

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 104 (2000), p. 59-62

[http://www.numdam.org/item?id=RSMUP\\_2000\\_\\_104\\_\\_59\\_0](http://www.numdam.org/item?id=RSMUP_2000__104__59_0)

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## Fitting Height of $A$ -Nilpotent Groups.

PAVEL SHUMYATSKY (\*)

ABSTRACT - Let  $A$  be a finite supersolvable group acting coprimely on a finite solvable group  $G$  in such a manner that the fixed point subgroup  $C_G(A)$  normalizes every  $A$ -invariant Sylow subgroup of  $G$ . Let  $h(G)$  and  $k(A)$  denote the Fitting height of  $G$  and the composition length of  $A$  respectively. It is shown that under certain assumptions on  $A$  the inequality  $h(G) \leq k(A) + 1$  holds.

Let  $A$  be a finite group acting coprimely on a finite group  $G$ . Following [1] we say that  $G$  is  $A$ -nilpotent if the fixed point subgroup  $C_G(A)$  normalizes every  $A$ -invariant Sylow subgroup of  $G$ . This generalizes the notion of the fixed-point-free action, i. e. the action with  $C_G(A) = 1$ . The results obtained in [1], [2], [3] show that the relation with the «fixed-point-free» case extends far beyond formal definitions.

Assume that  $G$  and  $A$  are solvable and let  $h(G)$  and  $k(A)$  denote the Fitting height of  $G$  and the composition length of  $A$  respectively. (Thus,  $k(A)$  is the number of primes dividing  $|A|$  counting multiplicities.) There is a conjecture that if  $C_G(A) = 1$  then  $h(G) \leq k(A)$ . This has been confirmed in many cases (see [5]). In particular, A. Turull showed that the conjecture is true if  $A$  is supersolvable and no section of  $A$  is isomorphic to  $\mathbb{Z}_r \wr \mathbb{Z}_s$  or to  $GN(p^{q^e})$ , where  $p$  is a prime dividing  $|G|$  (see [5] for the necessary definitions).

The question on the Fitting height of an  $A$ -nilpotent group was considered by E. Jabara [2]. He proved that if  $A$  is cyclic of prime-power or-

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Research supported by FAPDF and CNPq.

1991 Mathematics Subject Classification: 20 D 10.

der then, under some additional assumptions on  $G$ , the inequality  $h(G) \leq k(A) + 1$  holds. The goal of the present paper is to show that the technique developed by A. Turull can be quite effective in the treatment of length problems for  $A$ -nilpotent groups. In fact most of the argument contained in this paper originates from Turull's work [4].

**THEOREM.** Let  $G$  be a solvable  $A$ -nilpotent group, where  $A$  is supersolvable and no section of  $A$  is isomorphic to  $\mathbb{Z}_r \wr \mathbb{Z}_s$  or to  $GN(p^{q^e})$  for any prime  $p$  dividing  $|G|$ . Then  $h(G) \leq k(A) + 1$ .

In fact the hypothesis that  $G$  is solvable is superfluous in the above theorem as it is shown in [1] that any  $A$ -nilpotent group is solvable. Our first lemma is quite obvious and so we omit the proof.

**LEMMA 1.** Let  $A$  act coprimely on a finite group  $G$  and assume that  $G$  is  $A$ -nilpotent. Let  $N$  be an  $A$ -invariant normal subgroup of  $G$  and  $H$  an  $A$ -invariant subgroup of  $G$ . Then  $A$  induces an action on  $G/N$  and  $H$  under which both groups  $G/N$  and  $H$  are  $A$ -nilpotent.

**LEMMA 2.** Let  $G$  be a finite solvable group with  $h = h(G) \geq 2$ . Suppose that  $A$  acts coprimely on  $G$  and  $G$  is  $A$ -nilpotent. Let  $M$  be a minimal  $A$ -invariant normal subgroup of  $G$  and assume that  $h(G/C_G(M)) = h - 1$ . Then  $C_M(A) = 1$ .

**PROOF.** Since  $G$  is solvable,  $M$  is an elementary abelian  $p$ -group for some prime  $p$ . Set  $C = C_G(M)$  and  $F/C = F(G/C)$ . Then  $F/C$  is a nilpotent  $p'$ -group. Let  $S$  be an  $A$ -invariant Sylow  $q$ -subgroup of  $F$  for some prime  $q \neq p$ . By the hypothesis  $C_M(A)$  normalizes  $S$  and therefore  $[S, C_M(A)] \leq M \cap S = 1$ . Since  $M$  is minimal and  $S$  does not centralize  $M$ , it follows that  $C_M(A) = 1$ .

We now require the notion of  $A$ -support of  $G$  as introduced by A. Turull.

