

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 104 (2000), p. 109-127

[http://www.numdam.org/item?id=RSMUP\\_2000\\_\\_104\\_\\_109\\_0](http://www.numdam.org/item?id=RSMUP_2000__104__109_0)

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## Regularity of Lipschitz Minima.

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The existence of Lipschitz minima was the fundamental step in Hilbert's proof of Dirichlet Principle (see [6] and [7]). Hilbert's Lemma in today's language would be written: «For any open, bounded set  $\Omega \subset \mathbb{R}^n$  and any Lipschitz function  $\gamma : \partial\Omega \rightarrow \mathbb{R}$  satisfying the B.S.C (Bounded Slope Condition), there exists a unique Lipschitz function  $w \in \text{Lip}(\mathbb{R}^n)$  satisfying (1) and  $w|_{\partial\Omega} = \gamma$ » (see [5]).

Its validity for a general  $F$  was remarked by A. Haar [8] in the 2-dimensional case. The results of Hilbert and Haar opened an interesting problem about the regularity of the minimizing functions according with the regularity of the function  $F$ . Hilbert conjectured (see the XIX Problem in [1]) that the Lipschitz minima are analytic if  $F$  is so.

E. Hopf [9] and C. B. Morrey [10] proved that the Hilbert conjecture was true for the  $C^1$ -minima.

The gap, from Lipschitz to  $C^1$ , was filled by the famous regularity result established by E. De Giorgi in 1957 [2].

What we do here, is to apply De Giorgi's method directly to the proof of Hilbert conjecture.

I wish to thank Mario Miranda, I am indebted to him for his advice. I am grateful to the Dipartimento di Matematica dell'Università Degli Studi di Trento for financial support and access to its facilities.

NOTATION. Throughout this paper the symbol  $\Omega$  means an open, bounded set in  $\mathbb{R}^n$  with  $n \geq 2$ . By  $\text{Lip}(\Omega, \mathbb{R}^k)$  we denote the set of Lip-

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schitz functions  $f : \Omega \rightarrow \mathbb{R}^k$  and by  $\text{Lip}_c(\Omega, \mathbb{R}^k)$  the set of functions in  $\text{Lip}(\Omega, \mathbb{R}^k)$  with compact support in  $\Omega$ .

If  $k = 1$  we write  $\text{Lip}(\Omega)$ ,  $\text{Lip}_c(\Omega)$  for  $\text{Lip}(\Omega, \mathbb{R})$  and  $\text{Lip}_c(\Omega, \mathbb{R})$ .

In this paper we shall prove the following:

**THEOREM 1.** *If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a strictly convex function of class  $C^2$  and  $w \in \text{Lip}(\mathbb{R}^n)$  satisfies*

$$(1) \quad \int_{\Omega} F(Dw + D\varphi) \, dx \geq \int_{\Omega} F(Dw) \, dx ,$$

for all  $\varphi \in \text{Lip}_c(\Omega)$ , then  $w$  has Hölder-continuous first derivatives in  $\Omega$ .

**PROOF.** We shall divide the proof in six steps.

**STEP 1.** We prove that the differential quotients of  $w$  satisfy the integro-differential equation (7), whose coefficients are defined by (7) and have the property (10).

The convexity assumption on  $F$  implies that (1) is equivalent to the following identity:

$$(2) \quad \int_{\Omega} DF(Dw) D\varphi \, dx = 0 , \quad \forall \varphi \in \text{Lip}_c(\Omega) .$$

For any  $e \in S^{n-1} = \{e \in \mathbb{R}^n : |e| = 1\}$  and  $|t| < \text{dist}(\text{spt } \varphi, \partial\Omega)$ , we get

$$\int_{\Omega} DF(Dw) D\varphi(x - te) \, dx = 0$$

and

$$(3) \quad \int_{\Omega} DF(Dw(x + te)) D\varphi \, dx = 0 .$$

If we subtract (2) from (3), we obtain

$$(4) \quad \int_{\Omega} D[F(Dw(x + te)) - F(Dw)] D\varphi \, dx = 0 .$$

Since

$$DF(Dw(x + te)) - DF(Dw) = DF(Dw + s[Dw(x + te) - Dw]) \Big|_{s=0}^{s=1} ,$$

we have

$$\begin{aligned}
 (5) \quad D_i[F(Dw(x + te)) - F(Dw)] &= \\
 &= \int_0^1 \sum_{j=1}^n D_i D_j F(Dw + s[Dw(x + te) - Dw]) D_j[w(x + te) - w] ds \\
 &= \sum_{j=1}^n a_{ij}(x) D_j[w(x + te) - w], \quad \forall i = 1, \dots, n
 \end{aligned}$$

where

$$(6) \quad a_{ij}(x) = \int_0^1 D_i D_j F(Dw + s[Dw(x + te) - Dw]) ds$$

are bounded, Lebesgue measurable functions for all  $i, j = 1, \dots, n$ .

