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On a Functional Depending on Curvature and Edges.

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1. Introduction.

Recently the Mumford-Shah functional for image segmentation has been greatly investigated. Its minima are functions u jumping through a set S_u which is regular enough as some results by Ambrosio, Fusco & Pallara (see [7]) and by David & Semmes (see [13], [14]) prove. This leads to a difficulty to recognize edges that could be present in the contour of the images. In order to overcome this difficulty, in a previous paper (see [21]), a new criterion, which we refer to as (C), has been introduced following Gauch, Pien and Shah [19]. The idea in [21] is to look for curves, exhibiting edges, which are close enough to the jump set S_u. The existence of such curves minimizing (C) has been proved in the framework of the geometric measure theory. In (C) the closeness between the closure $\overline{S}_u$ of S_u and the minimizing curve γ is represented by the term $\int_\gamma \operatorname{dist}(x, \overline{S}_u)\, d\mathcal{H}^1(x)$ which makes the functional, in some sense, asymmetric. To take into account this phenomenon, in the present paper (C) is modified also by introducing the new term: $\int_{\overline{S}_u} \operatorname{dist}(x, \gamma)\, d\mathcal{H}^1(x)$ and the new functional (3.1) is proposed for minimization. The proof of the existence of minima for (3.1) relies on the varifolds theory as in [21], but the proof of the lower semicontinuity of the new term requires the monotonicity formula (see [28]) and non trivial technical details to apply it.

A last remark concerns the regularity of minima. The special classes of varifolds, introduced by J. E. Hutchinson and C. Mantegazza (see [22], [25]), where our minima are contained are locally graphs of multiple

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valued functions (see [3]). A general selection result for $C^{1,1}$ multiple valued functions is proved in [20], and a sharper result for $C^{1,\alpha}$ functions will appear in a work by C. De Lellis, P. Tilli and the author himself. As a conclusion, the minima of functional (3.1) are finite union of graphs of $C^{1,\alpha}$ curves.

2. Preliminaries and notations.

In this section we recall some results of varifolds theory which we will use later. For a more exhaustive exposition of the basic theory we refer to Simon (see [28]), for the notions regarding more specifically curvature and boundary varifolds we refer to Hutchinson [23], [22] and Mantegazza [24], [25].

Let $n, m \in \mathbb{N}$ with $n < m$ and let $G_{n,m}$ be the Grassmannian of the n-dimensional linear subspaces of $\mathbb{R}^m$. Given $T \in G_{n,m}$, $\pi^T : \mathbb{R}^m \to \mathbb{R}^m$ denotes the orthogonal projection on T. π^T may be identified with a $m \times m$-matrix $P^T = (P^T_{ij})$.

$G_{n,m}$ is a topological compact space endowed with the Euclidean metric in $\mathbb{R}^{m^2}$, up to the identification $T \to P^T_{ij}$ (see [28]).

Given $\Omega \subset \mathbb{R}^m$ we set $G_n(\Omega) = \Omega \times G_{n,m}$.

DEFINITION 2.1. *A n-dimensional varifold γ in an open set $\Omega \subset \mathbb{R}^m$ is a Radon measure on $G_n(\Omega)$.*

Let $\pi : G_n(\Omega) \to \Omega$ be the projection on the first component: $\pi(x, T) = x$. We introduce the *weight measure* μ_γ *of the varifold* γ by setting

$$\mu_\gamma(B) = \gamma(\pi^{-1}(B))$$

where B is a Borel set in Ω. We will deal only with varifolds such that the support of their weight measure is a rectifiable set of $\mathbb{R}^m$. To be more precise we will need some definitions.

DEFINITION 2.2. *Let $M \subset \mathbb{R}^m$ be $\mathcal{H}^n$-measurable and $\theta : M \to$ $\to (0, +\infty)$ locally integrable with respect to $\mathcal{H}^n \llcorner M$. We say that a linear subspace $P \in G_{n,m}$ is the approximate tangent space $apT_{x_0}M$ to M*

at x_0 with respect to the function θ if

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda^n}\int_M f\left(\frac{z-x_0}{\lambda}\right)\theta(z)\,d\mathcal{H}^n(z)=\theta(x_0)\int_P f(y)\,d\mathcal{H}^n(y)$$

for all $f\in C_c(\mathbb{R}^m)$.

REMARK 2.3. *If M is countably n-rectifiable, there exists $\mathcal{H}^n$-almost everywhere the approximate tangent space to M (see* [28]).

We denote by $Id\times P$ the function, defined $\mathcal{H}^n$- almost everywhere,

$$Id\times P: x\in M\subset\Omega\to(x,\, apT_xM)\in G_n(\Omega)\,.$$

We can define a Radon measure on $G_n(\Omega)$ by setting:

$$\gamma_{M,\theta}(B)=(\theta\mathcal{H}^n\llcorner M)(Id\times P)^{-1}(B)$$

for every $B\subset G_n(\Omega)$ Borel set. $\gamma_{M,\theta}$ is a varifold in Ω to which we refer as a *rectifiable varifold.* The weight measure of $\gamma_{M,\theta}$ is $\theta\mathcal{H}^n\llcorner M$: indeed, set $\gamma=\gamma_{M,\theta}$,

$$\mu_\gamma(B)=\gamma(\pi^{-1}(B))=(\theta\mathcal{H}^n\llcorner M)(Id\times P)^{-1}(\pi^{-1}(B))=(\theta\mathcal{H}^n\llcorner M)(B)$$

for every B Borel set in Ω.

