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Functionals Depending on Curvatures with Constraints.

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ABSTRACT - We deal with a family of functionals depending on curvatures and we prove for them compactness and semicontinuity properties in the class of closed and bounded sets which satisfy a uniform exterior and interior sphere condition. We apply the results to state an existence theorem for the Nitzberg and Mumford problem under this additional constraint.

1. Introduction.

In this paper we are dealing with geometrical functionals of the form

$$F(E) = \int_{\partial E} \varphi(K_1, \ldots, K_{n-1}) \, d\mathcal{H}^{n-1}, \tag{1.1}$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a given convex function, E varies in a class of sufficiently regular closed subsets of $\mathbb{R}^n$, $K_1, \ldots, K_{n-1}$ denote the elementary symmetric curvatures of ∂E (see (4.1)), and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

In [3] BELLETTINI, DAL MASO, PAOLINI studied the functional F in the case $n = 2$ and $\varphi(\kappa) = 1 + |\kappa|^p$, where κ denotes the curvature of ∂E, and remarked that F does not have the right compactness properties in its natural class of definition, composed of all closed sets E whose boun-

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dary is of class $W^{2,p}$: simple examples show that there exist sets of class C^∞, except for a finite number of cusp points (the functional then is not naturally defined on them), which can be approximated by a sequence of sets of class C^∞, whose boundaries have bounded curvature. Moreover, they show that the lower semicontinuous envelope, $\overline{F}$, of F with respect to the L^1-topology cannot be represented as an integral of the form

$$\int_{\partial E} f(\kappa)\, d\mathcal{H}^1,$$

and that the fact that a set E belongs to the domain of $\overline{F}$ depends on the global structure of E. For instance, if ∂E is smooth except for a finite number k of cusp points, then $\overline{F}(E) < +\infty$ if and only if k is even.

The idea of this work is to modify the domain of F by introducing some suitable constraints.

Fixed $R > 0$, we choose as domain of F the class

$$\mathcal{U}_R = \{E \subset \mathbb{R}^n,\ E \text{ closed and bounded: } \forall p \in \partial E\ \exists p',\, p'':$$
$$p \in \partial B(p', R) \cap \partial B(p'', R),\ B(p', R) \subset E,\ B(p'', R) \cap E = \emptyset\}, \tag{1.2}$$

where $B(q, R)$ denotes the open ball centred at q of radius R; we will say, equivalently, that $\mathcal{U}_R$ is the class of all closed and bounded subsets of $\mathbb{R}^n$, which satisfy *the exterior and interior sphere condition* with radius R at every point of the boundary. Note that the introduced constraint has a nonlocal effect on the thickness, which cannot be too small, and a local effect on the curvatures, which are bounded from above by a constant depending only on R. Remark also that this upper bound on the curvatures goes to infinity, when R tends to 0.

In the class $\mathcal{U}_R$ the pathological phenomena described above cannot occur; indeed, they are related to the existence of approximating sequences of sets having regions with vanishing thickness or different connected components whose distance goes to 0.

In Section 2 we study the regularity of sets belonging to $\mathcal{U}_R$, showing that the functional in (1.1) is well defined. In Sections 3 and 4 we prove compactness and semicontinuity results for F in $\mathcal{U}_R$. In Section 5 we consider the case $n = 2$ and we apply the theorems of Sections 3 and 4 to show the existence of a solution to the variational problem

$$\min\left\{\sum_{i=1}^{k}\left(\alpha\int_{E_i'\cap\Omega}|g-g_{E_i'\cap\Omega}|^2\,dx+\beta\mathcal{L}^2(E_i)+\gamma\int_{\partial E_i}\varphi(\kappa)\,d\mathcal{H}^1\right)+\right.$$
$$\left.+\alpha\int_{\Omega\setminus\bigcup_{i=1}^{k}E_i}|g-g_{\Omega\setminus\bigcup_{i=1}^{k}E_i}|^2\,dx : E_1,\ldots,E_k\in\mathcal{U}_R\right\}, \tag{1.3}$$

where Ω is a bounded subset of $\mathbb{R}^2$, α, β, γ are positive parameters, $E_i' := E_i \setminus \bigcup_{j=1}^{i-1} E_j$, g is a function in $L^2(\Omega)$. This functional was proposed by NITZBERG and MUMFORD as a variant of the MUMFORD and SHAH image segmentation model, allowing regions to overlap (for further information about this model, see [9]). In this framework the constant R can be interpreted as a resolution parameter of the segmented image: the thickness of the reconstructed objects has to be greater or equal to $2R$. We conclude the section by giving an example of non trivial minimizer for a functional of the form as in (1.3).

2. Preliminary results.

In this section we investigate the regularity of sets belonging to the class $\mathcal{U}_R$ introduced in (1.2) and we show that the functional (1.1) is well defined in this class.

Let us fix first some notation. If E belongs to $\mathcal{U}_R$ and $p\in\partial E$, we denote the centres of the interior and exterior balls associated to p by p' and p'' respectively, as in (1.2); moreover, we call S_E^p the class of all coordinate systems centred at p such that the vector $\frac{1}{2R}(p''-p')$ coincides with the n-th vector of the coordinate basis.

PROPOSITION 2.1. *There exists a constant $\varrho>0$ (depending only on R), such that for every $E\in\mathcal{U}_R$ and for every $p_0\in\partial E$, if we call C the cylinder $\{x\in\mathbb{R}^{n-1}: |x|<\varrho\}\times]-R,R[$ expressed with respect to a coordinate system belonging to $S_E^{p_0}$, then $\partial E\cap C$ is the subgraph of a function f belonging to $W^{2,\infty}(\{x\in\mathbb{R}^{n-1}: |x|<\varrho\})$. Moreover, the $W^{2,\infty}$-norm of f is bounded by a constant depending only on R (independent of p_0, of E and of the choice of the coordinate system in $S_E^{p_0}$).*

PROOF. We first perform the proof in the case $n=2$ showing that $\varrho=\sqrt{3}R/2$ is a good choice.

Let E be in $\mathcal{U}_R$ and let p_0 belong to ∂E. Let us consider a coordinate system belonging to $S_E^{p_0}$. We can reduce to work in the cylinder $C^+ := = [0, \sqrt{3}R/2[\times]-R, R[$. For the proof we need the following lemma.

LEMMA 2.2. *Let $\overline{p}=(\overline{x},\overline{y})$ be in $\partial E\cap C^+$. If we call $\alpha(\overline{p})$ the angle in $[0,\pi[$ between the x-axis and the tangent line to $B(\overline{p}', R)$ and $B(\overline{p}'', R)$ at $\overline{p}$, then*

$$|\cos\alpha(\overline{p})| \geqslant \frac{\sqrt{R^2-\overline{x}^2}}{R}. \tag{2.1}$$

Moreover, either $B(\overline{p}', R)$ or $B(\overline{p}'', R)$ contains the whole segment $\{\overline{x}\}\times]\overline{y}, R-\sqrt{R^2-\overline{x}^2}]$.

PROOF. Let us suppose by contradiction that (2.1) does not hold; hence,

$$\sin\alpha(\overline{p}) > \frac{\overline{x}}{R}. \tag{2.2}$$

The point $q:=(\overline{x}-R\sin\alpha(\overline{p}), \overline{y}+R\cos\alpha(\overline{p}))$ must coincide either with $\overline{p}'$ or with $\overline{p}''$. To get the contradiction it is enough to show that

$$|p_0'-q|<2R \quad \text{and} \quad |p_0''-q|<2R\,; \tag{2.3}$$

indeed, if (2.3) is true, $B(q, R)$ intersects both $B(p_0', R)$ and $B(p_0'', R)$, while it must be contained either in E or in the complement of E. Let us compute the distance between p_0' and q:

$$\begin{aligned}|p_0'-q|^2 &= (\overline{x}-R\sin\alpha(\overline{p}))^2+(\overline{y}+R\cos\alpha(\overline{p})+R)^2\\ &= 2R^2+\overline{x}^2+\overline{y}^2-2R\overline{x}\sin\alpha(\overline{p})+2R^2\cos\alpha(\overline{p})+2R(1+\cos\alpha(\overline{p}))\,\overline{y}.\end{aligned}$$

Using the estimate $-R+\sqrt{R^2-\overline{x}^2}\leqslant\overline{y}\leqslant R-\sqrt{R^2-\overline{x}^2}$, the absurd assumption and (2.2), we obtain

$$\begin{aligned}|p_0'-q|^2 &\leqslant 6R^2-4R\sqrt{R^2-\overline{x}^2}-2R\overline{x}\sin\alpha(\overline{p})+2R(2R-\sqrt{R^2-\overline{x}^2})\cos\alpha(\overline{p})\\ &< 6R^2-4R\sqrt{R^2-\overline{x}^2}-2\overline{x}^2+2(2R-\sqrt{R^2-\overline{x}^2})\sqrt{R^2-\overline{x}^2}\\ &= 4R^2.\end{aligned}$$

By similar computations one can estimate the distance between p_0'' and q.