Thanks to (5) the integral equation (4) becomes

$$\int_{\Omega} \sum_{i,j} a_{ij}(x) D_j[w(x + te) - w] D_i \varphi dx = 0 .$$

Dividing by  $t \neq 0$ , we obtain the following integral equation

$$(7) \quad \int_{\Omega} \sum_{i,j} a_{ij}(x) D_j u D_i \varphi dx = 0 ,$$

where

$$u(x) = u(x, t, e) = \frac{w(x + te) - w(x)}{t}$$

is a differential quotient of  $w$ .

Putting

$$p = Dw + s[Dw(x + te) - Dw],$$

with  $s \in [0, 1]$  and denoting by  $K$  the Lipschitz constant of  $w$ , we have

$$(8) \quad |p| = |Dw + s[Dw(x + te) - Dw]| \leq (1 - s) K + sK = K ,$$

for all  $s \in [0, 1]$  and for all  $x, t, e$ .

From (6) and (8), since  $F$  is of class  $C^2$  and strictly convex, we obtain

$$(9) \quad \mu_1 |\lambda|^2 \leq \sum_{i,j} a_{ij}(x) \lambda_i \lambda_j \leq \mu_2 |\lambda|^2, \quad \forall \lambda \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n,$$

where

$$\mu_1 = \min \left\{ \sum_{i,j} D_i D_j F(p) \lambda_i \lambda_j : |p| < K, |\lambda| = 1 \right\} > 0,$$

$$\mu_2 = \max \left\{ \sum_{i,j} D_i D_j F(p) \lambda_i \lambda_j : |p| < K, |\lambda| = 1 \right\}.$$

REMARK 2. For any  $\varepsilon > 0$ , put  $\Omega_{-\varepsilon} = \{x \in \Omega : \text{dist}(x, (\mathbb{R}^n - \Omega)) > \varepsilon\}$ , we have that

$$u(x, t, e) = \frac{w(x + te) - w(x)}{t}, \quad \forall x \in \Omega_{-\varepsilon}, \quad e \in S^{n-1}, \quad |t| < \varepsilon$$

is a family of bounded functions (where the bound is the Lipschitz constant  $K$  of  $w$ ) that satisfies the integro-differential equation

$$(10) \quad \int_{\Omega_{-\varepsilon}} \sum_{i,j} a_{ij}(x) D_j u D_i \varphi \, dx = 0, \quad \forall \varphi \in \text{Lip}_c(\Omega_{-\varepsilon}),$$

where  $\{a_{ij}(x)\}$  are defined by (6) and satisfy (9).

For notational convenience the same letter  $\Omega$  will be used to denote the set  $\Omega_{-\varepsilon}$ .

STEP 2. As consequence of (10) and (9) we obtain the so-called Caccioppoli inequalities (see [2]), i.e. for all  $y \in \Omega$ ,  $0 < \varrho_1 < \varrho_2 < \text{dist}(y, \partial\Omega)$ ,  $\gamma = \frac{\mu_2}{\mu_1}$  and  $k \in \mathbb{R}$ , we have

$$(11a) \quad \frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_{A(k, \varrho_2)} (u(x) - k)^2 \, dx \geq \int_{A(k, \varrho_1)} |Du|^2 \, dx,$$

$$(11b) \quad \frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_{B(\varrho_2) - A(k)} (u(x) - k)^2 \, dx \geq \int_{B(\varrho_1) - A(k)} |Du|^2 \, dx,$$

where

$$B(\varrho) = B_y(\varrho) = \{x \in \mathbb{R}^n : |x - y| < \varrho\},$$

$$A(k) = \{x \in \mathbb{R}^n : u(x) > k\},$$

$$A(k, \varrho) = A(k) \cap B(\varrho).$$

We restrict ourselves to the proof of (11a), since (11b) can be derived from the application of (11a) to the function  $-u$ .

