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EDGAR E. ENOCHS
OVERTOUN M. G. JENDA
LUIS OYONARTE

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λ and μ-Dimensions of Modules.

EDGAR E. ENOCHS (*) - OVERTOUN M. G. JENDA (**)
LUIS OYONARTE (***)

ABSTRACT - Bourbaki [1] defined λ-dimension using finite presentations. In this paper, we extend this definition by replacing finite presentations with resolutions obtained by using either $\mathcal{F}$-precovers, or $\mathcal{F}$-precovers $\varphi : F \to M$ such that φ is an epimorphism and $\mathrm{Ker}(\varphi)$ is orthogonal to $\mathcal{F}$, where $\mathcal{F}$ is a class of modules closed under direct sums. The aim of this paper is to study these λ-dimensions. As an application, we prove the existence of Gorenstein flat covers over n-Gorenstein rings.

1. Introduction.

Throughout this paper, R will denote an associative ring with unity, R-module will mean a left R-module, and $\mathcal{F}$ will denote a class of R-modules closed under finite direct sums.

We recall that if M is an R-module, then a morphism $\varphi : F \to M$ is called an $\mathcal{F}$-precover of M if $F \in \mathcal{F}$ and $\mathrm{Hom}(F', F) \to \mathrm{Hom}(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. If moreover, any morphism $f : F \to F$ such that

(*) Indirizzo dell'A.: Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA. E-mail address: enochs@ms.uky.edu

(**) Indirizzo dell'A.: Department of Discrete and Statistical Sciences, 120 Mathematics Annex, Auburn University, AL 36849-5307, USA. E-mail address: jendaov@mail.auburn.edu

(***) Indirizzo dell'A.: Department of Algebra and Mathematical Analysis, University of Almería, 04120 Almería, Spain. E-mail address: oyonart@ual.es

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$\varphi = \varphi \circ f$ is an automorphism of F, then $\varphi : F \to M$ is called an $\mathcal{F}$-cover of M. $\mathcal{F}$-preenvelope and $\mathcal{F}$-envelope $M \to F$ are defined dually. If $\mathcal{F}$-covers and $\mathcal{F}$-envelopes exist, then they are unique up to isomorphism. If every R-module has an $\mathcal{F}$-(pre)cover, $\mathcal{F}$-(pre)envelope, we say that $\mathcal{F}$ is *(pre)covering, (pre)enveloping*, respectively.

We note that $\mathcal{F}$-precovers are not necessarily epimorphisms. But if $\mathcal{F}$ contains all the projective R-modules, then φ is an epimorphism. Similarly, if $\mathcal{F}$ contains all the injective R-modules, then an $\mathcal{F}$-preenvelope $\varphi : M \to F$ is a monomorphism.

A (partial) complex $M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 (n \geq 2)$ of R-modules is said to be $\operatorname{Hom}(\mathcal{F}, -)$ *exact* if the sequence

$$\cdots \operatorname{Hom}(F, M_n) \to \operatorname{Hom}(F, M_{n-1}) \to \cdots \to \operatorname{Hom}(F, M_1) \to \operatorname{Hom}(F, M_0)$$

is exact for all $F \in \mathcal{F}$. By a *left $\mathcal{F}$-resolution* of an R-module M we mean an $\operatorname{Hom}(\mathcal{F}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (not necessarily exact) with each $F_i \in \mathcal{F}$. A *right $\mathcal{F}$-resolution* of M is defined dually. We note that Eilenberg-Moore [2] call these resolutions *projective (injective) resolutions of M for the class $\mathcal{F}$*. A finite $\operatorname{Hom}(\mathcal{F}, -)$ exact complex $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathcal{F}$ is called a *partial left $\mathcal{F}$-resolution of M of length n. Partial right resolutions* are defined similarly.