The function θ is called *multiplicity* or *density* of the varifold M. Its name is due to the following result (see section 3.2 in [24]):

$$\lim_{\varrho\to 0}\frac{\mu_\gamma(B_\varrho(x))}{\omega_n\varrho^n}=\theta(x)\qquad\text{for }\mathcal{H}^n\text{a.e. }x\in M\tag{2.1}$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

DEFINITION 2.4. *If the density function of a rectifiable n-varifold $\gamma_{M,\theta}$ is integer valued, we say that $\gamma_{M,\theta}$ is an integer varifold.*

We are going to deal only with integer varifolds. It will be useful the following equality which holds for every bounded Borel function $\psi(x,P)\colon G_n(\mathbb{R}^m)\to\mathbb{R}$ (see [25]):

$$\int_{G_n(\mathbb{R}^m)}\psi(x,P)\,d\gamma_{M,\theta}(x,P)=\int_M\psi(x,\,apT_xM)\,\theta(x)\,d\mathcal{H}^n(x)\,.\tag{2.2}$$

In order to introduce the curvature and the boundary for a varifold we need the definition of gradient and divergence of a field $X\in$

$\in C_c^1(\Omega, \mathbb{R}^k)$, with respect to a linear n-dimensional subspace $P \in \mathbb{R}^m$. If we denote by π_P: $\mathbb{R}^m \to \mathbb{R}^m$ the orthogonal projection over P, for a function $f \in C^1(\mathbb{R}^m)$ we set:

$$\nabla^P f(x) = \pi_P \nabla f(x), \qquad \operatorname{div}^P X(x) = \sum_{i=1}^{m} \langle \nabla^P X_i(x), e_i \rangle .$$

More generally, if M is a smooth n-dimensional manifold in $\mathbb{R}^m$, we define

$$\nabla^M f(x) = \nabla^{T_x M} f(x), \qquad \operatorname{div}^M X(x) = \sum_{i=1}^{m} \langle \nabla^M X_i(x), e_i \rangle .$$

The last expression is called *tangential divergence* of the field X with respect to the manifold M. It can be seen, after a change of base, that the above expression is the divergence referred to an orthonormal base of the tangent space to M.

The following definitions of Allard's varifold and boundary varifold are given via integration by parts formulae. Let $\Omega \subset \mathbb{R}^m$ be an open set and γ an n-varifold in Ω. For each vector field $X \in C_c^1(\Omega, \mathbb{R}^m)$ we set

$$F(X) = \int_{G_n(\Omega)} \operatorname{div}^P X(x)\, d\gamma(x, P) .$$

If F is a linear locally bounded functional, Riesz's theorem (see [17]) gives the existence of a Radon vector measure $\delta\gamma$ in Ω such that:

$$F(X) = -\int_{\Omega} \langle X, d\delta\gamma \rangle \qquad \forall X \in C_c^1(\Omega) .$$

By Lebesgue decomposition theorem there exists a function H^γ: $\Omega \to \mathbb{R}^m$, $H^\gamma \in L^1_{\mathrm{loc}}(\mu_\gamma)$ such that $\delta\gamma = H^\gamma \mu_\gamma + \sigma_\gamma$, where σ_γ is the singular part of $\delta\gamma$ with respect to μ_γ. Therefore there exists a vector function ν_γ with $|\nu_\gamma| = 1$, $|\sigma_\gamma|$-a.e., such that:

$$\text{(2.3)} \qquad \int_{G_n(\Omega)} \operatorname{div}^P X(x)\, d\gamma(x, P) =$$

$$= -\int_{\Omega} \langle X, H^\gamma \rangle\, d\mu_\gamma - \int_{\Omega} \langle X, \nu_\gamma \rangle\, d|\sigma_\gamma| \quad \forall X \in C_c^1(\Omega) .$$

Following Allard (see [1], [2], [28]) we recall the definition:

DEFINITION 2.5. *A varifold γ in Ω satisfying equation* (2.3) *will be called Allard's varifold or varifold with locally bounded first variation. The support of the measure σ_γ is the generalized boundary of γ; H^γ, ν_γ are respectively the generalized mean curvature and the generalized inner unit co-normal to the generalized boundary.*

Allard's varifolds have good compactness properties but they exhibit structure defects. In fact they may have a very irregular boundary and no rectifiability result can be proved (see [24] for some examples). A sharper description of the curvature of a varifold was given by Hutchinson (see [23], [22]) who introduced the notion of generalized second fundamental form rather than mean curvature. It turns out that Allard's mean curvature is actually the trace of Hutchinson's second fundamental form, as in the classical case for manifolds. Nevertheless Hutchinson's varifolds are necessarily without boundary. For this reason Mantegazza introduced in [24] the notion of curvature varifolds with boundary which extends both Allard's and Hutchinson's definitions.

Given a function $\psi(x, P): \Omega \times \mathbb{R}^{m^2} \to \mathbb{R}$ we denote by $D_j \psi$, $j = 1, \ldots, m$ the partial derivatives with respect to $x \in \Omega$ and by $D^*_{ij} \psi$, $i, j = 1, \ldots, m$ the partial derivatives with respect to the variables $P \in \mathbb{R}^{m^2}$.

DEFINITION 2.6. *An integer n-varifold γ in Ω is called curvature varifold with boundary if there exist functions $A^\gamma_{ijl} \in L^1_{\mathrm{loc}}(G_n(\Omega), \gamma)$ and a Radon vector measure $\partial\gamma$ in $G_n(\Omega)$ with values in $\mathbb{R}^m$ such that, for every $\psi \in C^1_c(\Omega \times \mathbb{R}^{m^2})$,*

$$\sum_{j,l=1}^{m} \int_{G_n(\Omega)} \{P_{ij} D_j \psi(x,P) + D^*_{jl} \psi(x,P) A^\gamma_{ijl}(x,P) + \psi(x,P) A^\gamma_{jij}(x,P)\}\, d\gamma(x,P) =$$

$$= - \int_{G_n(\Omega)} \psi(x, P)\, d\partial\gamma_i(x, P) \qquad \forall i = 1, \ldots m\,.$$

$\partial\gamma$ is called «boundary measure» of the varifold γ.

We shall denote by $AV^p_n(\Omega)$ the class of the n-dimensional curvature varifolds with boundary such that $A^\gamma_{ijl} \in L^p_{\mathrm{loc}}(G_n(\Omega), \gamma)$. We remark that the functions A^γ_{ijl} are the components of a tensor A^γ which is related to the generalized second fundamental form of γ. Indeed, in the classical case when the varifold γ is actually $\gamma_{M,1}$ where M is a C^2-manifold, we have $A^{\gamma}_{ijl} = \langle \nabla^M P_{jl}, e_i \rangle$, with P projection on the tangent space to M. The mean curvature H is related to the curvature tensor A^{γ} by the following equality:

$$H^{\gamma}_i = \sum_j A^{\gamma}_{jij} \qquad \forall i = 1, \dots k$$

which is proved in [23].