Let be

$$\bar{\varphi} = \eta^2[(u - k) \vee 0],$$

where  $\eta \in \text{Lip}_c(\Omega)$ . The function  $\bar{\varphi}$  is in  $\text{Lip}_c(\Omega)$ , moreover it vanishes in  $\mathbb{R}^n - A(k)$ . Hence substituting in the integral equation (7) the function  $\bar{\varphi}$  in the place of  $\varphi$ , we obtain

$$\int_{A(k)} \sum_{i,j} a_{ij}(x) D_i u D_j \bar{\varphi} \, dx = 0,$$

that is,

$$\int_{A(k)} \sum_{i,j} a_{ij}(x) D_i (u - k) D_j [\eta^2(u - k)] \, dx = 0.$$

By a simple computation, we get

$$(12) \quad \int_{A(k)} \sum_{i,j} a_{ij}(x) \{D_i [\eta(u - k)] D_j [\eta(u - k)] - (u - k)^2 D_i \eta D_j \eta - (u - k) D_i \eta D_j [\eta(u - k)] + (u - k) D_i [\eta(u - k)] D_j \eta\} \, dx = 0.$$

Since the matrix  $A = \{a_{ij}(x)\}$  is symmetric, we have

$$\int_{A(k)} \sum_{i,j} a_{ij}(x) \{D_i [\eta(u - k)] D_j [\eta(u - k)] - (u - k)^2 D_i \eta D_j \eta\} \, dx = 0,$$

then by (9),

$$(13) \quad \begin{aligned} \mu_1 \int_{A(k)} |D[\eta(u - k)]|^2 \, dx &\leq \int_{A(k)} \sum_{i,j} a_{ij}(x) \{D_i [\eta(u - k)] D_j [\eta(u - k)]\} \, dx \\ &= \int_{A(k)} \sum_{i,j} (u - k)^2 a_{ij}(x) D_i \eta D_j \eta \, dx \\ &\leq \mu_2 \int_{A(k)} (u - k)^2 |D\eta|^2 \, dx. \end{aligned}$$

Making for  $\eta$  the following choice

$$\eta(x) = 1, \quad \forall x \in B(\varrho_1),$$

$$\eta(x) = 0, \quad \forall x \in \Omega - B(\varrho_2),$$

$$\eta(x) = \frac{\varrho_2 - |x - y|}{\varrho_2 - \varrho_1}, \quad \forall x \in B(\varrho_2) - B(\varrho_1),$$

we get

$$\mu_1 \int_{A(k, \varrho_1)} |D(u - k)|^2 dx \leq \frac{\mu_2}{(\varrho_2 - \varrho_1)^2} \int_{A(k, \varrho_2)} (u - k)^2 dx,$$

that is (11a).

STEP 3. Here we recall the definition of «perimeter of a set» following [3] and prove two inequalities which will be useful later.

Let  $E$  be a Lebesgue measurable set and  $A$  an open set in  $\mathbb{R}^n$ , we define the *perimeter of  $E$  in  $A$*  as

$$P(E, A) = \sup \left\{ \int_E \operatorname{div} g \, dx : g \in \operatorname{Lip}_c(A, \mathbb{R}^n), |g(x)| \leq 1 \right\}.$$

If  $A = \mathbb{R}^n$ , we shall write  $P(E)$  for  $P(E, \mathbb{R}^n)$ .

It is useful now to remember the global and local isoperimetric inequalities (see [3], [4], [5]).

For any Lebesgue measurable  $E \subset \mathbb{R}^n$

$$(14a) \quad \min \{ |E|, |\mathbb{R}^n - E| \}^{\frac{n-1}{n}} \leq (n\omega_n^{1/n})^{-1} P(E),$$

and (see [2])

$$(14b) \quad \min \{ |E \cap B(\varrho)|, |(\mathbb{R}^n - E) \cap B(\varrho)| \}^{\frac{n-1}{n}} \leq \beta_1(n) P(E, B(\varrho)),$$

with

$$\beta_1(n) = [(1 - 2^{-1/n}) n\omega_n^{1/n}]^{-1},$$

where  $\omega_n$  is the Lebesgue measure of the unit ball of  $\mathbb{R}^n$ .

The following Lemma establishes a connection between the gradient and the perimeter of level sets of Lipschitz functions.

LEMMA 3. For all  $f \in \text{Lip}(\mathbb{R}^n)$  and for all  $\varrho, t \in \mathbb{R}$  with  $\varrho > 0$  we have

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx \geq P(A(t), B(\varrho)).$$

PROOF. Given  $\varepsilon > 0$ , let us denote with  $f_\varepsilon(x)$  the function defined by

$$f_\varepsilon(x) = \varepsilon^{-1} \{ [t \vee f(x) \wedge (t + \varepsilon)] - t \}.$$

Let us observe that

$$(15) \quad |Df_\varepsilon(x)| = \varepsilon^{-1} |Df(x)|, \quad \text{for almost all } x \in [A(t, \varrho) - A(t + \varepsilon, \varrho)],$$

and

$$(16) \quad |Df_\varepsilon(x)| = 0, \quad \text{for almost all } x \in B(\varrho) \setminus [A(t, \varrho) - A(t + \varepsilon, \varrho)].$$