**DEFINITION.** Let a group  $A$  act on a finite solvable group  $G$ . A subgroup  $P \leq G$  is called a generating  $A$ -support subgroup of  $G$  if:

- 1)  $P$  is normal in  $AG$  and  $P$  is  $p$ -group for some prime  $p$ .
- 2) There are  $AG$ -invariant subgroups  $P_1$  and  $H$  such that
  - A)  $P_1 \leq Z(P)$ ,  $P/P_1$  is elementary abelian and  $AG$ -completely reducible,

- B)  $H \leq C_G(P_1)$ ,
- C)  $H/H \cap C_G(P/P_1)$  is elementary abelian for some prime  $r$ ,
- D)  $H$  acts non-trivially on each  $H$ -chief factor of  $P/P_1$ .

Then the  $A$ -support of  $G$  (denoted by  $\text{supp}_A(G)$ ) is the subgroup generated by all normal in  $AG$  subgroups  $S \leq G$  such that  $S$  is either abelian or a generating  $A$ -support.

LEMMA 3 ([4, 4.3]). Let  $G$  be a finite solvable group and  $A$  act on  $G$ . Then

1.  $\cap C_G(X) \leq F(G)$ , where  $X$  runs through the  $AG$ -chief factors of  $\text{supp}_A(G)$ . In particular  $C_G(\text{supp}_A(G)) \leq F(G)$ .
2. If  $N \leq G$  and  $N$  is normal in  $AG$  then  $\text{supp}_A(G)N/N \leq \text{supp}_A(G/N)$ .
3. If  $B \leq A$  and  $(|A|, |G|) = 1$  then  $\text{supp}_B(G) \geq \text{supp}_A(G)$ .
4.  $C_A(\text{supp}_A(G)) \leq C_A(G/F(G))$ .

The next lemma is immediate from Theorem 4.6 of [4].

LEMMA 4. Let  $AG$  be a solvable finite group with  $G$  normal in  $AG$  and  $A$  supersolvable without sections isomorphic to  $\mathbb{Z}_r \wr \mathbb{Z}_s$  or to  $GN(p^{q^e})$  for any  $p$  dividing  $|G|$ . Let  $k$  be a field of characteristic not dividing  $|A|$  and  $M$  an irreducible  $kAG$ -module. Assume

1.  $M$  is faithful for  $G$ ;
2.  $B_1 > B_2$  are normal subgroups of  $A$  with  $|B_1/B_2|$  a prime;
3.  $C_M(B_1) = 0$  and  $C_M(B_2) \neq 0$ .

If  $S = \text{supp}_A(G)$ , we have  $C_S(B_1) = C_S(B_2)$  and  $C_{G/F(G)}(B_1) = C_{G/F(G)}(B_2)$ .

THEOREM. Let  $G$  be a solvable  $A$ -nilpotent group, where  $A$  is supersolvable and no section of  $A$  is isomorphic to  $\mathbb{Z}_r \wr \mathbb{Z}_s$  or to  $GN(p^{q^e})$  for any prime  $p$  dividing  $|G|$ . Then  $h(G) \leq k(A) + 1$ .

PROOF. Let  $1 = A_0 < A_1 < \dots < A_n = A$  be a chief series of  $A$ . Set  $h = h(G)$ . By Lemma 3, we can choose an  $AG$ -chief factor  $F_1$  of  $\text{supp}_A(G)$  such that  $h(G/C_G(F_1)) = h - 1$ . If  $h - 1 > 0$  we can choose an  $AG$ -chief factor  $F_2$  of  $\text{supp}_A(G/C_G(F_1))$  such that  $h(G/C_G(F_2)) = h - 2$ . Continuing this process we obtain  $AG$ -chief factors  $F_1, F_2, \dots, F_h$  of  $G$  such that for any  $1 \leq i < j \leq h$  either  $F_j$  is a factor of  $\text{supp}_A(G/C_G(F_i))$  or  $F_j$  is a factor

of  $(G/C_G(F_i))/F(G/C_G(F_i))$ . Lemma 2 shows that if  $i \leq h-1$  then  $C_{F_i}(A) = 1$ . We therefore can define the map

$$f : \{1, \dots, h-1\} \rightarrow \{1, \dots, n\},$$

where  $f(i)$  is the smallest  $k$  such that  $C_{F_i}(A_k) = 1$ .

If  $k = f(i) = f(j)$  for some  $j > i$  then, by Lemma 4 applied with  $A_{k-1}$  and  $A_k$  in place of  $B_2$  and  $B_1$  respectively, we have  $C_{F_j}(A_k) = C_{F_j}(A_{k-1})$ . But then  $f(j) \leq k-1$ , a contradiction. Therefore  $f$  is one-to-one and  $h \leq n+1$ .

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Manoscritto pervenuto in redazione l'1 settembre 1998.