We say that $\lambda_{\mathcal{F}}(M) = -1$ if M does not have an $\mathcal{F}$-precover. If $n \geq 0$, we say that $\lambda_{\mathcal{F}}(M) = n$ if there is a partial left $\mathcal{F}$-resolution $F_n \to \cdots \to \to F_1 \to F_0 \to M \to 0$ of M of length n and if there is no longer such complex. We say $\lambda_{\mathcal{F}}(M) = \infty$ if there exists a partial left $\mathcal{F}$-resolution for each $n \geq 0$. Dually, we say that $\mu_{\mathcal{F}}(M) = -1$ if M does not have an $\mathcal{F}$-preenvelope, and $\mu_{\mathcal{F}}(M) = n$ with $0 \leq n < \infty$ if there is a partial right $\mathcal{F}$-resolution $0 \to M \to F^0 \to \cdots \to F^n$ of length n and if there is no longer such complex. $\mu_{\mathcal{F}}(M) = \infty$ if there is such a complex for each $n \geq 0$. $\lambda_{\mathcal{F}}(M)$ is called the λ-dimension of M relative to $\mathcal{F}$ and is denoted $\lambda(M)$ if $\mathcal{F}$ is understood. Similarly, $\mu_{\mathcal{F}}(M)$ (or simply $\mu(M)$) is called the μ-dimension of M relative to $\mathcal{F}$.

In this paper, we will study properties of λ-dimensions. It is natural to ask whether $\lambda(M) = \infty$ implies that there is an infinite left $\mathcal{F}$-resolution $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ of M. We will show that this is indeed the case (Corollary 2.6). We will also show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules such that $0 \to \operatorname{Hom}(F, M') \to \to \operatorname{Hom}(F, M) \to \operatorname{Hom}(F, M'') \to 0$ is exact for all $F \in \mathcal{F}$, then $\lambda(M'') \geq \geq \min(\lambda(M') + 1, \lambda(M))$, $\lambda(M) \geq \min(\lambda(M'), \lambda(M''))$ and $\lambda(M') \geq$

$\geqslant \min(\lambda(M), \lambda(M'') - 1)$ (Theorem 2.10). We note that if $\mathcal{F}$ is the class of finitely generated projectives, then $\lambda(M) \geqslant 0$ if and only if M is finitely generated, and $\lambda(M) \geqslant 1$ if and only if M is finitely presented. In this case, the λ-dimension defined above is the λ-dimension of Bourbaki [1, page 41], and Theorem 2.10 corresponds to their Exercise 6. In Section 3, we will obtain results corresponding to the ones in Section 2 for λ-dimensions relative to $\mathcal{F}$-precovers $\varphi : F \to M$ such that φ is an epimorphism and $\mathrm{Ext}^1(F, \mathrm{Ker}\,\varphi) = 0$ for all $F \in \mathcal{F}$. All the results in Sections 2 and 3 have their counterparts concerning μ-dimensions. For each of these the proof is just the dual of the proof of the corresponding result and hence we will not state them here. Finally, in Section 4 we use λ-dimensions to prove that the class of Gorenstein flat R-modules is covering over n-Gorenstein rings (Theorem 4.3) which is a result of Xu-Enochs [7].

As usual, *inj. dim M, proj. dim M* will denote injective and projective dimensions of M respectively.

It is well known that if $0 \to C' \to C \to C'' \to 0$ is an exact sequence of complexes then we have an associated long exact sequence of homology. We will frequently use this result and its concomitant implications about the exactness of C', C, C'' at the various terms of these complexes. We also recall that given a map $f : C \to D$ of complexes we have the mapping cone $M(f)$ of f and the associated exact sequence $0 \to D \to M(f) \to$ $\to C[-1] \to 0$ of complexes.

A partial complex will often be thought of as a complex with the extra terms being zero.

2. λ-dimensions.

We start with the following easy

LEMMA 2.1. *If M is an R-module and $F \in \mathcal{F}$, then $F \oplus M$ has an $\mathcal{F}$-precover if and only if M has an $\mathcal{F}$-precover.*

PROOF. If $G \to M$ is an $\mathcal{F}$-precover, then easily so is $F \oplus G \to F \oplus M$. Conversely, if $\varphi : G \to F \oplus M$ is an $\mathcal{F}$-precover, then so is $\pi_2 \circ \varphi : G \to M$ where $\pi_2 : F \oplus M \to M$ is the projection map. ■

The following is called Schanuel's lemma when $\mathcal{F}$ is the class of projective R-modules.