Before we state the compactness theorem we remind what we mean by varifolds convergence: we say that a sequence of varifolds (γ_n) converges to a varifold γ_0 if it converges in the sense of weak convergence of measures; we will write $\gamma_n \rightharpoonup \gamma_0$.

The following theorem extends Allard's compactness theorem for locally bounded first variation varifolds to the case of curvature boundary varifolds (for a proof see [25]).

THEOREM 2.7. *Suppose (γ_k) is a sequence of $AV^p_n(\Omega)$ varifolds, with $p > 1$. If there exists a constant $c(W)$ such that*

$$\sup_{k \in \mathbb{N}} \mu_{\gamma_k}(W) + \int_{G_n(W)} |A^{\gamma_k}|^p \, d\gamma_k + |\partial \gamma_k|(G_n(W)) < \tag{2.4}$$

$$< c(W) < +\infty \qquad \forall W \subset\subset \Omega ,$$

then there exists a subsequence, which for simplicity we still denote by (γ_k), and a varifold $\gamma \in AV^p_n(\Omega)$, such that $\gamma_k \rightharpoonup \gamma$, $A^{\gamma_k}_{ijl} \llcorner \gamma_k$ converge weakly to $A_{ijl} \llcorner \gamma$ and $\partial \gamma_k$ converge weakly to $\partial \gamma$. Moreover, for every convex and lower semicontinuous function $f : \mathbb{R}^{m^3} \to [0, +\infty]$ the following inequality holds:

$$\int_{G_n(\Omega)} f(A^{\gamma}_{ijl}) \, d\gamma \leq \liminf_{k \to \infty} \int_{G_n(\Omega)} f(A^{\gamma_k}_{ijl}) \, d\gamma_k .$$

Before we state the structure theorem for curvature boundary varifolds, we have to recall the definition of tangent varifold, which extends the definition 2.2.

DEFINITION 2.8. *Given γ, n-varifold in Ω, $\varrho > 0$ and $x \in \operatorname{spt}(\mu_\gamma)$, consider the functions $\eta_{\varrho, x}(y) = \dfrac{y-x}{\varrho}$. For every $B \subset G_n(\mathbb{R}^m)$ Borel set, let*

$$E^{-1}_{\varrho, x}(B) = \{(y, P) \in G_n(\Omega) : (\eta_{\varrho, x}(y), (d\eta_{\varrho, x})_y(P)) \in B\}$$

and $\gamma_{\varrho,x}(B) = \frac{1}{\varrho^n}\gamma(E^{-1}_{\varrho,x}(B))$. *We call tangent varifold to* γ *at* x *and we write* $\operatorname{Var\,Tan}(\gamma, x)$, *the set of all possible weak limits for the sequence of varifolds* $(\gamma_{\varrho,x})$ *when* $\varrho \to 0$.

THEOREM 2.9. *Let* $\gamma \in AV_1^p(\Omega)$ *with* $p > 1$ *and let* $x_0 \in \operatorname{spt}(\mu_\gamma)$. *Then the following statements hold:*

1. *The density exists at* x_0, *i.e. there exists* $\lim_{\varrho \to 0} \frac{\mu_\gamma(B_\varrho(x_0))}{\omega_n \varrho^n}$.

2. *There exists the tangent varifold to* γ *in* x_0. *If* x_0 *is not a boundary point then the tangent varifold is a finite union of lines (with their multiplicity), otherwise is a finite union of lines and half-lines.*

3. *Let* T *be a line or an half-line tangent to the varifold* γ *in* x_0. *Then there exists a neighborhood* U *of* x_0 *such that* $\operatorname{spt}(\mu_\gamma) \cap U$ *is the graph, in* U, *of a* $C^{1,\alpha}$ *multiple valued function defined on* T.

4. *The rectifiable set* M *associated to* γ *is closed up to a* $\mathcal{H}^1$*-negligible set and the density function is upper semicontinuous in the points not belonging to the boundary of* γ.

The description of the varifolds in $AV_1^p(\Omega)$ will not be complete without the following result concerning the boundary structure:

THEOREM 2.10. *Let* $\gamma \in AV_1^p(\Omega)$, $p > 1$ *and let* $\partial\gamma$ *be its boundary measure. Then there exists a set at most countable* $\{x_j\} \subset \Omega$ *such that*

$$\partial\gamma = \sum_{j=1}^{\infty} \delta_{x_j} \times \nu_{x_j} \sigma_{x_j}$$

where δ_{x_j} *are Dirac's delta in* $\mathbb{R}^m$, σ_{x_j} *are measures in* $G_{1,m}$ *which are finite sums of Dirac's delta in* $G_{1,m}$ *and* ν_{x_j} *are unit vectors.*

For the definition of multiple valued function see [3], [5]; for the proof of theorems 2.9 and 2.10 see [24].

We remark that the good structure properties stated in the above theorem hold true only for one-dimensional varifolds. For this reason our work doesn't extend to higher dimensional objects.

The last tool we need is a corollary of the monotonicity formula for varifolds (see [28]).

THEOREM 2.11. *Let γ be a n-varifold in $\Omega \subset \mathbb{R}^m$ with locally bounded first variation, without boundary and mean curvature H^γ. Given $x \in \Omega$, set $\Gamma = \left(\int_{B_R(x)} |H^\gamma|^p d\mu_\gamma \right)^{1/p}$, where $\overline{B}_R(x) \subset \Omega$ and $p > n$, then:*

$$\left(\frac{\mu_\gamma(B_\sigma(x))}{\sigma^n} \right)^{1/p} \leqslant \left(\frac{\mu_\gamma(B_\varrho(x))}{\varrho^n} \right)^{1/p} + \frac{\Gamma}{p-n}(\varrho^{1-n/p} - \sigma^{1-n/p})$$

whenever $0 < \sigma < \varrho \leqslant R$.