Hence we have

$$(17) \quad \int_{B(\varrho)} |Df_\varepsilon(x)| \, dx = \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx.$$

The functions  $f_\varepsilon(x) \in \text{Lip}(\mathbb{R}^n)$  for all  $\varepsilon > 0$ . Moreover it is obvious that

$$0 \leq f_\varepsilon(x) \leq 1, \quad \forall x \in \mathbb{R}^n$$

and

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = \chi_{A(t)}(x), \quad \forall x \in \mathbb{R}^n,$$

where  $\chi_{A(t)}$  is the characteristic function of  $A(t)$ .

Let  $g \in \text{Lip}_c(B(\varrho), \mathbb{R}^n)$  be such that  $|g(x)| \leq 1$ . Then

$$\int_{A(t)} \text{div } g \, dx = \int \chi_{A(t)} \text{div } g \, dx = \lim_{\varepsilon \rightarrow 0^+} \int f_\varepsilon(x) \text{div } g \, dx.$$

Since

$$\int f_\varepsilon(x) \text{div } g \, dx = - \int g Df_\varepsilon(x) \, dx \leq \int_{B(\varrho)} |Df_\varepsilon(x)| \, dx,$$



we get

$$\int_{A(t)} \operatorname{div} g \, dx \leq \liminf_{\varepsilon \rightarrow 0^+} \int_{B(\varrho)} |Df_\varepsilon| \, dx .$$

On taking the supremum over all such  $g$  and recalling (17), we have

$$\begin{aligned} P(A(t), B(\varrho)) &\leq \liminf_{\varepsilon \rightarrow 0^+} \int_{B(\varrho)} |Df_\varepsilon(x)| \, dx = \\ &= \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx . \quad \blacksquare \end{aligned}$$

Lemma 3 leads us to two important inequalities (see [2]).

**FIRST INEQUALITY.** *For all  $f \in \operatorname{Lip}(\mathbb{R}^n)$  and for all  $k, \lambda, \varrho \in \mathbb{R}$  with*

$$k < \lambda , \quad \varrho > 0 ,$$

*we have*

$$(18) \quad \beta_1 \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, dx \geq (\lambda - k) [\tau(k, \lambda, \varrho)]^{\frac{n-1}{n}} ,$$

*where  $\tau(k, \lambda, \varrho) = \min \{ |A(\lambda, \varrho)| , |B(\varrho) - A(k, \varrho)| \}$  and  $\beta_1$  is the constant in the isoperimetric inequality (14).*

**PROOF.** The isoperimetric inequality (14b) implies

$$\beta_1 P(A(t), B(\varrho)) \geq \min \{ |A(t, \varrho)| , |B(\varrho) - A(t, \varrho)| \}^{\frac{n-1}{n}} ,$$

then it follows from Lemma 3 that

$$\begin{aligned} (19) \quad &\beta_1 \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx \geq \\ &\geq \min \{ |A(t, \varrho)| , |B(\varrho) - A(t, \varrho)| \}^{\frac{n-1}{n}} . \end{aligned}$$

If

$$(20) \quad 2 |A(t, \varrho)| \leq |B(\varrho)| , \quad \forall t \geq k ,$$

the inequality (19) becomes

$$(21) \quad \beta_1 \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx \geq |A(t, \varrho)|^{\frac{n-1}{n}}, \quad \forall t \geq k.$$

The function  $F(t) = \int_{A(t, \varrho)} |Df(x)| \, dx$  is a non-increasing function of  $t$  and then

$$\begin{aligned} \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, dx &\geq - \int_k^\lambda \frac{dF(t)}{dt} \, dt = \\ &= \int_k^\lambda \left[ \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| \, dx \right] dt, \end{aligned}$$

for all  $k, \lambda \in \mathbb{R}$  with  $k < \lambda$ .