LEMMA 2.2. *If $F \to M$, $G \to M$ are $\mathcal{F}$-precovers with kernels K and L respectively, then $K \oplus G \cong L \oplus F$.*

PROOF We consider the pullback diagram

$$\begin{array}{ccc} P & \to & G \\ \downarrow & & \downarrow \\ F & \to & M \end{array}$$

The map $G \to M$ has a factorization $G \to F \to M$ since $F \to M$ is an $\mathcal{F}$-precover. So $P \to G$ has a section and thus $P \cong K \oplus G$ since $\operatorname{Ker}(P \to G) \cong \operatorname{Ker}(F \to M) = K$. Similarly, $P \cong L \oplus F$ and so we are done. ∎

PROPOSITION 2.3. *Let $n \geq 0$ and $F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ and $G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be partial left $\mathcal{F}$-resolutions of M. If $K = \operatorname{Ker}(F_n \to F_{n-1})$ and $L = \operatorname{Ker}(G_n \to G_{n-1})$ where $F_{-1} = G_{-1} = M$, then*

$$K \oplus G_n \oplus F_{n-1} \oplus \cdots \cong L \oplus F_n \oplus G_{n-1} \oplus \cdots$$

PROOF. By induction on n. The case $n = 0$ is Lemma 2.2 above. If $n > 0$, the complexes $F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \oplus G_0 \to K \oplus G_0 \to 0$ and $G_n \to G_{n-1} \to \cdots \to G_2 \to G_1 \oplus F_0 \to L \oplus F_0 \to 0$ are partial left $\mathcal{F}$-resolutions by Lemma 2.1. Furthermore, $K \oplus G_0 \cong L \oplus F_0$ by the above. So an appeal to the induction hypothesis gives the result. ∎

PROPOSITION 2.4. *$\lambda(F \oplus M) = \lambda(M)$ for all $F \in \mathcal{F}$.*

PROOF. We prove that for $n \geq -1$, $\lambda(F \oplus M) \geq n$ if and only if $\lambda(M) \geq n$. This is trivial if $n = -1$. It is true for $n = 0$ by Lemma 2.1. Now let $n > 0$.

Suppose $\lambda(M) \geq n$. If $F_n \to \cdots \to F_0 \to M \to 0$ is a partial left $\mathcal{F}$-resolution, then so is the complex $F \oplus F_n \to \cdots \to F \oplus F_0 \to F \oplus M \to 0$. Thus $\lambda(F \oplus M) \geq n$.

Conversely suppose $\lambda(F \oplus M) \geq n$ and let $G_n \to \cdots \to G_0 \to F \oplus M \to 0$ be a partial left $\mathcal{F}$-resolution of $F \oplus M$. We know that $\lambda(M) \geq 0$ and so let $F_0 \to M$ be an $\mathcal{F}$-precover. Set $K = \operatorname{Ker}(F_0 \to M)$ and $L = \operatorname{Ker}(G_0 \to F \oplus M)$. Then $F \oplus F_0 \to F \oplus M$ is also an $\mathcal{F}$-precover with kernel K and so $L \oplus F \oplus F_0 \cong K \oplus G_0$ by Proposition 2.3. But $\lambda(L) \geq n-1$ and so $\lambda(L \oplus F \oplus F_0) \geq n-1$. But then $\lambda(K \oplus G_0) \geq n-1$ which means that $\lambda(K) \geq n-1$ by induction. Hence $\lambda(M) \geq n$. ∎

THEOREM 2.5. *Suppose* $\lambda(M) \geqslant n > k \geqslant 0$. *If* $F_k \to F_{k-1} \to \cdots \to$ $\to F_0 \to M \to 0$ *is a partial left* $\mathcal{F}$*-resolution of* M *and* $K = \operatorname{Ker}(F_k \to$ $\to F_{k-1})$ *where* $F_{-1} = M$, *then* $\lambda(K) \geqslant n - k - 1$. *In particular if* $\lambda(M) = n$, *then* $\lambda(K) = n - k - 1$.

PROOF If $\lambda(M) \geqslant n$, then there is a partial left $\mathcal{F}$-resolution $G_n \to$ $\to \cdots \to G_0 \to M$. Let $L = \operatorname{Ker}(G_k \to G_{k-1})$. Then $\lambda(L) \geqslant n - k - 1$. By Proposition 2.3, $L \oplus F_k \oplus G_{k-1} \oplus \cdots \cong K \oplus G_k \oplus F_{k-1} \oplus \cdots$ and so $\lambda(L) =$ $= \lambda(K)$ by Proposition 2.4. Hence $\lambda(K) \geqslant n - k - 1$. ■

COROLLARY 2.6. *If* $\lambda(M) = \infty$, *then there is an infinite left* $\mathcal{F}$*-resolution* $\cdots \to F_1 \to F_0 \to M \to 0$ *of* M.