3. The main result.

Let us consider the functional

$$F_B(\gamma) = \int_B \operatorname{dist}(x, K)\, d\mu_\gamma(x) + \int_K \operatorname{dist}(x, \gamma)\, d\mathcal{H}^1(x) + \int_B \phi(A^\gamma)\, d\mu_\gamma + \tag{3.1}$$

$$+ \mathcal{H}^1(K) \operatorname{diam}(\Omega) \int_B d\pi_\# |\partial\gamma|$$

where $B \subset \mathbb{R}^m$ is an open set, $\Omega \subset \mathbb{R}^m$ is an open bounded set, γ is an $AV_1^p(\mathbb{R}^m)$ varifold with boundary, $p > 1$, $\phi : \mathbb{R}^{m^3} \to [0, \infty)$ and K is a fixed closed subset of $\mathbb{R}^m$ such that $\mathcal{H}^1(K) < +\infty$. We will denote by γ both the varifold as measure in $G_1(\mathbb{R}^m)$ and the support of its weight measure μ_γ in $\mathbb{R}^m$. We prove existence of minima γ for $F_{\mathbb{R}^m}$ in $AV_1^p(\mathbb{R}^m)$, with the constraint $\operatorname{spt}(\mu_\gamma) \subset \overline{\Omega}$. Indeed we have the following:

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^m$ be an open bounded set and let $p > 1$. If ϕ is a lower semicontinuous convex function such that $\phi(t) \geqslant c|t|^p$ $\forall t \in \mathbb{R}^{m^3}$ for some constant $c > 0$, then the problem*

$$\min \{ F_{\mathbb{R}^m}(\gamma) : \gamma \in AV_1^p(\mathbb{R}^m),\ \operatorname{spt}(\mu_\gamma) \subset \overline{\Omega} \} \tag{3.2}$$

has at least one solution.

In order to prove this theorem we will need the following lemmas.

LEMMA 3.2. *Let (γ_n) be a minimizing sequence for (3.2). Then there exist a subsequence (γ_{k_n}) and $\gamma_0 \in AV_1^p(\mathbb{R}^m)$ such that $\mu_{\gamma_{k_n}} \rightharpoonup \mu_{\gamma_0}$ and $\operatorname{spt}(\mu_{\gamma_0}) \subset \overline{\Omega}$.*

PROOF. We first observe that the minimizing property of (γ_n) implies

$$(3.3)\qquad \sup_n \int_{\mathbb{R}^m} \operatorname{dist}(x,K)\, d\mu_{\gamma_n}(x) + \int_{\mathbb{R}^m} \phi(A^{\gamma_n})\, d\mu_{\gamma_n} + \int_{\mathbb{R}^m} d\pi_{\#}|\partial\gamma_n| \doteq M < +\infty$$

hence we can use the compactness result proved in Remark 2 of [21]. We obtain a subsequence of (γ_n), which we still denote by (γ_n), converging in $G_1(\mathbb{R}^m)$ to $\gamma_0 \in AV_1^p(\mathbb{R}^m)$. By remark 1.7.1 in [8] we know that $\mu_{\gamma_n} \rightharpoonup \mu_{\gamma_0}$ as measures in $\mathbb{R}^m$, then, since $\overline{\Omega}$ is compact and $\operatorname{spt}(\mu_{\gamma_n}) \subset \overline{\Omega}$, we have

$$\mu_{\gamma_0}(\mathbb{R}^m) \leq \liminf_{n\to\infty} \mu_{\gamma_n}(\mathbb{R}^m) \leq$$
$$\leq \limsup_{n\to\infty} \mu_{\gamma_n}(\mathbb{R}^m) = \limsup_{n\to\infty} \mu_{\gamma_n}(\overline{\Omega}) \leq \mu_{\gamma_0}(\overline{\Omega}). \qquad \blacksquare$$

Let us denote by $N_n \subset \overline{\Omega}$ the sets where the projections of the boundary measures $\pi_{\#}|\partial\gamma_n|$ are concentrated; set $N = \bigcup_{n=1}^{\infty} N_n$ and $L = \{y \in \mathbb{R}^m\colon y = \lim_{n\to\infty} z_n,\ z_n \in N_n\ \forall n \in \mathbb{N}\}$.

LEMMA 3.3. *Let (γ_n) be as in lemma 3.2. Then, up to a subsequence, the set L is finite and contains all cluster points of N.*

PROOF. By regularity results on one dimensional varifolds (see [24]), we know that N_n is discrete. Eventually extracting further subsequences, we can suppose that N_n is non empty for every $n \in \mathbb{N}$. In fact, if $N_n = \emptyset\ \forall n \geq \overline{n}$ then N is finite and the set of its cluster points is empty. We observe that, by setting $s = \operatorname{int}(M)$ the integer part of the real number M defined in (3.3), we have $\operatorname{card}(N_n) \leq s$. Hence we can set $N_n = \{x_n^{(1)}, \ldots, x_n^{(s_n)}\}$ with $s_n \leq s$. For simplicity we set $x_n^{(j)} = x_n^{(s_n)}$ if $s_n \leq j \leq s$.