Let $p = (\bar{x}, y)$ with $\bar{y} < y \leq R - \sqrt{R - \bar{x}^2}$ and let us suppose that

$$\cos \alpha(\bar{p}) \geq \frac{\sqrt{R^2 - \bar{x}^2}}{R}. \tag{2.4}$$

We want to check that

$$|p - q|^2 \leq R^2,$$

which, by easy computations, is equivalent to

$$y - \bar{y} < 2R \cos \alpha(\bar{p}).$$

By assumption we know that

$$\begin{aligned} y - \bar{y} &\leq R - \sqrt{R^2 - \bar{x}^2} - \bar{y} \\ &\leq 2R - 2\sqrt{R^2 - \bar{x}^2} \\ &< 2\sqrt{R^2 - \bar{x}^2} \leq 2R \cos \alpha(\bar{p}), \end{aligned}$$

where the two last inequalities follow by the hypothesis $|\bar{x}| < \sqrt{3}R/2$ and by (2.4). ■

Fixed x_1 in $[0, \sqrt{3}R/2[$, let us suppose by contradiction that the straight line $x = x_1$ intersects $\partial E \cap C^+$ in two distinct points p_1 and q_1. Then, if we call p_1 the point with smallest y-coordinate, by Lemma 2.2 it follows that either $B(p_1', R)$ or $B(p_1'', R)$ must contain the point q_1 and this is impossible. Therefore, we can conclude that $\partial E \cap C^+$ is the graph of a function f.

Since f is between the functions $-R + \sqrt{R^2 - x^2}$ and $R - \sqrt{R^2 - x^2}$, which are both differentiable at $x = 0$ with null derivative, f is differentiable at $x = 0$ with derivative equal to 0. By a change of coordinates, we can repeat the same argument at every point belonging to $[0, \sqrt{3}R/2[$; therefore, f is differentiable in $[0, \sqrt{3}R/2[$ and the tangent line to the graph of f at any point coincides with the tangent line to the spheres associated to the same point. From here, we obtain by Lemma 2.2 the following bound on the norm of the derivative of f:

$$\left| \frac{df}{dx}(x) \right| \leqslant \frac{|x|}{\sqrt{R^2 - x^2}}, \tag{2.5}$$

for every $x \in [0, \sqrt{3}R/2[$.

To conclude the proof of the proposition in the case $n = 2$, it is sufficient to check that the derivative of f is Lipschitz with constant depending only on R. First, we observe that, by (2.5),

$$\left| \frac{df}{dx}(x) \right| \leqslant \frac{2|x|}{R}, \tag{2.6}$$

for every $x \in [0, \sqrt{3}R/2[$. Given $p_1 = (x_1, f(x_1))$ and $p_2 = (x_2, f(x_2))$, we consider the following change of coordinates:

$$\begin{cases} \tilde{x} = \dfrac{1}{1 + \left(\dfrac{df}{dx}(x_1)\right)^2} \left(x - x_1 + (z - f(x_1)) \dfrac{df}{dx}(x_1) \right) \\ \tilde{z} = \dfrac{1}{1 + \left(\dfrac{df}{dx}(x_1)\right)^2} \left(-\dfrac{df}{dx}(x_1)(x - x_1) + z - f(x_1) \right), \end{cases}$$

which transforms the point p_1 in the origin and the tangent line to ∂E at p_1 in the $\tilde{x}$-axis. With respect to the new coordinates, ∂E is locally the graph of a function $\tilde{f}$ and the point p_2 has coordinates $(\tilde{x}_2, \tilde{f}(\tilde{x}_2))$; then, by (2.6),

$$\left| \frac{d\tilde{f}}{d\tilde{x}}(\tilde{x}_2) \right| \leqslant \frac{2|\tilde{x}_2|}{R}, \tag{2.7}$$

if

$$|\tilde{x}_2| < \frac{\sqrt{3}}{2} R. \tag{2.8}$$

If we denote by L the Lipschitz constant of f in $[0, \sqrt{3}R/2[$, we have that

$$|\tilde{x}_2| \leqslant |p_1 - p_2| \leqslant \sqrt{1 + L^2}\,|x_1 - x_2|. \tag{2.9}$$

Therefore, the condition (2.8) is satisfied if $|x_1 - x_2| \leqslant \frac{\sqrt{3}R}{2\sqrt{1+L^2}} =: \lambda$.

By the relation

$$\frac{df}{dx}(x_2) - \frac{df}{dx}(x_1) = \left(1 + \frac{df}{dx}(x_1)\frac{df}{dx}(x_2)\right)\frac{d\tilde{f}}{d\tilde{x}}(\tilde{x}_2),$$

by (2.7), and (2.9), it follows that

$$\text{(2.10)} \quad \left|\frac{df}{dx}(x_2) - \frac{df}{dx}(x_1)\right| \leqslant \left|1 + \frac{df}{dx}(x_1)\frac{df}{dx}(x_2)\right| \left|\frac{d\tilde{f}}{d\tilde{x}}(\tilde{x}_2)\right|$$

$$\leqslant \frac{2\sqrt{1+L^2}}{R}\left|1 + \frac{df}{dx}(x_1)\frac{df}{dx}(x_2)\right| |x_1 - x_2|.$$

By the boundedness of the derivative of f, we can conclude that there exists a positive constant c, depending only on R, such that, if $|x_1 - x_2| \leqslant \lambda$, then

$$\left|\frac{df}{dx}(x_2) - \frac{df}{dx}(x_1)\right| \leqslant c|x_1 - x_2|.$$

In the case $|x_1 - x_2| > \lambda$, we can find a finite number of points $y_0 := x_1 < y_1 < \ldots < y_{k-1} < y_k := x_2$ such that $|y_{j+1} - y_j| \leqslant \lambda$ for every $j = 0, \ldots, k-1$. Then, we obtain

$$\left|\frac{df}{dx}(x_1) - \frac{df}{dx}(x_2)\right| \leqslant \sum_{j=0}^{k-1}\left|\frac{df}{dx}(y_{j+1}) - \frac{df}{dx}(y_j)\right|$$

$$\leqslant c\sum_{j=0}^{k-1}|y_{j+1} - y_j| = c|x_1 - x_2|.$$

The proposition in the case $n = 2$ is proved.

In the case $n \geqslant 3$ we can reduce to the 2-dimensional one by a slicing argument. For simplicity we sketch the proof only for $n = 3$; the general case can be treated in the same way.

From now on, we will write the coordinates of a point $p \in \mathbb{R}^3$ as a pair $(x, z) \in \mathbb{R}^2 \times \mathbb{R}$. Given $E \in \mathcal{U}_R$ and $p \in \partial E$, we denote by Π_p^E the projection on the plane which is tangent at p to the balls $B(p', R)$ and $B(p'', R)$.