Using this inequality and integrating (21) with respect to  $t$  from  $k$  to  $\lambda$  ( $k < \lambda$ ), we get

$$\beta_1 \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, dx \geq \int_k^\lambda |A(t, \varrho)|^{\frac{n-1}{n}} \, dt.$$

Since  $|A(t, \varrho)|^{\frac{n-1}{n}}$  is a non-increasing function of  $t$ , we have

$$(22) \quad \beta_1 \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, dx \geq (\lambda - k) |A(\lambda, \varrho)|^{\frac{n-1}{n}} = (\lambda - k) [\tau(k, \lambda, \varrho)]^{\frac{n-1}{n}}.$$

If

$$(23) \quad 2 |A(t, \varrho)| \geq |B(\varrho)|, \quad \forall t \leq \lambda,$$

considering the non-decreasing function  $|B(\varrho) - A(t, \varrho)|^{\frac{n-1}{n}}$  and proceeding as before, we obtain

$$(24) \quad \beta_1 \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, dx \geq (\lambda - k) |B(\varrho) - A(k, \varrho)|^{\frac{n-1}{n}} = (\lambda - k) [\tau(k, \lambda, \varrho)]^{\frac{n-1}{n}}.$$

If neither inequality (20) nor inequality (23) are satisfied then there must exist  $\bar{t} \in [k, \lambda]$  such that

$$2 |A(t, \varrho)| \leq B(\varrho), \quad \forall t \geq \bar{t}$$

and

$$2 |A(t, \varrho)| > B(\varrho), \quad \forall t < \bar{t}.$$

Applying (22) over the interval  $[\bar{t}, \lambda]$  and (24) over  $[k, \bar{t}]$ , we obtain again (18). ■

SECOND INEQUALITY. For all  $f \in \text{Lip}(\mathbb{R}^n)$  and for all  $k, \varrho \in \mathbb{R}$  with

$$(25) \quad \varrho > 0, \quad 0 < 2 |A(k, \varrho)| \leq |B(\varrho)|,$$

we have

$$(26) \quad \int_{A(k, \varrho)} (f(x) - k)^2 dx \leq \beta_2 |A(k, \varrho)|^{2/n} \int_{A(k, \varrho)} |Df(x)|^2 dx,$$

where  $\beta_2 = 4\beta_1^2$ .

PROOF. Lemma 3 and the isoperimetric inequality (14b) imply

$$(27) \quad \beta_1 \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| dx \geq \\ \geq \min \{ |B(\varrho) - A(t, \varrho)|, |A(t, \varrho)| \}^{\frac{n-1}{n}}.$$

Since

$$|A(t, \varrho)| \leq |A(k, \varrho)|, \quad \forall t \geq k,$$

we have by (25) and (27)

$$\beta_1 \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| dx \geq |A(t, \varrho)|^{\frac{n-1}{n}}, \quad \forall t \geq k,$$

so that

$$\beta_1 \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t + \varepsilon, \varrho)} |Df(x)| dx \geq |A(k, \varrho)|^{-1/n} |A(t, \varrho)|, \quad \forall t \geq k.$$

If we integrate this last inequality with respect to  $t$  over the interval  $(k, +\infty)$ , we get

$$\beta_1 |A(k, \varrho)|^{1/n} \int_{A(k, \varrho)} |Df(x)| dx \geq \int_k^{+\infty} |A(t, \varrho)| dt,$$

that is,

$$(28) \quad \beta_1 |A(k, \varrho)|^{1/n} \int_{A(k, \varrho)} |D(f(x) - k)| \delta x \geq \int_{A(k, \varrho)} (f(x) - k) dx .$$

Being

$$\begin{aligned} A(k, \varrho) &= \{x \in B(\varrho) : f(x) > k\} = \{x \in B(\varrho) : (f(x) - k) > 0\} \\ &= \{x \in B(\varrho) : [(f(x) - k) \vee 0]^2 > 0\}, \end{aligned}$$

we can apply (28) to the function  $[(f(x) - k) \vee 0]^2$ , obtaining

$$\beta_1 |A(k, \varrho)|^{1/n} \int_{A(k, \varrho)} |D(f(x) - k)^2| dx \geq \int_{A(k, \varrho)} (f(x) - k)^2 dx .$$

Since

$$[D(f(x) - k)]^2 = 2(f(x) - k) Df(x),$$

we have

$$\beta_1 |A(k, \varrho)|^{1/n} \int_{A(k, \varrho)} 2(f(x) - k) |Df(x)| dx \geq \int_{A(k, \varrho)} (f(x) - k)^2 dx .$$

Now, using Schwarz-Hölder inequality, we get

$$\begin{aligned} &\int_{A(k, \varrho)} (f(x) - k)^2 dx \leq \\ &\leq 2\beta_1 |A(k, \varrho)|^{1/n} \left[ \int_{A(k, \varrho)} |Df(x)|^2 dx \right]^{1/2} \left[ \int_{A(k, \varrho)} (f(x) - k)^2 dx \right]^{1/2}, \end{aligned}$$

then

$$\left[ \int_{A(k, \varrho)} (f(x) - k)^2 dx \right]^{1/2} \leq 2\beta_1 |A(k, \varrho)|^{1/n} \left[ \int_{A(k, \varrho)} |Df(x)|^2 dx \right]^{1/2} .$$

By taking the square, we can conclude

$$\int_{A(k, \varrho)} (f(x) - k)^2 dx \leq \beta_2 |A(k, \varrho)|^{2/n} \int_{A(k, \varrho)} |Df(x)|^2 dx ,$$

where  $\beta_2 = 4\beta_1^2$ . ■

STEP 4. What we are going to prove now is an estimate for the oscillation of the differential quotients  $u$ , directly derived from (11a) and (11b).