We are going to extract iteratively s subsequences from (γ_n) in order to obtain a new sequence of sets N_{k_n} with the required property. By virtue of the constraint $\operatorname{spt}(\mu_{\gamma_n}) \subset \overline{\Omega}$ which is compact, we can find a first subsequence $(\gamma_{k_n^1})$ such that $(x_{k_n^1}^{(1)})$ converges to an element $y^{(1)} \in \overline{\Omega}$. Restarting from this new sequence we extract new subsequences $(\gamma_{k_n^j})_{n\in\mathbb{N}}$ from $(\gamma_{k_n^{j-1}})_{n\in\mathbb{N}}$ such that $(x_{k_n^j}^{(j)})_{n\in\mathbb{N}}$ converges to $y^{(j)}$ for every $j \in \{1, \ldots, s\}$. Let $\widetilde{N} = \bigcup_{n=1}^{\infty} N_{k_n^s}$ and $\widetilde{L} = \{y \in \mathbb{R}^m\colon y = \lim_{n\to\infty} z_n,\ z_n \in N_{k_n^s}\ \forall n \in \mathbb{N}\}$. Choose now $y \in \widetilde{L}$ and $z_n \to y$ with $z_n \in N_{k_n^s}\ \forall n \in \mathbb{N}$. We have that

$z_n = x_{k_n^s}^{(j_n)}$ for a suitable choice of j_n. There exists at least an index $j \leq s$ such that $\operatorname{card}(\{n \in \mathbb{N}: j_n = j\}) = \infty$. Considering the subsequence $\{z_n: j_n = j\}$ we have that $y = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_{k_n^s}^{(j)} = \lim_{n\to\infty} x_{k_n^j}^{(j)} = y^{(j)}$, hence $\operatorname{card}(\tilde{L}) \leq s$. Let now $\bar{z}$ be a cluster point for $\tilde{N}$. By contradiction we suppose that $\bar{z} \notin \tilde{L}$. There exists $\varepsilon > 0$ such that $B_\varepsilon(\bar{z}) \cap \left(\bigcup_{j=1}^{s} B_\varepsilon(y^{(j)})\right) = \emptyset$. But there exist $n_j \in \mathbb{N}$ such that if $n \geq n_j$ then $x_{k_n^s}^{(j)} \in B_\varepsilon(y^{(j)})$ for $j = 1, \ldots, s$. Hence in $B_\varepsilon(\bar{z})$ there are only a finite number of elements of $\tilde{N}$ that contradicts the assumption made on $\bar{z}$. This proves that $\bar{z} \in \tilde{L}$ and that $\tilde{N}$ have at most s cluster points. ■

In the sequel we use N and L instead of $\tilde{N}$ and $\tilde{L}$ and we suppose directly that $\gamma_{k_n^s} = \gamma_n$ so that the sequences $x_n^{(j)}$ are converging to $y^{(j)}$ and consequently N has at most s cluster points.

REMARK 3.4. *The set $N \cup L$ is compact.*

PROOF. Since L is finite, the cluster points for $N \cup L$ are also cluster points for N, hence $N \cup L$ is closed. The compactness follows from the inclusion $(N \cup L) \subset \overline{\Omega}$. ■

To prove lower semicontinuity for the functional $F_{\mathbb{R}^m}$, by using the remark 2 of [21], we have just to prove that the term $\int_K \operatorname{dist}(x, \gamma)\, d\mathcal{H}^1(x)$ is lower semicontinuous. This is, in general, not true, as the following example shows.

EXAMPLE 3.5. Let γ_n the varifold constructed over the rectifiable set $\{(x, y) \in \mathbb{R}^2: 0 \leq x \leq 1,\ y = 2\} \cup \{(x, y) \in \mathbb{R}^2: 0 \leq x \leq \frac{1}{n},\ y = 1\}$. It is easy to see that the sequence (γ_n) converges to the varifold γ_0 constructed on the set $\{(x, y) \in \mathbb{R}^2: 0 \leq x \leq 1,\ y = 2\}$. If we set $K = \{(x, y) \in \mathbb{R}^2: 0 \leq x \leq 1,\ y = 0\}$ then, for every $x \in K$, we have $\operatorname{dist}(x, \gamma_0) = 2$, $\operatorname{dist}(x, \gamma_n) \leq \sqrt{2}$. Hence $\liminf_{n\to\infty} \int_K \operatorname{dist}(x, \gamma_n)\, d\mathcal{H}^1 < \int_K \operatorname{dist}(x, \gamma_0)\, d\mathcal{H}^1$.

The idea is to pull away from γ_n those components which are useless because they tend to shrink and disappear in the limit varifold γ_0, giving lack of lower semicontinuity. Therefore we are going to consider a new sequence (γ'_n) of varifolds in $AV_1^p(U)$, with $U \subset \mathbb{R}^m$ suitable open set, which converges to a varifold $\gamma'_0 \in AV_1^p(U)$. We will prove lower semicontinuity for the functional F_U on the sequence (γ'_n) and theorem 3.1 will easily follow.

LEMMA 3.6. *Let (γ_n) and L be as above. Set*

$$\Lambda = \left\{ y^{(j)} \in L : \exists \varrho_j > 0 : \lim_{n \to \infty} \mu_{\gamma_n}(\overline{B}_{\varrho_j}(y)) = 0 \right\}.$$

Then we can find $r_j < \varrho_j$ such that $(L \setminus \Lambda) \subset U \doteqdot \mathbb{R}^m \setminus \bigcup_{y^{(j)} \in \Lambda} \overline{B}_{r_j}(y^{(j)})$. If we consider the varifolds $\gamma'_n = \gamma_n \llcorner G_1(U)$ then $\gamma'_n \in AV_1^p(U)$, $\operatorname{spt}(\gamma_0) \subset \subset G_1(U)$ and $\gamma'_n \rightharpoonup \gamma_0$ in $G_1(U)$.

PROOF. Since the set L is finite, the existence of r_j is trivial. It is also easy to see that $\gamma'_n \in AV_1^p(U)$. Moreover, since $\gamma_n \rightharpoonup \gamma_0$ as measures in $G_1(\mathbb{R}^m)$, we have $\mu_{\gamma_n} \rightharpoonup \mu_{\gamma_0}$ as measures in $\mathbb{R}^m$. Set $V = \mathbb{R}^m \setminus \bigcup_{y^{(j)} \in \Lambda} B_{\eta_j}(y^{(j)})$, with $r_j < \eta_j < \varrho_j$. If j is such that $y^{(j)} \in \Lambda$ then $\lim_{n \to \infty} \mu_{\gamma_n}(\overline{B}_{\varrho_j}(y^{(j)})) = 0$ hence, because of the lower semicontinuity:

$$\mu_{\gamma_0}(B_{r_j}(y^{(j)})) \leq \mu_{\gamma_0}(B_{\eta_j}(y^{(j)})) \leq \liminf_{n \to \infty} \mu_{\gamma_n}(B_{\varrho_j}(y^{(j)})) = 0 .$$

Therefore the set $(\mathbb{R}^m \setminus V) \times G_{1,m}$ is negligible for the measure γ_0, hence $\operatorname{spt}(\gamma_0) \subset G_1(V) \subset G_1(U)$ (V is a closed set).

