', Y')\, X'', Y''\rangle \leq \frac{1}{2(n-2)}\left(2r - \frac{sn}{n-1}\right),$$

which cannot happen, hence $\dim M = 1$.

In the case of theorem 4, following section 2 of [F-R-T],

$$\langle R^N(X', Y')\, X'', Y''\rangle = \langle R^N(X, Y)\, X, Y\rangle + \langle R^N(JX, JY)\, JX, JY\rangle + \langle R^N(X, JY)\, X, JY\rangle + \langle R^N(JX, Y)\, JX, Y\rangle + 2\langle R^N(JX, X)\, JY, Y\rangle \neq 0 ,$$

according to Lemma 1. ■

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