If $q \in \partial E$ we define

$$a_p^E(q) := \Pi_p^E\left(\frac{1}{2R}(q'' - q')\right),$$

$$b_p^E(q) := \sqrt{1 - (a_p^E(q))^2}.$$

LEMMA 2.3. *There exist two constants $\delta > 0$, $M > 0$ such that, for every $E \in \mathcal{U}_R$ and for every $p, q \in \partial E$ with $|\Pi_p^E(p - q)| < \delta$, it results that $b_p^E(q) > M$.*

PROOF. Let us suppose by contradiction that for every $h \in \mathbb{N}$ there exist $E_h \in \mathcal{U}_R$, p_h, $q_h \in \partial E_h$ such that

$$|\Pi_{p_h}^{E_h}(p_h - q_h)| \leqslant \frac{1}{h}, \qquad 0 \leqslant b_{p_h}^{E_h}(q_h) \leqslant \frac{1}{h}. \tag{2.11}$$

Up to rototranslations, we can suppose that $p_h = (0, 0)$, $p_h' = (0, -R)$, and $p_h'' = (0, R)$. If we denote by (x_h, z_h) the coordinates of q_h, we obtain that

$$|p_h'' - q_h'|^2 = |(x_h, 0) + Ra_{p_h}^{E_h}(q_h)|^2 + (z_h - Rb_h(q_h) - R)^2.$$

Since by (2.11) the right-hand side tends to $2R^2$ as $h \to \infty$, for h large the ball $B(p_h'', R)$ intersects $B(q_h', R)$, which is impossible. ■

Now we are in a position to prove the crucial lemma which allows us to perform the two-dimensional reduction.

LEMMA 2.4. *Let $\delta > 0$ and $M > 0$ as in Lemma 2.3. Let E be in $\mathcal{U}_R$, $p \in \partial E$ and choose a coordinate system in S_E^p. Then, for every $(\overline{x}, 0)$ with $|\overline{x}| < \delta$ the section of E with any vertical plane γ passing through $(\overline{x}, 0)$ satisfies in γ the exterior and interior sphere condition with radius MR at every point of $\partial E \cap C$, where $C := \{x \in \mathbb{R}^{n-1}: |x| < \delta\} \times$ $\times]-R, R[$.*

PROOF. Let γ be a vertical plane passing through $(\overline{x}, 0)$ and let $(v, 0)$ be a unit normal vector to γ. Let $q \in \partial E \cap C \cap \gamma$. By Lemma 2.3, we have that

$$|a_p^E(q)| = \sqrt{1 - (b_p^E(q))^2} < \sqrt{1 - M^2};$$

hence, if we call α the angle in $[0, \pi[$ between $q'' - q'$ and $(v, 0)$, then

$$|\cos \alpha| = |a_p^E(q) \cdot (v, 0)| < \sqrt{1 - M^2}. \tag{2.12}$$

Then the point q satisfies the exterior and interior sphere condition in γ with radius $R \sin \alpha$, which by (2.12) is greater than MR. ■

Now we can prove the proposition in the case $n = 3$.

Given $E \in \mathcal{U}_R$ and $p \in \partial E$, we choose a coordinate system in $\mathcal{S}_E^p$ and we call C the cylinder $\{x \in \mathbb{R}^2: |x| < \bar{\delta}\} \times]-R, R[$, where $\bar{\delta} := \min\{\delta, \sqrt{3}MR/2\}$, and δ, M are as in Lemma 2.3. Applying the 2-dimensional result to the sections of E with the vertical planes passing through the point p, by Lemma 2.4 we obtain that $\partial E \cap C$ is the graph of a function f defined in $\{x \in \mathbb{R}^2: |x| < \bar{\delta}\}$.

To show the differentiability of f, we can repeat the same argument as in the 2-dimensional case. Moreover, Lemma 2.3 gives a uniform bound on the norm of the gradient of f.

Using Lemma 2.4, the 2-dimensional result, and Lemma 2.3, we can find $\varrho \in]0, \bar{\delta}]$ and $N > 0$ such that in $\{x \in \mathbb{R}^2: |x| < \varrho\}$ the restriction of f to any straight line is a function of class $W^{2,\infty}$ with $W^{2,\infty}$-norm less than N.

To conclude, we define the function

$$g(x_1, x_2) := \lim_{h \to \infty} h \left[\partial_{x_1} f \left(x_1 + \frac{1}{h}, x_2 \right) - \partial_{x_1} f(x_1, x_2) \right],$$

for a.e. $x = (x_1, x_2)$. By the above remark, g is defined a.e. and belongs to L^∞ with L^∞-norm less than N. Using the absolute continuity of $\partial_{x_1} f$ on the straight lines $x_2 =$ constant, it is easy to check that g coincides with the second distributional derivative $\partial_{x_1}^2 f$. Analogously, we can prove that there exists $\partial_{x_2}^2 f$ in the distributional sense, and that it belongs to L^∞ with L^∞-norm less than N. To show that $\partial_{x_1} \partial_{x_2} f$ exists and belongs to L^∞ with L^∞-norm less than N, one can argue in a similar way, by considering the restriction of f to the straight lines $x_1 - x_2 =$ = constant. ■

LEMMA 2.5. *Let $\{E_h\}_h$ be a sequence of connected sets in $\mathcal{U}_R$ such that $\lim_{h \to \infty} \operatorname{diam}(E_h) = +\infty$. Then*

$$\lim_{h \to \infty} \mathcal{L}^n(E_h) = +\infty.$$

PROOF. Since $\lim_{h\to\infty} \operatorname{diam}(E_h) = +\infty$, for every $h \in \mathbb{N}$ we can find $p_1^h, \dots, p_{m_h}^h \in \partial E_h$, where m_h is the integer part of $\operatorname{diam}(E_h)/4R$, such that $|p_i^h - p_j^h| \geq 4R$ for every $i \neq j$. We clearly have that $\{B((p_h^i)', R)\}_{i=1,\dots,m_h}$ is a family of disjoint balls all contained in E_h; hence,

$$\mathcal{L}^n(E_h) \geq m_h \mathcal{L}^n(B(0, R)),$$

and the second term goes to infinity as $h \to \infty$. ■

3. The compactness result.

In the sequel, if $\{f_j\}_j$ is a sequence in $W^{2,\infty}(\Omega)$ and f is a function in $W^{2,\infty}(\Omega)$, we mean by the notation $f_j \rightharpoonup f$ in w*-$W^{2,\infty}(\Omega)$ that the sequence $\{f_j\}_j$ converges to f in the weak*-topology of $W^{2,\infty}(\Omega)$. Given $E \subset \mathbb{R}^n$, we denote the characteristic function of E by χ_E. If ∂E is sufficiently regular, we denote the unit outer normal vector to ∂E at the point p by $\nu_{\partial E}(p)$.

We start by recalling two notions of set-convergence.

DEFINITION 3.1. *Let $\{E_h\}_h$ and E be measurable subsets of $\mathbb{R}^n$. We say that the sequence $\{E_h\}_h$ converges to E a.e. if $\chi_{E_h} \to \chi_E$ a.e., and that $\{E_h\}_h$ converges to E in L^1 if $\chi_{E_h} \to \chi_E$ in $L^1(\mathbb{R}^n)$.*

DEFINITION 3.2. *Let $\{E_h\}_h$ and E be closed subsets of $\mathbb{R}^n$. We say that the sequence $\{E_h\}_h$ converges to E in the sense of Kuratowski (and we write $E_h \xrightarrow{\mathcal{K}} E$) if*

i) $p_h \in E_h$, $\exists p_{h_k} \to p \Rightarrow p \in E$;

ii) $\forall p \in E$, $\exists p_h \in E_h$: $p_h \to p$.

It is well known that on the space of equibounded compact sets, the Kuratowski convergence is induced by the Hausdorff distance.

THEOREM 3.3. *Let $\{E_h\}_h$ be an equibounded sequence of sets belonging to $\mathcal{U}_R$. Then there exist $E \in \mathcal{U}_R$ and a subsequence $\{E_{h_j}\}_j$ such that*

a) $E_{h_j} \xrightarrow{\mathcal{K}} E$ *and* $E_{h_j} \to E$ in L^1;

b) $\partial E_{h_j} \xrightarrow{\mathcal{K}} \partial E$ *and* $\lim_{j\to\infty} \mathcal{H}^{n-1}(\partial E_{h_j}) = \mathcal{H}^{n-1}(\partial E)$;

c) *there exists a constant* $\eta \in]0, 1[$ (*depending only on* R), *such that for every* $p \in \partial E$, *if we call* C^η *the cylinder* $\{x \in \mathbb{R}^{n-1}: |x| \leq \eta R\} \times \times [-\eta R, \eta R]$ *expressed with respect to any coordinate system belonging to* $\mathcal{S}_E^p$, *and* S^η *the section* $C^\eta \cap \{z = 0\}$, *then* $\partial E \cap C^\eta$ *is the graph of a function* $f \in W^{2,\infty}(S^\eta)$, *and* $\partial E_{h_j} \cap C^\eta$ *is definitively the graph of a function* $f_j \in W^{2,\infty}(S^\eta)$. *Moreover,* $f_j \rightharpoonup f$ *in* w^*-$W^{2,\infty}(S^\eta)$.