PROOF If $F_n \to \cdots \to F_0 \to M \to 0$ is a partial left $\mathcal{F}$-resolution and $K = \operatorname{Ker}(F_n \to F_{n-1})$, then $\lambda(K) = \infty$. So this complex can be extended to a partial left $\mathcal{F}$-resolution $F_{n+1} \to \cdots \to F_0 \to M \to 0$. Continuing in this manner we get the desired complex. ■

LEMMA 2.7. *If* $M_1 \to M_2$ *is a linear map such that the induced* $\operatorname{Hom}(F, M_1) \to \operatorname{Hom}(F, M_2)$ *is an isomorphism for all* $F \in \mathcal{F}$, *then* $\lambda(M_1) = \lambda(M_2)$.

PROOF. If $\lambda(M_1) \geqslant n$ and $F_n \to \cdots \to F_0 \to M_1 \to 0$ is a partial left $\mathcal{F}$-resolution, then so is $F_n \to \cdots \to F_0 \to M_2 \to 0$ where $F_0 \to M_2$ is the composition $F_0 \to M_1 \to M_2$. Hence $\lambda(M_2) \geqslant n$.

If $\lambda(M_2) \geqslant n$ and $F_n \to \cdots \to F_0 \to M_2 \to 0$ is a partial left $\mathcal{F}$-resolution, then by hypothesis, $F_0 \to M_2$ has a lifting $F_0 \to M_1$ and so $F_0 \to M_2$ has a factorization $F_0 \to M_1 \to M_2$. But $\operatorname{Hom}(F_1, M_1) \to \operatorname{Hom}(F_1, M_2)$ is an isomorphism and $F_1 \to F_0 \to M_2$ is 0. So $F_1 \to F_0 \to M_1$ is a complex. Thus we see that $F_n \to \cdots \to F_1 \to F_0 \to M_1 \to 0$ is a partial left $\mathcal{F}$-resolution. That is, $\lambda(M_1) \geqslant n$. ■

COROLLARY 2.8. *If a complex* $0 \to M' \to M \to M'' \to 0$ *of* R*-modules is* $\operatorname{Hom}(\mathcal{F}, -)$ *exact and* $K = \operatorname{Ker}(M \to M'')$, *then the map* $M' \to K$ *is such that* $\operatorname{Hom}(F, M') \to \operatorname{Hom}(F, K)$ *is an isomorphism for all* $F \in \mathcal{F}$. *Hence* $\lambda(M') = \lambda(K)$ *by Lemma* 2.7 *above.*

LEMMA 2.9 (Horseshoe Lemma). *Let* $0 \to M' \to M \to M'' \to 0$ *be a* $\operatorname{Hom}(\mathcal{F}, -)$ *exact complex of left* R*-modules. If* $\cdots \to F_1' \to F_0' \to M' \to 0$ *and* $\cdots \to F_1'' \to F_0'' \to M'' \to 0$ *are left* $\mathcal{F}$*-resolutions, then there exists a commutative diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_1' \arrow[r] \arrow[d] & F_1' \oplus F_1'' \arrow[r] \arrow[d] & F_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0' \arrow[r] \arrow[d] & F_0' \oplus F_0'' \arrow[r] \arrow[d] & F_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that the middle column is a left $\mathcal{F}$-resolution of M.

PROOF. This is standard. ■

THEOREM 2.10. *Let $0 \to M' \to M \to M'' \to 0$ be a* $\operatorname{Hom}(\mathcal{F}, -)$ *exact complex of left R-modules, then*

1) $\lambda(M'') \geqslant \min(\lambda(M') + 1, \lambda(M))$
2) $\lambda(M) \geqslant \min(\lambda(M'), \lambda(M''))$
3) $\lambda(M') \geqslant \min(\lambda(M), \lambda(M'') - 1)$

PROOF. We start with (1). We only need prove that if $n \geqslant -1$ is an integer and $\min(\lambda(M') + 1, \lambda(M)) \geqslant n$, then $\lambda(M'') \geqslant n$. If $n = -1$, this is trivially true. If $n = 0$, then $\lambda(M) \geqslant 0$ means M has an $\mathcal{F}$-precover $F \to \to M$. By hypothesis, $\operatorname{Hom}(\mathrm{G}, \mathrm{M}) \to \operatorname{Hom}(\mathrm{G}, \mathrm{M}'') \to 0$ is exact if $G \in \mathcal{F}$. So $\operatorname{Hom}(G, F) \to \operatorname{Hom}(G, M) \to \operatorname{Hom}(G, M'')$ is surjective. Thus $F \to M''$ is an $\mathcal{F}$-precover and so $\lambda(M'') \geqslant 0$.