Let $\psi \in C_c(U \times \mathbb{R}^{m^2})$, then:

$$\int_{G_1(U)} \psi(x,P)\, d\gamma'_n = \int_{G_1(\mathbb{R}^m)} \psi(x,P)\, d\gamma_n \to \int_{G_1(\mathbb{R}^m)} \psi(x,P)\, d\gamma_0 = \int_{G_1(U)} \psi(x,P)\, d\gamma_0$$

which concludes the proof. ∎

COROLLARY 3.7. $F_{\mathbb{R}^m}(\gamma_0) = F_U(\gamma_0)$.

PROOF. This is obvious because $\operatorname{spt}(\mu_{\gamma_0}) \subset (U)$ and $\operatorname{spt}(\pi_\# |\partial\gamma_0|) \subset \subset U$. ∎

LEMMA 3.8. *Let (γ_n) and N_n be as above. There exists $c > 0$ independent of n such that for every $x \in \operatorname{spt}(\mu_{\gamma_n}) \setminus N_n$ and ϱ with $0 < \varrho < \min\{c, \operatorname{dist}(x, N_n)\}$ it results that:*

$$\mu_{\gamma_n}(B_\varrho(x)) \geq \varrho .$$

PROOF. For each $n \in \mathbb{N}$ let $V_n = \mathbb{R}^m \setminus N_n$; $\gamma_n \llcorner G_1(V_n)$ is a curvature varifold without boundary and we can apply theorem 2.11 obtaining:

$$\left(\frac{\mu_{\gamma_n}(B_\sigma(x))}{\sigma}\right)^{1/p} \leqslant \left(\frac{\mu_{\gamma_n}(B_\varrho(x))}{\varrho}\right)^{1/p} + \frac{\Gamma_n}{p-1}(\varrho^{1-1/p} - \sigma^{1-1/p})$$

where $\Gamma_n = \left(\int_{V_n} |H^{\gamma_n}|^p d\mu_{\gamma_n}\right)^{1/p}$, H^{γ_n} is the mean curvature of γ_n and

$$0 < \sigma < \varrho < \operatorname{dist}(x, N_n).$$

By passing to the limit when $\sigma \to 0$, recalling (2.1) and that $\omega_1 = 2$, we have:

$$(2\theta_{\gamma_n}(x))^{1/p} \leqslant \left(\frac{\mu_{\gamma_n}(B_\varrho(x))}{\varrho}\right)^{1/p} + \frac{\Gamma_n}{p-1}\varrho^{1-1/p};$$

moreover, by using the minimizing property of the integer varifolds γ_n, set $\Gamma = \sup_n \Gamma_n \leqslant M < +\infty$, we have

$$\frac{\mu_{\gamma_n}(B_\varrho(x))}{\varrho} \geqslant \left(2^{1/p} - \frac{\Gamma_n}{p-1}\varrho^{1-1/p}\right)^p \geqslant \left(2^{1/p} - \frac{\Gamma}{p-1}\varrho^{1-1/p}\right)^p.$$

Hence, if we choose $c > 0$ such that $c < 2^{\frac{1}{p-1}}\left(\frac{p-1}{\Gamma}\right)^{\frac{p}{p-1}}$ than for every ϱ such that $0 < \varrho < \min\{c, \operatorname{dist}(x, N_n)\}$

$$\mu_{\gamma_n}(B_\varrho(x)) \geqslant \varrho \qquad \forall x \in \operatorname{spt}(\mu_{\gamma_n}) \setminus N_n. \tag{3.4}$$ ■

LEMMA 3.9. *Let U, γ'_n and γ_0 be as in lemma 3.6, then $F_U(\gamma_0) \leqslant \liminf_{n\to\infty} F_U(\gamma'_n)$.*

PROOF. By the lower semicontinuity result in [21] the only term in the functional that needs to be checked is the integral $\int_K \operatorname{dist}(x, \gamma'_n)\, d\mathcal{H}^1(x)$. By Fatou's lemma, it will be enough to prove that for every $x \in \mathbb{R}^m$ it results

$$\operatorname{dist}(x, \gamma_0) \leqslant \liminf_{n\to\infty} \operatorname{dist}(x, \gamma'_n).$$

By contradiction, we fix $x \in \mathbb{R}^m$ and we suppose that:

$$\operatorname{dist}(x, \gamma_0) > \liminf_{n\to\infty} \operatorname{dist}(x, \gamma'_n).$$

There exists $\varepsilon > 0$ and a subsequence of (γ'_n), which we still denote by

(γ'_n), such that

$$\operatorname{dist}(x, \gamma'_n) < \operatorname{dist}(x, \gamma_0) - 8\varepsilon .$$

We will use the same notations as in the proof of lemma 3.3 with N, L, N_n instead of $\widetilde{N}, \widetilde{L}, N_{k_n^s}$, therefore $N_n = \{x_n^{(j)}: j = 1, \dots, s\}$, $\lim_{n \to \infty} x_n^{(j)} = y^{(j)}$ and $L = \{y^{(j)}: j = 1, \dots, s\}$. We denote by N'_n the support of the boundary measure $\pi_\# |\partial \gamma'_n|$. Since the varifolds γ'_n are considered in the open set $U \subset \mathbb{R}^m$ it results that $N'_n = N_n \cap U$. We set $L' = L \cap U$ and $N' = \bigcup_{n=1}^{\infty} N'_n$. We recall now that $\mu_{\gamma'_n} = (\theta_n \mathcal{H}^1 \llcorner \gamma'_n)$ where θ_n and γ'_n are respectively the density and the support of the measure $\mu_{\gamma'_n}$, hence $\mu_{\gamma'_n}(N' \cup L') = 0$ since the set $N' \cup L'$ is at most countable. By theorem 2.9, the density exists at every point of γ'_n, and $\operatorname{dist}(x, \gamma'_n) = \inf_{z \in \gamma'_n} |x - z|$, then we can choose $z_n \in \gamma'_n \setminus (N' \cup L') \subset \overline{\Omega}$ such that

$$\text{(3.5)} \quad |x - z_n| \leqslant d(x, \gamma'_n) + 4\varepsilon < d(x, \gamma_0) - 4\varepsilon \leqslant |x - y| - 4\varepsilon \quad \forall y \in \gamma_0 .$$