PROOF. Since $\{E_h\}_h$ is equibounded, there exist a compact set E and a subsequence, which we denote again by $\{E_h\}_h$, such that

$$E_h \overset{\mathcal{K}}{\to} E . \tag{3.1}$$

Let us prove that $E \in \mathcal{U}_R$.

First of all, we remark that if $\{p_h\}_h$ is a sequence such that $\operatorname{dist}(p_h, E_h) > c > 0$ for every $h \in \mathbb{N}$, then every limit point p of $\{p_h\}_h$ belongs to $\complement E$. Indeed, let us suppose by contradiction that there exists $\{p_{h_k}\}_k$ which converges to $p \in E$; then, by ii) in Definition 3.2, for every $h \in \mathbb{N}$ there is $q_h \in E_h$ such that $\{q_h\}_h$ converges to p and so, $|q_{h_k} - p_{h_k}| \to \to 0$, in contradiction with the initial assumption.

CLAIM 1. Every point $p \in \partial E$ is the limit of a sequence $\{p_h\}_h$ such that $p_h \in \partial E_h$ for every $h \in \mathbb{N}$.

Let $p \in \partial E$. By ii) in Definition 3.2 there exists $p_h \in E_h$ such that $\{p_h\}_h$ converges to p; clearly, it is enough to show that $\operatorname{dist}(p_h, \partial E_h) \to 0$. If by contradiction there exists a subsequence $\{p_{h_k}\}_k$ such that $\operatorname{dist}(p_{h_k}, \partial E_{h_k}) > c > 0$ for every $k \in \mathbb{N}$, then the ball $B(p_{h_k}, c)$ is contained in E_{h_k}. Since for every $q \in B(p, c)$ we can find $q_k \in B(p_{h_k}, c)$ such that $q_k \to q$, then by i) in Definition 3.2, $q \in E$. Therefore $B(p, c) \subset E$, hence $p \in \operatorname{Int} E$, which contradicts our initial assumption.

CLAIM 2. If $p_h \in \partial E_h$ for every $h \in \mathbb{N}$ and there is a subsequence $\{p_{h_k}\}_k$ converging to a point p, then $p \in \partial E$. Moreover, there exist p', p'' such that

$$p \in \partial B(p', R) \cap \partial B(p'', R), \qquad B(p', R) \subset E, \qquad B(p'', R) \cap E = \emptyset .$$

Since $E_{h_k} \in \mathcal{U}_R$ for every $k \in \mathbb{N}$, there exist p_k', p_k'' such that the balls $B(p_k', R)$, $B(p_k'', R)$ are contained respectively in E_{h_k} and in $\complement E_{h_k}$. Up to subsequences, we can suppose that $\{p_k'\}_k$ and $\{p_k''\}_k$ converge to p' and

p'' respectively. Therefore,

$$\overline{B(p_k', R)} \overset{\mathcal{K}}{\to} \overline{B(p', R)}, \qquad \overline{B(p_k'', R)} \overset{\mathcal{K}}{\to} \overline{B(p'', R)},$$

and, since $\{p_{h_k}\} = \partial B(p_k', R) \cap \partial B(p_k'', R)$, we have that

$$\{p\} = \partial B(p', R) \cap \partial B(p'', R). \tag{3.2}$$

If $q \in B(p', R)$, then q is the limit of a sequence $\{q_k\}_k$ such that $q_k \in \in B(p_k', R) \subset E_{h_k}$; by (3.1) and i) in Definition 3.2, it follows that $q \in E$; this means that $B(p', R)$ is contained in E.

Let $q \in B(p'', R)$ and let $q_k := q - p'' - p_k''$ for every $k \in \mathbb{N}$. It is clear that $q_k \in B(p_k'', R)$, there exists a constant $c > 0$ such that $\operatorname{dist}(q_k, E_{h_k}) = = c$, and the sequence $\{q_k\}_k$ converges to q. Thus, as remarked before, $q \in \complement E$. We can conclude that $B(p'', R)$ is contained in the complement of E.

By (3.2), it follows that $p \in \partial E$ and this concludes the proof of the claim.

By Claim 1 and 2, we can deduce that $E \in \mathcal{U}_R$ and also

$$\partial E_h \overset{\mathcal{K}}{\to} \partial E. \tag{3.3}$$

To show the convergence in L^1, it is enough to prove the pointwise convergence of $\{\chi_{E_h}\}_h$ to χ_E for every $p \notin \partial E$; indeed, by the regularity of E, we have that $\mathcal{L}^n(\partial E) = 0$. If $p \in \operatorname{Int} E$, then by (3.1) and (3.3) there exists $p_h \in \operatorname{Int} E_h$ such that $\operatorname{dist}(p_h, \partial E_h) > c > 0$ and $p_h \to p$. Then p definitively belongs to $B(p_h, c)$, which is contained in $\operatorname{Int} E_h$; hence, $\chi_{E_h}(p) = = 1$ for h large and so, $\{\chi_{E_h}(p)\}_h$ obviously converges to $\chi_E(p)$. If $p \in \complement E$ and by contradiction there exists a subsequence $\{h_k\}_k$ such that $p \in E_{h_k}$, then by i) in Definition 3.2 $p \in E$, which is absurd.

Let us prove the third part of the proposition.

Let $p \in \partial E$. By (3.3), there is a sequence $\{p_h\}_h$ such that $p_h \in \partial E_h$ for every $h \in \mathbb{N}$ and $p_h \to p$. From now on, we will work in a coordinate system belonging to $\mathcal{S}_E^p$. By Proposition 2.1, there exists $\delta \in]0, 1[$, depending only on R, such that, if we set $C := \{x \in \mathbb{R}^{n-1}: |x| < \delta R\} \times]- - R, R[$, then $\partial E \cap C$ is the graph of a function f defined on the base of C and of class $W^{2,\infty}$. Let us denote by C_h the cylinder obtained by translating the centre of C in p_h and by rotating the axis of C in such a way that it is directed along $\nu_{\partial E_h}(p_h)$. By Proposition 2.1, $\partial E_h \cap C_h$ is the graph of a function f_h defined on the base of C_h and of class $W^{2,\infty}$. We recall that

$$\|f_h\|_\infty \leq (1-\sqrt{1-\delta^2})\,R\,, \tag{3.4}$$

(see the proof of Proposition 2.1). Since $\nu_{\partial E_h}(p_h)$ is parallel to the vector $p_h''-p_h'$, $\{p_h'\}_h$ converges to p', $\{p_h''\}_h$ converges to p'' (see the proof of Claim 2), and $\nu_{\partial E}(p)$ is parallel to the vector $p''-p'$, we have that

$$\nu_{\partial E_h}(p_h) \to \nu_{\partial E}(p)\,. \tag{3.5}$$

By the convergence of $\{p_h\}_h$ to p and by (3.5), it follows that for h sufficiently large C_h contains the cylinder $C^\eta = \{x \in \mathbb{R}^{n-1}\colon |x| \leq \eta R\} \times \times [-\eta R, \eta R]$, where $\eta \in]1-\sqrt{1-\delta^2}, \delta[$. Using (3.5), (3.4) and the equiboundedness of $\{\nabla f_h\}_h$, one can easily check that for h large enough $\partial E_h \cap C^\eta$ can be expressed as the graph of a new function $\tilde{f}_h$ defined on the base of C^η.

Using again (3.5) and the equiboundedness of $\{f_h\}_h$ in $W^{2,\infty}$-norm, it is easy to see that $\tilde{f}_h \in W^{2,\infty}(S^\eta)$ and the $W^{2,\infty}$-norm of $\tilde{f}_h$ is bounded by a constant depending only on R. Then there exist a subsequence $\{\tilde{f}_{h_k}\}_k$ and a function $\tilde{f} \in W^{2,\infty}(S^\eta)$ such that $\{\tilde{f}_{h_k}\}_k$ converge to $\tilde{f}$ in w*-$W^{2,\infty}(S^\eta)$ (and then in C^1-norm). It remains to prove that $\tilde{f}$ coincides with f on S^η.