We now suppose $n > 0$. We have $\lambda(M') \geqslant n - 1 \geqslant 0$ and $\lambda(M) \geqslant n$ by assumption. So we have partial left $\mathcal{F}$-resolutions $F'_{n-1} \to \cdots \to F'_0 \to \to M' \to 0$ and $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & F'_{n-1} & \to \cdots \to & F'_0 & \to & M' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
F_n & \to & F_{n-1} & \to \cdots \to & F_0 & \to & M & \to & 0
\end{array}$$

A mapping cone then gives rise to the complex $F_n \oplus F'_{n-1} \to F_{n-1} \oplus$

$\oplus F'_{n-2} \to \cdots \to F_1 \oplus F'_0 \to F_0 \oplus M' \to M \to 0$ which is Hom$(\mathcal{F}, -)$ exact.

But then we have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & M' & \longrightarrow & M' & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
F_n \oplus F'_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 \oplus F'_0 & \longrightarrow & F_0 \oplus M' & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
F_n \oplus F'_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 \oplus F'_0 & \longrightarrow & F_0 & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

where the rows are Hom$(\mathcal{F}, -)$ exact complexes. We now apply the additive functor Hom$(F, -)$ with any $F \in \mathcal{F}$ to the diagram above. Then, using the long exact sequence associated with the short exact sequence of complexes we see that $F_n \oplus F'_{n-1} \to F_{n-1} \oplus F'_{n-2} \to \cdots \to F_1 \oplus F'_0 \to$ $\to F_0 \to M'' \to 0$ is also Hom$(\mathcal{F}, -)$ exact. Hence $\lambda(M'') \geqslant n$.

The proof of (3) is similar. We need to argue that if $\min(\lambda(M), \lambda(M'') - 1) \geqslant n$, then $\lambda(M') \geqslant n$. We can assume $n \geqslant 0$. Then we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & F_n & \to & \cdots & \to & F_0 & \to & M \to 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
F''_{n+1} & \to & F''_n & \to & \cdots & \to & F''_0 & \to & M'' \to 0
\end{array}$$

and the complex $F''_{n+1} \oplus F_n \to \cdots \to F''_1 \oplus F_0 \to F''_0 \oplus M \to M'' \to 0$. But then we get a commutative diagram

$$\begin{array}{ccccccccccc}
F''_{n+1} \oplus F_n & \longrightarrow & \cdots & \longrightarrow & F''_1 \oplus F_0 & \longrightarrow & F''_0 \oplus M & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & M'' & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

The kernel of the corresponding map of complexes is the complex $F''_{n+1} \oplus F_n \to \cdots \to F''_1 \oplus F_0 \to P \to 0$ where $P = \text{Ker}(F''_0 \oplus M \to M'')$. So

$$\begin{array}{ccc}
P & \to & M \\
\downarrow & & \downarrow \\
F''_0 & \to & M''
\end{array}$$

is a pullback diagram. Hence by our hypothesis on $0 \to M' \to M \to M'' \to$ $\to 0$, we see that the map $F''_0 \to M''$ has a lifting $F''_0 \to M$. But by the property of a pullback this means $P \to F''_0$ has a section. Hence $P \cong F''_0 \oplus K$

where $K = \mathrm{Ker}(M \to M'')$. But as in the argument for (1), we see that $F''_{n+1} \oplus F_n \to \cdots \to F''_1 \oplus F_0 \to P \to 0$ is $\mathrm{Hom}(\mathcal{F}, -)$ exact. This means $\lambda(P) \geq n$. But since $P \cong F''_0 \oplus K$ we get that $\lambda(K) \geq n$ by Proposition 2.4. But then by Lemma 2.7 and Corollary 2.8, we get $\lambda(M') \geq n$.