Therefore, up to a subsequence which we still denote by (z_n), there exists $\bar{z} \in \overline{\Omega}$ such that $z_n \to \bar{z}$. Passing to the limit in (3.5), we have

$$|x - \bar{z}| \leqslant |x - y| - 4\varepsilon \quad \forall y \in \gamma_0$$

therefore

$$|y - \bar{z}| \geqslant |x - y| - |x - \bar{z}| \geqslant 4\varepsilon \quad \forall y \in \gamma_0$$

and consequently

$$\text{(3.6)} \qquad B_{4\varepsilon}(\bar{z}) \cap \gamma_0 = \emptyset .$$

To contradict (3.6) we will distinguish several cases:

1) $\bar{z} \notin L' \cup N'$.

By remark 3.4, $L' \cup N'$ is compact, hence it results that $\delta \doteqdot \operatorname{dist}(\bar{z}, L' \cup N') > 0$. We are going to evaluate $\operatorname{dist}(\bar{z}, N_n)$. If $L' = L$ then $U = \mathbb{R}^m$, $N'_n = N_n$ and we have the estimate $\operatorname{dist}(\bar{z}, N_n) \geqslant \operatorname{dist}(\bar{z}, N' \cup L') = \delta$. If $L' \neq L$ we shall consider separately the points of N_n which are outside or inside of U. Let us fix j such that $y^{(j)} \in L'$; then $y^{(j)} \in U$, $x_n^{(j)} \in N'_n$, for $n \gg 1$ and

$$|\bar{z} - x_n^{(j)}| \geqslant \operatorname{dist}(\bar{z}, N') \geqslant \operatorname{dist}(\bar{z}, N' \cup L') = \delta > 0 .$$

Now set

$$\delta_n = \min\{r_j - |x_n^{(j)} - y^{(j)}| : 1 \leq j \leq s,\ y^{(j)} \notin L'\}$$

with r_j as in the definition of U. Since $\lim_{n\to\infty} x_n^{(j)} = y^{(j)}$, there exists $\overline{\delta} > 0$ such that $\delta_n \geq \overline{\delta}$ for n large enough. We recall that $z_n \in U \forall n \in \mathbb{N}$, hence $\overline{z} \in \overline{U} \setminus (L' \cup N')$. Let us fix $j \in \{1, \ldots, s\}$ such that $y^{(j)} \notin L'$ (this means that the points of U are far from $y^{(j)}$). Therefore we have:

$$|\overline{z} - x_n^{(j)}| \geq |\overline{z} - y^{(j)}| - |y^{(j)} - x_n^{(j)}| \geq r_j - |y^{(j)} - x_n^{(j)}| \geq \delta_n \geq \overline{\delta}.$$

Collecting the inequalities above we obtain that, for n large enough,

$$\operatorname{dist}(\overline{z}, N_n) \geq \min\{\delta, \overline{\delta}\} > 0. \tag{3.7}$$

If we set $\delta' = \min\left\{\frac{\delta}{2}, \frac{\overline{\delta}}{2}, \varepsilon\right\}$, it results $|\overline{z} - z_n| < \delta'$ definitively. As a consequence of lemma 3.8 we have that there exists $\overline{n} \in \mathbb{N}$ such that

$$\mu_{\gamma_n}(B_\varrho(z_n)) \geq \varrho \quad \text{if} \quad 0 < \varrho < \min\{\delta', c\} \quad \forall n \geq \overline{n}, \tag{3.8}$$

therefore, by upper semicontinuity of measure convergence on compact sets we have:

$$0 < \varrho \leq \limsup_{n\to\infty} \mu_{\gamma_n}(\overline{B}_\varrho(z_n)) \leq \limsup_{n\to\infty} \mu_{\gamma_n}(B_{\varrho+\delta'}(\overline{z})) \leq$$

$$\leq \limsup_{n\to\infty} \mu_{\gamma_n}(\overline{B}_{2\varepsilon}(\overline{z})) \leq \mu_{\gamma_0}(\overline{B}_{2\varepsilon}(\overline{z}))$$

which contradicts (3.6).

2) $\overline{z}$ is a cluster point for N'.

This means that $\overline{z} \in L'$. By construction of (γ'_n) we have that, for every $r > 0$, it results $\mu_{\gamma'_n}(\overline{B}_r(\overline{z})) \not\to 0$. The upper semicontinuity over compact sets implies that:

$$0 < \limsup_{n\to\infty} \mu_{\gamma'_n}(\overline{B}_r(\overline{z})) \leq \mu_{\gamma'_0}(\overline{B}_r(\overline{z})) = \mu_{\gamma_0}(\overline{B}_r(\overline{z}))$$

which contradicts condition (3.6).

3) $\overline{z} \in N' \setminus L'$.

We will prove that $\overline{z} \in N'_n$ only for a finite number of $n \in \mathbb{N}$. By contradiction, let us suppose that there exists an increasing sequence (k_n) such that $\overline{z} \in N'_{k_n}$. It follows that $\overline{z} = x_{k_n}^{(j_n)}$ for a suitable choice of $j_n \in \{1, \ldots, s\}$, hence, extracting a further subsequence, we can suppose $\overline{z} = x_{k_n}^{(j)}$ for a fixed j. For the construction of the set N' then it must be $\overline{z} = y^{(j)} \in L'$ and this is not possible. We have proved that there exists

$n' \in \mathbb{N}$ such that, if $\bar{z} \in N_n'$ then $n \leqslant n'$. We have that $\delta = \operatorname{dist}\left(\bar{z}, L' \cup \bigcup_{n > n'} N_n'\right) > 0$ and we can conclude as in case 1).