Claim 3. It results that

$$\operatorname{graph} \tilde{f}_{h_k} \xrightarrow{\mathcal{H}} \operatorname{graph} \tilde{f} \tag{3.6}$$

and

$$\partial E_h \cap C^\eta \xrightarrow{\mathcal{H}} \partial E \cap C^\eta. \tag{3.7}$$

Let $p_k \in \operatorname{graph} \tilde{f}_{h_k}$ and let $\{p_{k_j}\}_j$ be a subsequence converging to a point p. The point p_k has coordinates $(x_k, \tilde{f}_{h_k}(x_k))$ with $|x_k| \leq \eta R$; up to subsequences, $\{x_k\}_k$ converges to a point x such that $|x| \leq \eta R$. By the uniform convergence of the functions, we obtain that $\{p_k\}_k$ tends to the point $(x, \tilde{f}(x))$, which belongs trivially to graph $\tilde{f}$. Then property i) in Definition 3.2 is proved. Let $p = (x, \tilde{f}(x)) \in \operatorname{graph} \tilde{f}$ with $|x| \leq \eta R$. The point $p_k := (x, \tilde{f}_{h_k}(x))$ belongs to graph $\tilde{f}_{h_k}$ and $\{p_k\}_k$ converges to p; hence, property ii) in Definition 3.2 is verified.

Since C^η is closed and by (3.3), property i) in Definition 3.2 is trivial. By (3.3) property ii) is easily verified for the points belonging to $\partial E \cap \cap \operatorname{Int} C^\eta$; if $p \in \partial E \cap \partial C^\eta$ and $p = (x, z)$ with $|x| = \eta R$ and $|z| \leq \eta R$, then it is enough to take the sequence $p_h = (x, \tilde{f}_h(x)) \in \partial E_h \cap \partial C^\eta$.

By Claim 3, since graph $\tilde{f}_{h_k} = \partial E_{h_k} \cap C^\eta$, it follows that graph $\tilde{f}$ coincides with $\partial E \cap C^\eta$. Then, $\tilde{f} = f$ on C^η and the whole sequence $\{\tilde{f}_h\}_h$ converges to f in w*-$W^{2,\infty}(S^\eta)$.

Let us prove the second part of b).

By point c), for every $p \in \partial E$ there exists a cylinder C centred at p, with base a $(n-1)$-dimensional sphere S, such that $\partial E \cap C$ is the graph of a function $f \in W^{2,\infty}(S)$, for h large $\partial E_h \cap C$ is the graph of a function $f_h \in W^{2,\infty}(S)$, and $f_h \rightharpoonup f$ in w*-$W^{2,\infty}(S)$. We can recover ∂E with a finite number of these cylinders $C_1, \ldots, C_m$. Let us call f_h^i the function such that graph $f_h^i = \partial E_h \cap C_i$, and f^i the function such that graph $f^i = \partial E \cap \cap C_i$.

Let $\varepsilon > 0$ be such that

$$(\partial E)_\varepsilon := \{p \in \mathbb{R}^n \colon \operatorname{dist}(p, \partial E) \leq \varepsilon\} \subset \bigcup_{i=1}^m C_i.$$

We can consider a partition of unity associated to the recovering $\{C_1, \ldots, C_m\}$, i.e. a family of functions $\phi_i \in C_0^\infty(C_i)$ $(i = 1, \ldots, m)$ such that $0 \leq \phi_i \leq 1$,

$$\sum_{i=1}^m \phi_i = 1 \quad \text{on } (\partial E)_\varepsilon, \qquad \sum_{i=1}^m \phi_i \leq 1 \quad \text{on } \bigcup_{i=1}^m C_i.$$

By (3.3), for h large $\partial E_h \subset (\partial E)_\varepsilon$. Then,

$$\mathcal{H}^{n-1}(\partial E_h) = \sum_{i=1}^m \int_{\partial E_h \cap C_i} \phi_i \, d\mathcal{H}^{n-1} = \sum_{i=1}^m \int_{\operatorname{graph} f_h^i} \phi_i \, d\mathcal{H}^{n-1}.$$

Using the Area Formula and the C^1-convergence of $\{f_h^i\}_h$ to f^i, it is easy to see that for every $i = 1, \ldots, m$

$$\lim_{h \to \infty} \int_{\operatorname{graph} f_h^i} \phi_i \, d\mathcal{H}^{n-1} = \int_{\operatorname{graph} f^i} \phi_i \, d\mathcal{H}^{n-1}.$$

Therefore,

$$\mathcal{H}^{n-1}(\partial E) = \sum_{i=1}^{m} \int_{\text{graph } f^i} \phi_i \, d\mathcal{H}^{n-1} = \lim_{h\to\infty} \sum_{i=1}^{m} \int_{\text{graph } f_h^i} \phi_i \, d\mathcal{H}^{n-1} =$$

$$= \lim_{h\to\infty} \mathcal{H}^{n-1}(\partial E_h). \qquad \blacksquare$$

4. The semicontinuity result.

Given $E \in \mathcal{U}_R$, we think ∂E oriented by the outer normal field (all the results we will state still remain true if we choose the opposite orientation). We denote the principal curvatures (i.e. the eigenvalues of the second fundamental quadratic form) of ∂E at the point x by $\kappa_i(x)$ with $i = 1, \ldots, n-1$, and the p^{th}-elementary symmetric function of the principal curvatures, called p^{th}-elementary symmetric curvature, by

$$K_p(x) = \binom{n-1}{p}^{-1} \sum_{1 \le i_1 < \ldots < i_p \le n-1} \kappa_{i_1}(x) \ldots \kappa_{i_p}(x) \tag{4.1}$$

for $p = 1, \ldots, n-1$. We also use the notation

$$H := K_1 \quad \text{and} \quad K := K_{n-1}$$

for the mean curvature and the Gauss curvature respectively. In the case $n = 2$ we simply denote the curvature by κ.

It is well known from differential geometry (see [10]) that the p^{th}-elementary symmetric curvature is the coefficient of the term of degree $n - 1 - p$ of the characteristic polynomial of the second fundamental quadratic form. If ∂E is locally the graph of a function f, then the second fundamental quadratic form is given by the product $G^{-1}B$, where $G = (g_{ij})$ is the matrix defined by

$$g_{ij} = \begin{cases} 1 + (\partial_{x_i} f)^2 & \text{if } i = j\,, \\ \partial_{x_i} f \partial_{x_j} f & \text{if } i \neq j\,, \end{cases}$$

while $B = (b_{ij})$ is the matrix

$$b_{ij} = \frac{\partial_{x_i} \partial_{x_j} f}{\sqrt{1 + |\nabla f|^2}}\,.$$

By induction, it is easy to prove that for every $p = 1, \ldots, n-1$ there exists a continuous function $\psi_p = \psi_p(s, \zeta)$, linear with respect to ζ, such that

$$K_p(x, f(x)) = \psi_p(\nabla f(x), M(\nabla^2 f(x))),$$

where $M(\nabla^2 f(x))$ is the vector of the determinants of all the minors of $\nabla^2 f(x)$.

In the sequel we will consider functionals of the form

$$F(E) := \int_{\partial E} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1},$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a given convex function. Functionals of this type arise in different contexts; for instance:

- the Willmore's functional (see [5, 11]), $F(E) = \int_{\partial E} |H(x)|^{n-1} d\mathcal{H}^{n-1}(x)$;
- $F(E) = \int_{\partial E} [H^2 - K_2]^{(n-1)/2} d\mathcal{H}^{n-1}$, studied in [12];
- $F(E) = \int_{\partial E} \varphi\left(\sum_{i=1}^{n-1} \kappa_i^2(x)\right) d\mathcal{H}^{n-1}(x)$; in the case $n = 2$ and $\varphi(x) = 1 + x$, we find the functional considered in [3]:

$$F(E) = \int_{\partial E} (1 + \kappa^2)\, d\mathcal{H}^1.$$

THEOREM 4.1. *Let $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a convex function. If $E \in \mathcal{U}_R$ and $\{E_h\}_h$ is a sequence in $\mathcal{U}_R$ such that $E_h \to E$ in L^1, then*

$$\int_{\partial E} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} \leq \liminf_{h \to \infty} \int_{\partial E_h} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}.$$

For the proof of the theorem we need the following lemma.