We now prove (2). We assume $\lambda(M')$, $\lambda(M'') \geq n \geq 0$ and argue $\lambda(M) \geq n$. Let $F'_n \to \cdots \to F'_0 \to M' \to 0$ and $F''_n \to \cdots \to F''_0 \to M'' \to 0$ be partial left $\mathcal{F}$-resolutions of M' and M'' respectively. Then by Horseshoe Lemma 2.9, we get a partial left $\mathcal{F}$-resolution of M of length n. Hence $\lambda(M) \geq n$. ■

3. $\bar{\lambda}$-dimensions and special $\mathcal{F}$-precovers.

We recall that the class of modules C such that $\mathrm{Ext}^1(F, C) = 0$ for all $F \in \mathcal{F}$ is denoted by $\mathcal{F}^{\perp}$. It is easy to see that $\mathcal{F}^{\perp}$ is closed under extensions. Furthermore, if the sequence $0 \to C \to F \to M \to 0$ is exact with $C \in \mathcal{F}^{\perp}$ and $F \in \mathcal{F}$, then for each $G \in \mathcal{F}$, we have an exact sequence $\mathrm{Hom}(G, F) \to \mathrm{Hom}(G, M) \to \mathrm{Ext}^1(G, C) = 0$ and so $F \to M$ is an $\mathcal{F}$-precover.

DEFINITION 3.1. *An $\mathcal{F}$-precover $\varphi : F \to M$ is said to be a special $\mathcal{F}$-precover if φ is an epimorphism and* Ker $\varphi \in \mathcal{F}^{\perp}$. *For example, if R is n-Gorenstein, that is, R is left and right noetherian and has self injective dimension at most n on both sides, then every R-module has a Gorenstein projective precover $\varphi : C \to M$ such that $K = \mathrm{Ker}(\varphi)$ has projective dimension at most n. Furthermore,* $\mathrm{Ext}^1(C', K) = 0$ *for all Gorenstein projective R-modules C' (see Enochs-Jenda* [4]). *Hence in this case, if $\mathcal{F}$ is the class of Gorenstein projective R-modules, then every R-module has a special $\mathcal{F}$-precover. Dually, if $\mathcal{F}$ is the class of Gorenstein injective R-modules, then every R-module has a special $\mathcal{F}$-preenvelope over n-Gorenstein rings (see Enochs-Jenda-Xu* [6]).

DEFINITION 3.2. *For an R-module M, we say $\bar{\lambda}_{\mathcal{F}}(M) = -1$ if M does not have a special $\mathcal{F}$-precover. If there is an exact sequence $F_n \to \cdots \cdots \to F_0 \to M \to 0$ where $F_0 \to M$, $F_i \to K_{i-1}$ ($K_0 = \mathrm{Ker}(F_0 \to M)$ and $K_{i-1} = \mathrm{Ker}(F_{i-1} \to F_{i-2})$ for $i \geq 2$) are special $\mathcal{F}$-precovers for $i > 0$, and if there is no longer such sequences we say that $\bar{\lambda}(M) = n$. We say that $\bar{\lambda}(M) = \infty$ if there is such a sequence for each $n \geq 0$.*

PROPOSITION 3.3. *If $\mathcal{F}$ is such that $\lambda(M) \geqslant 0$ implies $\bar{\lambda}(M) \geqslant 0$ for all R-modules M, then $\lambda(M) = \bar{\lambda}(M)$ for all M.*

PROOF. Clearly $\lambda(M) \geqslant \bar{\lambda}(M)$. So we argue that $\lambda(M) \geqslant n$ implies $\bar{\lambda}(M) \geqslant n$ for $n \geqslant 0$. But this is true if $n = 0$ by assumption. So we suppose $\lambda(M) \geqslant n > 0$. Then $\bar{\lambda}(M) \geqslant 0$ and so let $F \to M$ be a special $\mathcal{F}$-precover with kernel K. Then $\lambda(K) \geqslant n - 1$ by Theorem 2.5. So $\bar{\lambda}(K) \geqslant n - 1$ by induction and hence $\bar{\lambda}(M) \geqslant n$. ■

The proofs of several results concerning $\bar{\lambda}$-dimensions are straightforward modifications of the corresponding results about λ-dimensions. These include Proposition 2.4, Theorem 2.5, and Corollary 2.6. We now prove results that correspond to Theorem 2.10.

We recall that if $\mathcal{F}$ contains all the projective modules then any $\mathcal{F}$-precover $F \to M$ is surjective. And in this case any $\operatorname{Hom}(\mathcal{F}, -)$ exact sequence is exact.