We have covered every possibility for the limit point $\bar{z}$ and this concludes the proof. ■

LEMMA 3.10. *$F_U(\gamma_n') \leqslant F_{\mathbb{R}^m}(\gamma_n)$ for n large enough.*

PROOF. Using the same notations used in the proof of the previous lemma, let us fix $j \in \{1, \ldots, s\}$ such that $y^{(j)} \notin L'$ (if such a j doesn't exist, then $L' = L$, $N' = N$, $U = \mathbb{R}^m$, $\gamma_n' = \gamma_n$ and there is nothing to prove). Since $x_n^{(j)} \to y^{(j)}$, there exists $\bar{n} \in \mathbb{N}$ such that if $n \geqslant \bar{n}$ then $x_n^{(j)} \notin U$. It follows that $N_n' \subsetneq N_n$ and, for theorem 2.10 $\left(\int_U d\pi_\# |\partial \gamma_n'|\right) + 1 \leqslant \int_{\mathbb{R}^m} d\pi_\# |\partial \gamma_n|$. Hence

$$F_U(\gamma_n') \leqslant \int_{\mathbb{R}^m} \operatorname{dist}(x, K)\, d\mu_{\gamma_n} + \int_K (\operatorname{dist}(x, \gamma_n) + \operatorname{diam}(\Omega))\, d\mathcal{H}^1(x) +$$

$$+ \int_{\mathbb{R}^m} \phi(A^{\gamma_n})\, d\mu_{\gamma_n} + \mathcal{H}^1(K) \operatorname{diam}(\Omega)\left(\int_{\mathbb{R}^m} d\pi_\# |\partial\gamma_n| - 1\right) \leqslant F_{\mathbb{R}^m}(\gamma_n). \qquad ■$$

Now we are ready to prove existence of minima for $F_{\mathbb{R}^m}$.

PROOF OF THEOREM 3.1. By corollary 3.7, lemma 3.9 and lemma 3.10 we get:

$$F_{\mathbb{R}^m}(\gamma_0) = F_U(\gamma_0) \leqslant \liminf_{n \to \infty} F_U(\gamma_n') \leqslant \liminf_{n \to \infty} F_{\mathbb{R}^m}(\gamma_n) =$$

$$= \inf\{F_{\mathbb{R}^m}(\gamma) \colon \gamma \in AV_1^p(\mathbb{R}^m),\ \operatorname{spt}(\mu_\gamma) \subset \overline{\Omega}\}. \qquad ■$$

4. Final remarks.

In the last section we proved the existence of solutions for the minimum problem (3.2). The question now is about the regularity of minima. What we know, from theorem 2.9, is that the varifolds we are dealing with, are locally the graph of multiple valued functions (for a definition of multiple valued function see [3], [5]). By recent results on regular selections for multiple valued functions (see [20]) we are able to prove that, locally, the support of the weight measure μ_γ for the varifold γ minimizing (3.2), is decomposable in the finite union of graphs of $C^{1,\alpha}$ curves (the result contained in [20] concerns, actually, $C^{1,1}$ graphs, but in a more recent work, to appear, C. De Lellis, P. Tilli and the author prove that the same result holds true for $C^{1,\alpha}$ functions too). Hence we can cover the support of μ_γ, which is compact because it is closed (see theorem 2.9) and Ω is bounded, with a finite union of balls, obtaining that it is globally the finite union of $C^{1,\alpha}$ arcs.

As a final remark we want to underline the role played by each single term in the functional (3.1). The first and the second term represent an integral distance between the sets K and γ. They prevent the varifold γ to be far from K, up to sets of small $\mathcal{H}^1$ measure. This means that it is possible, for a small connected component of γ, to be far from K (or vice versa) but, if this occurs, the third or the fourth term of the functional will increase. Indeed the third integral takes into account of the curvature, hence, if there are small connected components without boundary they must have a big curvature. For instance, if $\phi(A^\gamma) = |H^\gamma|^2$, then the integral $\int |H^\gamma|^2 d\mathcal{H}^1$ on a circle with radius r will be of order $\frac{1}{r}$. The fourth term counts the number of edges in the varifold, that is the number of points where the tangent jumps. Small connected components with boundary, like little segments, will pay a cost of at least two boundary points, hence the functional tends to prevent the growth of such phenomena.

We must pay particular attention to the coefficient $\mathcal{H}^1(K)\,\mathrm{diam}(\Omega)$ in front of the last term. It plays a crucial role in the proof of the lower semicontinuity. To be more precise, it is possible to modify the functional (3.1) including weights for its terms:

$$F_B(\gamma) = \alpha \int_B \mathrm{dist}(x, K)\, d\mu_\gamma(x) + \beta \int_K \mathrm{dist}(x, \gamma)\, d\mathcal{H}^1(x) +$$

$$+ \lambda \int_B \phi(A^\gamma)\, d\mu_\gamma + \sigma \int_B d\pi_\# |\partial\gamma| .$$

The proof is easily modified, but the ratio $\frac{\sigma}{\beta}$ have to be at least $\mathcal{H}^1(K)\,\mathrm{diam}(\Omega)$. This result is optimal for this functional as situations similar to example 3.5 show. We want to note that this constraint doesn't affect the freedom to balance the ratio between the others coefficients, especially the ratio $\frac{\lambda}{\sigma}$ which controls the trend of the functional to prefer sharp to smooth curves. If the value $\frac{\sigma}{\beta}$ is big, then the minimizing process is made, in some sense, in two steps: first we choose the right balance between edges and curvature, then we minimize the distance between γ and K among the curves with a fixed number of edges. We observe also that the number $\mathcal{H}^1(K)\,\mathrm{diam}\,(\Omega)$ depends by how much we want to allow the varifold γ to move around K: if we choose a set Ω too big we are wrong with the dimensional scale of the problem and the closeness to K doesn't make much sense.

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