LEMMA 4.2. *Let $\phi : \mathbb{R}^n \times \mathbb{R}^{n-1} \to [0, +\infty[$ be globally continuous and convex in the last $n-1$ variables. Let Ω be an open bounded subset* *of* $\mathbb{R}^{n-1}$ *and let* $f_h, f \in W^{2,\infty}(\Omega)$. *If* $f_h \rightharpoonup f$ *in* w^*-$W^{2,\infty}(\Omega)$, *then*

$$\int_{\text{graph} f} \phi(q, K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}(q) \leq$$

$$\leq \liminf_{h\to\infty} \int_{\text{graph} f_h} \phi(q, K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}(q).$$

PROOF It is not restrictive to assume that Ω is smooth.
As remarked above, for every $p = 1, \ldots, n-1$ and for every $x \in \Omega$ we have that

$$K_p(x, f_h(x)) = \psi_p(\nabla f_h(x), M(\nabla^2 f_h(x))),$$

where ψ_p is globally continuous and linear in the second variable.
Using the Area Formula, we can write

$$\text{(4.2)} \quad \int_{\text{graph} f_h} \phi(q, K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}(q) =$$

$$= \int_\Omega \phi'(x, f_h(x), \nabla f_h(x), \nabla^2 f_h(x))\, dx,$$

where

$$\phi'(x, z, s, \xi) := \phi((x, z), \psi_1(s, M(\xi)), \ldots, \psi_{n-1}(s, M(\xi)))\sqrt{1+|s|^2}$$

for every $x \in \Omega$, $z \in \mathbb{R}$, $s \in \mathbb{R}^n$, and $\xi \in \mathbb{M}^{n\times n}$. Let us define the function

$$\phi''(x, s, \xi) := \phi'(x, f(x), s, \xi)$$

for every $x \in \Omega$, $s \in \mathbb{R}^n$, and $\xi \in \mathbb{M}^{n\times n}$. Since ϕ'' is positive, globally continuous and polyconvex in ξ, by Theorem II.1 in [1], it follows that

$$\int_\Omega \phi''(x, \nabla f(x), \nabla^2 f(x))\, dx \leq \liminf_{h\to\infty} \int_\Omega \phi''(x, \nabla f_h(x), \nabla^2 f_h(x))\, dx,$$

that is

$$(4.3)\qquad \int_\Omega \phi'(x, f(x), \nabla f(x), \nabla^2 f(x))\,dx \leq$$

$$\leq \liminf_{h\to\infty} \int_\Omega \phi'(x, f(x), \nabla f_h(x), \nabla^2 f_h(x))\,dx\,.$$

Using the uniform continuity of ϕ' on bounded sets and the uniform convergence of $\{f_h\}_h$ to f, we have that

$$(4.4)\qquad \lim_{h\to\infty} \int_\Omega |\phi'(x, f_h(x), \nabla f_h(x), \nabla^2 f_h(x)) - \phi'(x, f(x), \nabla f_h(x), \nabla^2 f_h(x))|\,dx = 0\,.$$

By (4.2), (4.3), and (4.4), the thesis easily follows. ■

PROOF OF THEOREM 4.1. First of all, we observe that the sequence $\{E_h\}_h$ is equibounded; indeed, let $M > 0$ be such that $E \subset B(0, M)$ and let $\tilde{E}_h$ be the union of all the connected components of E_h which intersect $B(0, M)$. By the L^1-convergence of $\{E_h\}_h$ to E, it is clear that

$$\lim_{h\to\infty} \mathcal{L}^n(E_h \setminus \tilde{E}_h) = 0\,.$$

Recalling that, if $E \neq \emptyset$ belongs to $\mathcal{U}_R$, then $\mathcal{L}^n(E) \geq \mathcal{L}^n(B(0, R))$, we deduce that for h large $E_h = \tilde{E}_h$, i.e. all the connected components of E_h definitively intersect $B(0, M)$. The equiboundedness easily follows by Lemma 2.5, and allows to conclude that the sequence $\{E_h\}_h$ satisfies a), b), c) of Theorem 3.3. (Note that we have incidentally proved that in the class $\mathcal{U}_R$, L^1-convergence and Kuratowski convergence are actually equivalent).

Let us suppose for the moment that φ is positive. By Theorem 3.3, for every $p \in \partial E$ there exists a cylinder C centred at p, with base a $(n-1)$-dimensional sphere S, such that $\partial E \cap C$ is the graph of a function $f \in W^{2,\infty}(S)$, for h large $\partial E_h \cap C$ is the graph of a function $f_h \in W^{2,\infty}(S)$, and $f_h \rightharpoonup f$ in w*-$W^{2,\infty}(S)$. We can recover ∂E with a finite number of these cylinders $C_1, \ldots, C_m$. Let us call f_h^i the function such that graph $f_h^i = \partial E_h \cap C_i$, and f^i the function such that graph $f^i = \partial E \cap C_i$.

We can consider a partition of unity associated to the recovering $\{C_1, \ldots, C_m\}$, i.e. a family of functions $\phi_i \in C_0^\infty(C_i)$ $(i = 1, \ldots, m)$ such that $0 \leqslant \phi_i \leqslant 1$,

$$\sum_{i=1}^{m} \phi_i = 1 \text{ on } \partial E, \quad \sum_{i=1}^{m} \phi_i \leqslant 1 \text{ on } \bigcup_{i=1}^{m} C_i. \tag{4.5}$$

Then

$$\begin{aligned}
\int_{\partial E} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} &= \sum_{i=1}^{m} \int_{\partial E \cap C_i} \phi_i \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} \\
&= \sum_{i=1}^{m} \int_{\operatorname{graph} f^i} \phi_i \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} \\
&\leqslant \liminf_{h \to \infty} \sum_{i=1}^{m} \int_{\operatorname{graph} f_h^i} \phi_i \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} \\
&= \liminf_{h \to \infty} \sum_{i=1}^{m} \int_{\partial E_h \cap C_i} \phi_i \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} \\
&\leqslant \liminf_{h \to \infty} \int_{\partial E_h} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1},
\end{aligned}$$

where we used Lemma 4.2 and (4.5).

If φ is bounded from below by a constant $c \in \mathbb{R}$, we can apply the previous argument to the function $\varphi - c$, to conclude that

$$\begin{aligned}
&\int_{\partial E} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} - c\mathcal{H}^{n-1}(\partial E) \leqslant \\
&\leqslant \liminf_{h \to \infty} \left(\int_{\partial E_h} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} - c\mathcal{H}^{n-1}(\partial E_h) \right) \\
&\qquad = \liminf_{h \to \infty} \int_{\partial E_h} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} - c\mathcal{H}^{n-1}(\partial E),
\end{aligned}$$

where we used property b) in Theorem 3.3.

Finally, if φ is a generic convex function, let us set

$$c := \inf \left\{ \varphi(K_1(p), \ldots, K_{n-1}(p)) : p \in \bigcup_{h \in \mathbb{N}} \partial E_h \cup \partial E \right\},$$

which is finite by the equiboundedness of curvatures (see Proposition

2.1). If we define $\tilde{\varphi} := \varphi \vee c$, we have that $\tilde{\varphi}$ is a convex function bounded from below; hence,

$$\int_{\partial E} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1} = \int_{\partial E} \tilde{\varphi}(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}$$

$$\leq \liminf_{h\to\infty} \int_{\partial E_h} \tilde{\varphi}(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}$$

$$= \liminf_{h\to\infty} \int_{\partial E_h} \varphi(K_1, \ldots, K_{n-1})\, d\mathcal{H}^{n-1}. \qquad \blacksquare$$

In the following proposition, we study the asymptotic behaviour of F when R goes to 0, in the case $n = 2$ and $\varphi(\kappa) = 1 + |\kappa|^p$, showing the relationship with the relaxed functional introduced in [3].