THEOREM 3.4. *If $\mathcal{F}$ contains all the projective modules and if $0 \to M' \to M \to M'' \to 0$ is exact with $M' \in \mathcal{F}^{\perp}$ (so the sequence is also* $\operatorname{Hom}(\mathcal{F}, -)$ *exact) then*

$$\bar{\lambda}(M'') \geqslant \min(\bar{\lambda}(M') + 1, \bar{\lambda}(M))$$

PROOF. The argument is a straightforward modification of the proof of (1) of Theorem 2.10. ■

THEOREM 3.5. *If $0 \to M' \to M \to M'' \to 0$ is an* $\operatorname{Hom}(\mathcal{F}, -)$ *exact complex, then*

$$\bar{\lambda}(M) \geqslant \min(\bar{\lambda}(M'), \bar{\lambda}(M''))$$

PROOF. This argument is like that for (2) of Theorem 2.10. ■

DEFINITION 3.6. *The class $\mathcal{F}$ is said to be resolving if $\mathcal{F}$ contains all the projective modules and is closed under extensions, and if whenever $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, F' is also in $\mathcal{F}$.*

THEOREM 3.7. *If $\mathcal{F}$ is resolving and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules, then*

$$\bar{\lambda}(M') \geqslant \min(\bar{\lambda}(M), \bar{\lambda}(M'') - 1).$$

PROOF. We prove by induction on n that if $\bar{\lambda}(M) \geqslant n$ and $\bar{\lambda}(M'') \geqslant \geqslant n+1$ then $\bar{\lambda}(M') \geqslant n$.

Let $n=0$ and so $\bar{\lambda}(M'') \geqslant 1$ and $\bar{\lambda}(M) \geqslant 0$. So let $0 \to K_0'' \to F_0'' \to M'' \to 0$, $0 \to K_1'' \to F_1'' \to K_0'' \to 0$, and $0 \to K_0 \to F_0 \to M \to 0$ be exact sequences with $K_0, K_0'', K_1'' \in \mathcal{F}^\perp$ and $F_0'', F_1'', F_0 \in \mathcal{F}$.

We now form the pullback of $M \to M''$ and $F_0'' \to M''$ and get the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_0'' \arrow[r, "id"] \arrow[d] & K_0'' \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, "id"'] & H \arrow[r] \arrow[d] & F_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. We now consider the exact sequence $0 \to \to K_0'' \to H \to M \to 0$. Since $K_0'' \in \mathcal{F}^\perp$, this sequence is $\mathrm{Hom}(\mathcal{F}, -)$ exact. So by the Horseshoe Lemma, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1'' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1'' \arrow[r] \arrow[d] & F_1'' \oplus F_0 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0'' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Note that since $K_1'', K_0 \in \mathcal{F}^\perp$, we also have $K \in \mathcal{F}^\perp$. We now form the pullback of $M' \to H$ and $F_1'' \oplus F_0 \to H$. This gives us the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "id"] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & F_1'' \oplus F_0 \arrow[r] \arrow[d] & F_0'' \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & F_0'' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Since $F_1'' \oplus F_0$, $F_0'' \in \mathcal{F}$ and $\mathcal{F}$ is resolving, $F' \in \mathcal{F}$. As noted above, $K \in \mathcal{F}^{\perp}$. Hence $F' \to M'$ is a special $\mathcal{F}$-precover and so $\bar{\lambda}(M') \geqslant 0$.

Now assume $n > 0$ and use the construction above. Then by the exactness and Hom$(\mathcal{F}, -)$ exactness of $0 \to K_1'' \to K \to K_0 \to 0$ ($K_1'' \in \mathcal{F}^{\perp}$ gives the Hom$(\mathcal{F}, -)$ exactness), we get $\bar{\lambda}(K) \geqslant \min(\bar{\lambda}(K_1''), \bar{\lambda}(K_0))$ by Theorem 3.5. But $\min(\bar{\lambda}(K_1''), \bar{\lambda}(K_0)) \geqslant n-1$ by the $\bar{\lambda}$-dimension counterpart of Theorem 2.5 (or we can assume we chose K_1'' and K_0 so that the inequality holds). But then $\bar{\lambda}(K) \geqslant n-1$ implies $\bar{\lambda}(M') \geqslant n$.■

4. $\bar{\lambda}$-dimensions and Gorenstein flat modules.