LEMMA 4. For all  $\sigma \in (0, 1)$  we define

$$(29) \quad \theta = \theta(\sigma) = \min \left\{ \frac{(1 - \sigma)^n \omega_n}{2}, \left[ \frac{\sigma^2 2^{-n-2}}{1 + \gamma + \beta_2} \right]^n \right\},$$

and

$$(30) \quad c^2 = \varrho^{-n} \theta^{-1} \int_{A(k, \varrho)} (u(x) - k)^2 dx,$$

where  $\beta_2$  is the constant in (26) and  $\gamma$  is the constant in Caccioppoli inequalities. Then if  $k \in \mathbb{R}$  and

$$B(\varrho) \subset \Omega, \quad |A(k, \varrho)| \leq \varrho^n \theta,$$

we have

$$(31) \quad \sup_{B(\varrho - \sigma\varrho)} |u| \leq k + \sigma c.$$

PROOF. Putting

$$\varrho_h = \varrho - \sigma\varrho + 2^{-h} \sigma\varrho, \quad k_h = k + \sigma c - 2^{-h} \sigma c,$$

for all integer  $h \geq 0$ , we have

$$(32) \quad \varrho = \varrho_0 > \varrho_1 > \dots > \varrho_h > \dots > (1 - \sigma) \varrho$$

and

$$(33) \quad k = k_0 < k_1 < \dots < k_h < \dots < k + \sigma c.$$

The definition of  $\theta$  implies

$$\theta \leq \frac{1}{2} \omega_n (1 - \sigma)^n,$$

then

$$|A(k_h, \varrho_{h+1})| \leq \varrho_h^n \theta \leq \frac{1}{2} \omega_n \varrho_{h+1}^n, \quad \forall h \geq 0,$$

that is the condition for using inequality (26),

$$\int_{A(k_h, \varrho_{h+1})} (u(x) - k_h)^2 dx \leq \beta_2 |A(k_h, \varrho_{h+1})|^{2/n} \int_{A(k_h, \varrho_{h+1})} |Du|^2 dx, \quad \forall h \geq 0.$$

Writing (11a) for  $\varrho_1 = \varrho_{h+1} < \varrho_2 = \varrho_h$ ,  $k = k_h$ , we have

$$\gamma 4^{h+1} \int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 dx \geq \sigma^2 \varrho^2 \int_{A(k_h, \varrho_{h+1})} |Du|^2 dx, \quad h \geq 0.$$

Combining these two last inequalities, we obtain

$$(34) \quad \int_{A(k_h, \varrho_{h+1})} (u(x) - k_h)^2 dx \leq \beta_2 |A(k_h, \varrho_{h+1})|^{2/n} \gamma 4^{h+1} \sigma^{-2} \varrho^{-2} \cdot \int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 dx.$$

Noting that

$$\begin{aligned} \int_{A(k_h, \varrho_{h+1})} (u(x) - k_h)^2 dx &\geq \int_{A(k_{h+1}, \varrho_{h+1})} (u(x) - k_h)^2 dx \geq \\ &\geq \int_{A(k_{h+1}, \varrho_{h+1})} (u(x) - k_{h+1})^2 dx \end{aligned}$$

and

$$(u(x) - k_h)^2 \geq (k_{h+1} - k_h)^2 = \sigma^2 c^2 4^{-(h+1)}, \quad \forall x \in A(k_{h+1}, \varrho_{h+1}),$$

from (34) we obtain the following two inequalities

$$\begin{aligned} \sigma^2 c^2 4^{-(h+1)} |A(k_{h+1}, \varrho_{h+1})| &\leq \frac{\beta_2 |A(k_h, \varrho_h)|^{2/n} \gamma 4^{h+1}}{\sigma^2 \varrho^2} \int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 dx, \\ \int_{A(k_{h+1}, \varrho_{h+1})} (u(x) - k_{h+1})^2 dx &\leq \frac{\beta_2 |A(k_h, \varrho_h)|^{2/n} \gamma 4^{h+1}}{\sigma^2 \varrho^2} \int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 dx. \end{aligned}$$