PROPOSITION 4.3. *Let $\{F_R\}_{R>0}$ be the family of functionals*

$$F_R(E) := \begin{cases} \displaystyle\int_{\partial E} (1 + |\kappa|^p)\, d\mathcal{H}^1 & \textit{if } E \in \mathcal{M} \cap \mathcal{U}_R, \\ +\infty & \textit{otherwise in } \mathcal{M}, \end{cases}$$

where $p > 1$, and $\mathcal{M}$ is the class of measurable bounded sets in $\mathbb{R}^2$. Then, F_R, as $R \to 0$, Γ-converges (for the definition and the properties of Γ-convergence, see [6]*) with respect to the L^1-topology to the lower semicontinuous envelope, $\overline{F_0}$, of*

$$F_0(E) := \begin{cases} \displaystyle\int_{\partial E} (1 + |\kappa|^p)\, d\mathcal{H}^1 & \textit{if } E \in \mathcal{M} \cap C^2, \\ +\infty & \textit{otherwise in } \mathcal{M}. \end{cases}$$

PROOF. For every $E \in \mathcal{M}$, $\{E_h\}_h \to E$ in L^1 and $\{R_h\}_h \to 0^+$, we have to check that

$$\overline{F_0}(E) \leq \liminf_{h\to\infty} F_{R_h}(E_h).$$

We can suppose that

$$\liminf_{h\to\infty} F_{R_h}(E_h) < +\infty$$

and we can extract a subsequence $\{E_{h_k}\}_k$ such that $F_{R_{h_k}}(E_{h_k})$ is finite and

$$\liminf_{h\to\infty} F_{R_h}(E_h) = \lim_{k\to\infty} F_{R_{h_k}}(E_{h_k}).$$

Since E_{h_k} belongs to $\mathcal{U}_{R_{h_k}}$, by Corollary 3.2 in [3] it follows that

$$F_{R_{h_k}}(E_{h_k}) = \overline{F_0}(E_{h_k})$$

and then,

$$\overline{F_0}(E) \leq \lim_{k\to\infty} F_{R_{h_k}}(E_{h_k});$$

hence, the liminf inequality is proved.

To obtain the limsup inequality, fixed $E \in \mathfrak{M}$ and $\{R_h\}_h \searrow 0$, we have to find a sequence $\{E_h\}_h \to E$ in L^1 such that

$$\overline{F_0}(E) \geq \limsup_{h\to\infty} F_{R_h}(E_h).$$

We can assume $\overline{F_0}(E)$ finite; then, there exists a sequence $\{A_k\}_k$ such that A_k is in C^2, $\{A_k\}_k$ converges to E in L^1, as $k \to \infty$, and

$$\overline{F_0}(E) = \lim_{k\to\infty} F_0(A_k).$$

The smoothness of A_k implies that there is $r_k > 0$ such that A_k belongs to the class $\mathcal{U}_{r_k}$. Let us define by induction the following sequence of indices:

$$h_1 = \min\{h : R_h \leq r_1\}$$

$$h_k = \min\{h > h_{k-1} : R_h \leq r_k\}$$

and the sets

$$E_h := \begin{cases} A_1 & \text{for } h < h_1, \\ A_k & \text{for } h_k \leq h < h_{k+1}, \quad k \geq 1 . \end{cases}$$

It is easy to verify that $\{E_h\}_h$ is the required sequence. ■

5. A variational problem in Image Segmentation.

In this section we apply the results of the previous ones to state an existence theorem for the NITZBERG and MUMFORD problem in the class $\mathcal{U}_R$. For every $k \in \mathbb{N}$ and for every $E_1, \ldots, E_k \in \mathcal{U}_R$ let us define the following functional:

$$G_k(E_1, \ldots, E_k) := \alpha \int\limits_{\Omega \setminus \bigcup_{i=1}^k E_i} |g - g_{\Omega \setminus \bigcup_{i=1}^k E_i}|^2 \, dx + \tag{5.1}$$

$$+ \sum_{i=1}^{k} \left(\alpha \int\limits_{E_i' \cap \Omega} |g - g_{E_i' \cap \Omega}|^2 \, dx + \beta \mathcal{L}^2(E_i) + \gamma \int\limits_{\partial E_i} \varphi(\kappa) \, d\mathcal{H}^1 \right),$$

where α, β, γ are positive parameters, $E_i' := E_i \setminus \bigcup_{j=1}^{i-1} E_j$, g is a given function in $L^2(\Omega)$, $\varphi : \mathbb{R} \to \mathbb{R}$ is a given convex function, and κ denotes the curvature of ∂E. If we take

$$\varphi(\kappa) = \begin{cases} \nu + a\kappa^2 & \text{if } |\kappa| < \dfrac{b}{a}, \\[2ex] \nu + b|\kappa| & \text{if } |\kappa| \geq \dfrac{b}{a}, \end{cases}$$

we obtain exactly the original model proposed in [8].

THEOREM 5.1. *For every $R > 0$ and for every $k \in \mathbb{N}$ the problem*

$$\min \{ G_k(E_1, \ldots, E_k) : E_1, \ldots, E_k \in \mathcal{U}_R \} \tag{5.2}$$

admits a solution.

PROOF. For the sake of simplicity, we perform the proof only for $k = 1$; the general case follows by a similar argument, involving only some further difficulties of notation.

Let $\{E_m\}_m$ be a minimizing sequence in $\mathcal{U}_R$ for the functional G_1. We can suppose that all non-empty connected components of each E_m meet Ω; indeed, if we call $\widetilde{E}_m$ the union of the connected components of E_m which intersect Ω, we have that $G_1(\widetilde{E}_m) \leq G_1(E_m)$, and then, we can replace E_m by $\widetilde{E}_m$. By Lemma 2.5 the sequence results equibounded.

Applying Theorems 3.3 and 4.1 to the sequence $\{E_m\}_m$, we obtain a subsequence $\{E_{m_h}\}_h$ and a set $E \in \mathcal{U}_R$ such that

i) $E_{m_h} \to E$ in L^1 and a.e.;

ii) $\displaystyle\int\limits_{\partial E} \varphi(\kappa) \, d\mathcal{H}^1 \leq \liminf_{h \to \infty} \int\limits_{\partial E_{m_h}} \varphi(\kappa) \, d\mathcal{H}^1$.

We observe that

$$\left| \int_{\Omega \cap E_{m_h}} |g - g_{\Omega \cap E_{m_h}}|^2 dx - \int_{\Omega \cap E} |g - g_{\Omega \cap E}|^2 dx \right| \leq \int_{\Omega} \chi_{\Omega \cap E_{m_h}} \big| |g - g_{\Omega \cap E_{m_h}}|^2 - |g - g_{\Omega \cap E}|^2 \big| dx + \int_{\Omega} |\chi_{\Omega \cap E_{m_h}} - \chi_{\Omega \cap E}| \, |g - g_{\Omega \cap E}|^2 dx ;$$

hence, applying the Dominated Convergence Theorem to both addends, we can conclude that

$$\lim_{h \to \infty} \int_{\Omega \cap E_{m_h}} |g - g_{\Omega \cap E_{m_h}}|^2 dx = \int_{\Omega \cap E} |g - g_{\Omega \cap E}|^2 dx .$$

Analogously,

$$\lim_{h \to \infty} \int_{\Omega \setminus E_{m_h}} |g - g_{\Omega \setminus E_{m_h}}|^2 dx = \int_{\Omega \setminus E} |g - g_{\Omega \setminus E}|^2 dx .$$

At this point it is clear that

$$G_1(E) \leq \liminf_{h \to \infty} G_1(E_{m_h}) ,$$

and that E minimizes the functional. ∎

As explained in [9], the integer k is the number of depth levels of the reconstructed image; denoting by $(E_1, \ldots, E_k)$ the solution of (5.2), the set E_i represents all the objects at the i-th level. If k is not a priori fixed, we can consider the variational problem studied in the following theorem.

THEOREM 5.2. *For every $R > 0$ the problem*

$$\min \{ G_k(E_1, \ldots, E_k) : E_1, \ldots, E_k \in \mathcal{U}_R, \, k \in \mathbb{N} \}$$

admits a solution.