We recall that an R-module M is said to be Gorenstein flat if there exists an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with $M = \mathrm{Ker}(F^0 \to F^1)$ such that $E \otimes$— leaves the sequence exact whenever E is an injective R-module (see Enochs-Jenda-Torrecillas [5]). Clearly, the class of Gorenstein flat modules contains the flat modules. We recall from [5] that if R is n-Gorenstein, then M is Gorenstein flat if and only if $\mathrm{Tor}_i(L, M) = 0$ for all $i \geqslant 1$ and all right R-modules L of finite injective dimension.

We start with the following

THEOREM 4.1. *Let R be n-Gorenstein and $\mathscr{F}$ be the class of Gorenstein flat R-modules, then $\bar{\lambda}_{\mathscr{F}}(P) = \infty$ for every pure injective R-module P.*

PROOF. Let N be any right R-module and let $N \subset G$ be a Gorenstein injective envelope. Then we have the exact sequence $0 \to (G/N)^+ \to \to G^+ \to N^+ \to 0$ where G^+ is a Gorenstein flat left R-module (see [5] and [6]). But G/N has finite injective dimension. So if F is a Gorenstein flat left R-module, then $\mathrm{Ext}^1(F, (G/N)^+) \cong \mathrm{Tor}_1(F, G/N)^+ = 0$ by the remarks above. Hence $G^+ \to N^+$ is a special Gorenstein flat precover.

Now let P be a pure injective left R-module and set $N = P^+$. Then we have a special Gorenstein flat precover $G^+ \to N^+ = P^{++}$. Since P is pure injective, it is a direct summand of P^{++} and so P has a Gorenstein flat precover. But the class of Gorenstein flat modules is closed under direct limits (see [5]) and therefore P has a Gorenstein flat cover $F \to P$ by Enochs [3, Theorem 3.1]. So there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (G/N)^+ & \longrightarrow & G^+ & \longrightarrow & P^{++} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & P & \longrightarrow & 0
\end{array}$$

with exact rows and $P \to P^{++} \to P$ the identity on P. Since $F \to P$ is a flat cover, we see that F is isomorphic to a direct summand of G^+ and K is isomorphic to a direct summand of $(G/N)^+$. Since $(G/N)^+$ is pure injective, so is K. But $\mathrm{Ext}^1(F', (G/N)^+) = 0$ for F' Gorenstein flat. So $\mathrm{Ext}^1(F', K) = 0$ for all such F'. Hence $0 \to K \to F \to P \to 0$ is exact with $F \to P$ a special Gorenstein flat cover and K pure injective. But then we can repeat the argument with K replacing P. Proceeding in this manner we see that $\bar{\lambda}_{\mathscr{F}}(P) = \infty$. ∎

COROLLARY 4.2. *For every R-module L of finite injective dimension, $\bar{\lambda}_{\mathscr{F}}(L) = \infty$ where $\mathscr{F}$ is the class of Gorenstein flat R-modules.*

PROOF. If L is injective then L is pure injective and so the result holds by the theorem above. If *inj. dim* $L < \infty$, then we see that a repeated application of Theorem 3.7 gives the result noting that $\mathscr{F}$ is resolving. ∎

As an application, we use λ-dimensions and $\bar{\lambda}$-dimensions to prove the following now familiar result.

THEOREM 4.3 ([7, Theorem 3.2]). *If R is n-Gorenstein, then every R-module M has a Gorenstein flat cover $F \xrightarrow{\varphi} M$.*

PROOF. We will argue that for every left R-module M, $\bar{\lambda}_{\mathcal{F}}(M) = \infty$ with $\mathcal{F}$ the class of Gorenstein flat left R-modules. But every R-module has a special Gorenstein projective precover. That is, there is an exact sequence $0 \to L \to C \to M \to 0$ with C Gorenstein projective and *proj. dim* $L < \infty$. But *inj. dim* $L < \infty$ since R is n-Gorenstein. So by Corollary 4.2, $\bar{\lambda}_{\mathcal{F}}(L) = \infty$. But C is Gorenstein flat by [5] and so easily $\bar{\lambda}_{\mathcal{F}}(C) = \infty$. Then Theorem 2.10 says $\lambda_{\mathcal{F}}(M) = \infty$. So M has a Gorenstein flat precover. So since the class of Gorenstein flat modules is closed under direct limits ([5]), M has a Gorenstein flat cover ([3, Theorem 3.1]). ■

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