Thanks to these and the definitions of  $\theta$  and  $c$ , it is easy to prove, by in-

duction on  $h$ , that

$$(35) \quad |A(k_h, \varrho_h)| < \varrho^n \theta 2^{-2nh}, \quad \forall h \geq 0,$$

$$(36) \quad \int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 dx \leq \varrho^n \theta c^2 2^{-2nh}, \quad \forall h \geq 0.$$

Then (35) implies

$$(37) \quad |A(k + \sigma c, \varrho - \sigma \varrho)| = 0,$$

that is

$$(38) \quad \sup_{B(\varrho - \sigma \varrho)} u \leq k + \sigma c.$$

Considering  $-u$  and proceeding as before, we obtain (31). ■

STEP 5. We can study now the behaviour of the oscillation of  $u$  on  $B(\varrho)$  as  $\varrho \rightarrow 0$ .

LEMMA 5. *There exists a number  $\eta = \eta(n, \gamma) > 0$  such that, for all  $\varrho \in \mathbb{R}$  with*

$$0 < 4\varrho < \text{dist}(y, (\mathbb{R}^n - \Omega)),$$

*we have*

$$\text{osc}(u, B(\varrho)) \leq (1 - \eta) \text{osc}(u, B(4\varrho)),$$

*where*

$$\text{osc}(u, B(\varrho)) = \sup_{B(\varrho)} u - \inf_{B(\varrho)} u.$$

PROOF. Let us put

$$\omega = \text{osc}(u, B(4\varrho)) = \sup_{B(4\varrho)} u - \inf_{B(4\varrho)} u = \mu_1 - \mu_2,$$

and

$$\bar{\mu} = \frac{\mu_1 + \mu_2}{2}.$$

At least one of the following two inequalities must be true:

$$(39a) \quad 2 |A(\bar{\mu}, 2\varrho) | < |B(2\varrho) | ,$$

$$(39b) \quad 2 |B(2\varrho) - A(\bar{\mu}, 2\varrho) | < |B(2\varrho) | .$$

Assume that (39a) is true. If (39b) is satisfied we would use the same arguments for the function  $-u$ .

For all  $\lambda \leq \frac{\omega}{4}$ , we put

$$D(\lambda) = A(\mu_1 - 2\lambda, 2\varrho) - A(\mu_1 - \lambda, 2\varrho) .$$

From the Schwarz inequality we have

$$(40) \quad \left( \int_{D(\lambda)} |Du| dx \right)^2 \leq |D(\lambda)| \int_{D(\lambda)} |Du|^2 dx .$$

Noting that

$$|A(\mu_1 - \lambda, 2\varrho) | \leq |A(\bar{\mu}, 2\varrho) | \leq |B(2\varrho) - A(\mu_1 - 2\lambda, 2\varrho) | ,$$

and recalling (18), we obtain

$$(41) \quad \left( \int_{D(\lambda)} |Du| dx \right)^2 \geq \lambda^2 \beta_1^{-2} |A(\mu_1 - \lambda, 2\varrho) |^{\frac{2n-2}{n}} .$$

Since  $D(\lambda) \subset A(\mu_1 - 2\lambda, 2\varrho)$ , we have

$$\int_{D(\lambda)} |Du|^2 dx \leq \int_{A(\mu_1 - 2\lambda, 2\varrho)} |Du|^2 dx ,$$

then applying (11a) with  $\varrho_1 = 2\varrho < \varrho_2 = 4\varrho$  and  $k = \mu_1 - 2\lambda$ , we get

$$(42) \quad \int_{D(\lambda)} |Du|^2 dx \leq \gamma 4^{-1} \varrho^{-2} \int_{A(\mu_1 - 2\lambda, 4\varrho)} (u - \mu_1 + 2\lambda)^2 dx \leq \gamma \varrho^{-2} \lambda^2 |B(4\varrho)| .$$

Combining this last inequality with (40) and (41), we obtain

$$(43) \quad |D(\lambda)| \geq \frac{\varrho^2}{\beta_1^2 \gamma} |A(\mu_1 - \lambda, 2\varrho) |^{\frac{2n-2}{n}} |B(4\varrho) |^{-1}, \quad \forall \lambda \leq \frac{\omega}{4} .$$