PROOF. Let $\{(E_1^m, \ldots, E_{k_m}^m)\}_m$ be a minimizing sequence. Since for every $l \in \mathbb{N}$, $j \in \{1, \ldots, l-1\}$ and $A_1, \ldots, A_{l-1} \in \mathcal{U}_R$ we have that

$$G_l(A_1, \ldots, A_{j-1}, \emptyset, A_j, \ldots, A_{l-1}) = G_{l-1}(A_1, \ldots, A_{j-1}, A_j, \ldots, A_{l-1}),$$

we can suppose that $E_j^m \neq \emptyset$ for every m and for every $j \in \{1 \ldots, k_m\}$, and so,

$$G_{k_m}(E_1^m, \ldots, E_{k_m}^m) \geq \beta \sum_{j=1}^{k_m} \mathcal{L}^2(E_j^m) \geq \beta k_m \mathcal{L}^2(B(0, R)) ;$$

therefore, the sequence $\{k_m\}_m$ must be bounded and so admits a constant subsequence: now we can conclude by applying Theorem 5.1. ■

If we are interested not only in detecting contours, but also in cleaning and regularizing the image, we can consider the following variational problem:

$$\min\left\{\alpha \int_{\Omega\setminus\bigcup_{i=1}^k \partial E_i'} |u-g|^2 dx + \sum_{i=1}^k \left(\beta \mathcal{L}^2(E_i) + \gamma \int_{\partial E_i} \varphi(\kappa)\, d\mathcal{H}^1\right) + \right.$$
$$\left. + \delta \int_{\Omega\setminus\bigcup_{i=1}^k \partial E_i'} |\nabla u|^2 dx : u \in C^1\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right), E_1, \ldots, E_k \in \mathcal{U}_R\right\}, \tag{5.3}$$

where k is fixed in $\mathbb{N}$, δ is a positive parameter and we use the same notation as before.

THEOREM 5.3. *Let g be a function in $L^\infty(\Omega)$. Then, for every $R>0$ and for every $k \in \mathbb{N}$ the problem in* (5.3) *admits a solution.*

PROOF. We first look for a solution $(u, E_1, \ldots, E_k)$ where $u \in W^{1,2}\left(\Omega\setminus \bigcup_{i=1}^k \partial E_i'\right)$.

Let $\{(u_h, E_1^h, \ldots, E_k^h)\}_h$ be a minimizing sequence for the functional in (5.3). By a truncation argument we can suppose that $\|u_h\|_\infty \leq \|g\|_\infty$ and, as in the proof of Theorem 5.1, we can assume that $\{E_i^h\}_h$ is equibounded for every $i=1, \ldots, k$. By Theorem 3.3 there exist $E_1, \ldots, E_k$ belonging to $\mathcal{U}_R$ such that, up to subsequences,

$$E_i^h \xrightarrow{\mathcal{K}} E_i \quad \text{and} \quad E_i^h \to E_i \ \text{ in } L^1(\Omega).$$

Arguing as in the proof of Theorem 3.3, one can easily check that if U is an open subset compactly contained in $\Omega\setminus\bigcup_{i=1}^k \partial E_i'$, then for h large U is compactly contained in $\Omega\setminus\bigcup_{i=1}^k \partial (E_i^h)'$.

Since $\{u_h\}_h$ is equibounded in $W^{1,2}(U)$, up to subsequences, there exists $u \in W^{1,2}(U)$ such that $u_h \rightharpoonup u$ in w-$W^{1,2}(U)$. By the weakly lower semicontinuity of the L^2-norm, by Theorems 3.3 and 4.1, we obtain

$$\alpha \int_U |u-g|^2 dx + \sum_{i=1}^k \left(\beta \mathcal{L}^2(E_i) + \gamma \int_{\partial E_i} \varphi(\kappa)\, d\mathcal{H}^1\right) + \delta \int_U |\nabla u|^2 dx \leq$$

$$\leq \liminf_{h\to\infty} \alpha \int_U |u_h - g|^2 dx + \sum_{i=1}^k \left(\beta \mathcal{L}^2(E_i^h) + \gamma \int_{\partial E_i^h} \varphi(\kappa)\, d\mathcal{H}^1 \right) + \delta \int_U |\nabla u_h|^2 dx$$

$$\leq \liminf_{h\to\infty} \alpha \int_{\Omega \setminus \bigcup_{i=1}^k \partial(E_i^h)'} |u_h - g|^2 dx + \sum_{i=1}^k \left(\beta \mathcal{L}^2(E_i^h) + \gamma \int_{\partial E_i^h} \varphi(\kappa)\, d\mathcal{H}^1 \right)$$

$$+ \delta \int_{\Omega \setminus \bigcup_{i=1}^k \partial(E_i^h)'} |\nabla u_h|^2 dx\,.$$

Let us construct a sequence of open subsets compactly contained in $\Omega \setminus \bigcup_{i=1}^k \partial E_i'$ and increasing to it; the previous argument combined with a diagonal procedure allows us to conclude that there exists $u \in W^{1,2}\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right)$ such that $(u, E_1, \ldots, E_k)$ minimizes the functional.

Since $u - g \in L^\infty\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right)$, the regularity theory for elliptic equations ensures that $u \in W^{2,p}_{\mathrm{loc}}\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right)$ for every $p < \infty$, hence $u \in C^1\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right)$. ∎

Let us suppose now that k is not a priori fixed: arguing as in Theorem 5.2, we can prove the following result.

THEOREM 5.4. *Let g be a function in $L^\infty(\Omega)$. Then, for every $R > 0$ the problem*

$$\min\left\{ \alpha \int_{\Omega \setminus \bigcup_{i=1}^k \partial E_i'} |u - g|^2 dx + \sum_{i=1}^k \left(\beta \mathcal{L}^2(E_i) + \gamma \int_{\partial E_i} \varphi(\kappa)\, d\mathcal{H}^1 \right) + \right. \tag{5.4}$$

$$\left. + \delta \int_{\Omega \setminus \bigcup_{i=1}^k \partial E_i'} |\nabla u|^2 dx : u \in C^1\left(\Omega \setminus \bigcup_{i=1}^k \partial E_i'\right), E_1, \ldots, E_k \in \mathcal{U}_R, k \in \mathbb{N} \right\},$$

admits a solution.

We conclude this section by giving an example of non trivial (i.e. non empty) minimizer.

EXAMPLE 5.5. *Let us set $g := \chi_{B(0,R)}$ and assume $R > 1$. For a suitable choice of Ω and of the parameters α, β, γ, $B(0, R)$ minimizes the functional*

$$G_1(E) = \alpha \int_{\Omega \setminus E} |g - g_{\Omega \setminus E}|^2 dx + \alpha \int_{E \cap \Omega} |g - g_{E \cap \Omega}|^2 dx + \\ + \beta \mathcal{L}^2(E) + \gamma \int_{\partial E} (1 + |\kappa|^2)\, d\mathcal{H}^1 .$$

PROOF. It is known (see Theorem 5.7.3 in [4]) that for every smooth closed curve γ, it results that

$$2\pi \leq \int_{\gamma} |\kappa|\, d\mathcal{H}^1 . \tag{5.5}$$

Holder inequality and (5.5) imply that for every $E \in \mathcal{U}_R$, $E \neq \emptyset$, the following inequality holds:

$$\int_{\partial E} |\kappa|^2 d\mathcal{H}^1 \geq \frac{4\pi^2}{\mathcal{H}^1(\partial E)} ,$$

so that

$$G_1(E) \geq \beta\pi R^2 + \gamma \left(\mathcal{H}^1(\partial E) + \frac{4\pi^2}{\mathcal{H}^1(\partial E)} \right).$$

Since $\mathcal{H}^1(\partial E) \geq 2\pi R$ and $R > 1$,

$$G_1(E) \geq \beta\pi R^2 + \gamma \left(2\pi R + \frac{2\pi}{R} \right) = G_1(B(0, R)) .$$

Finally,

$$G_1(\emptyset) = \alpha \left(\pi R^2 - \frac{\pi^2 R^4}{\mathcal{L}^2(\Omega)} \right) \geq \beta\pi R^2 + \gamma \left(2\pi R + \frac{4\pi^2}{2\pi R} \right)$$

for a suitable choice of Ω and of the parameters. ■

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