Now, let  $h = h(n, \gamma) \in \mathbb{N}$  be the first integer such that

$$(44) \quad h \geq \frac{\gamma \beta_1^2 |B(4\varrho) | |B(2\varrho) | \left[ \theta \left( \frac{1}{2} \right) \right]^{\frac{2-2n}{n}}}{\varrho^{2n}} ,$$



and put

$$(45) \quad \eta = \eta(n, \gamma) = 2^{-(h+2)}.$$

For all  $\lambda = 2^m \eta \omega$  with  $m = 1, \dots, h$  we have  $\lambda \leq \frac{\omega}{4}$ , then from (43) it follows

$$\begin{aligned} |D(2^m \eta \omega)| &\geq \frac{\varrho^2}{\beta_1^2 \gamma} |A(\mu_1 - 2^m \eta \omega, 2\varrho)|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1} \\ &\geq \frac{\varrho^2}{\beta_1^2 \gamma} |A(\mu_1 - 2\eta \omega, 2\varrho)|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1}, \end{aligned}$$

for all  $m = 1, \dots, h$ .

To simplify the notation, let us put

$$k = \mu_1 - 2\eta \omega.$$

Being the sets  $\{D(2^m \eta \omega)\}_{1 \leq m \leq h}$  disjoint and contained in  $B(2\varrho)$ , we have

$$|B(2\varrho)| \geq \sum_{m=1}^h |D(2^m \eta \omega)| \geq h \frac{\varrho^2}{\beta_1^2 \gamma} |A(k, 2\varrho)|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1},$$

then, recalling (44), we obtain

$$|B(2\varrho)| \geq |B(2\varrho)| \varrho^{-2n+2} \left[ \theta \left( \frac{1}{2} \right) \right]^{\frac{2-2n}{n}} |A(k, 2\varrho)|^{\frac{2n-2}{n}},$$

from which

$$|A(k, 2\varrho)| \leq \theta \left( \frac{1}{2} \right) \varrho^n < \theta \left( \frac{1}{2} \right) (2\varrho)^n.$$

Being

$$(u - k)^2 \leq (\mu_1 - k)^2 = (2\eta \omega)^2, \quad \forall x \in A(k, 2\varrho),$$

we have

$$(46) \quad \int_{A(k, 2\varrho)} (u(x) - k)^2 dx \leq (2\eta \omega)^2 |A(k, 2\varrho)| < (2\eta \omega)^2 \theta \left( \frac{1}{2} \right) \varrho^n.$$

It follows then from Lemma 4

$$\sup_{B(2\varrho - \sigma\varrho)} |u| \leq k + \sigma c ,$$

where from (46),

$$c^2 = (2\varrho)^{-n} \left[ \theta \left( \frac{1}{2} \right) \right]^{-1} \left[ \int_{A(k, 2\varrho)} (u(x) - k)^2 dx \right] < (2\eta\omega)^2 .$$

Since

$$k + \sigma c \leq \mu_1 - \eta\omega \quad \text{and} \quad 2\varrho - \sigma\varrho = \frac{3}{2}\varrho > \varrho ,$$

we have

$$\sup_{B(\varrho)} |u| \leq \mu_1 - \eta\omega ,$$

then

$$\begin{aligned} \text{osc}(u, B(\varrho)) &\leq \sup_{B(\varrho)} u - \inf_{B(\varrho)} u \\ &\leq \sup_{B(\varrho)} u - \inf_{B(4\varrho)} u \\ &\leq \mu_1 - \eta\omega - \mu_2 = (1 - \eta) \text{osc}(u, B(4\varrho)) . \quad \blacksquare \end{aligned}$$

STEP 6. We are finally able to prove the following:

LEMMA 6. *There exists a number  $a = a(n, \gamma) \in (0, 1)$  such that, for all  $d \in \mathbb{R}$  with*

$$0 < 2d < \text{dist}(y, (\mathbb{R}^n - \Omega)) ,$$

we have

$$(47) \quad |u(x_1) - u(x_2)| \leq 2K \left( \frac{4|x_1 - x_2|}{d} \right)^a, \quad \forall x_1, x_2 \in B_y \left( \frac{d}{2} \right) ,$$

where  $K$  is the Lipschitz constant of  $w$ .

PROOF. Let  $x_1, x_2 \in B_y \left( \frac{d}{2} \right)$  and consider the ball  $B(\varrho) = B_{x_1}(\varrho)$  of

radius  $\varrho = |x_2 - x_1|$  centered in  $x_1$ . Let  $m$  be an integer such that

$$(48) \quad 4^{-m-1}d < \varrho \leq 4^{-m}d,$$

then Lemma 5 implies

$$(49) \quad \begin{aligned} \operatorname{osc}(u, B_{x_1}(\varrho)) &\leq \operatorname{osc}(u, B_{x_1}(4^{-m}d)) \leq (1-\eta)^m \operatorname{osc}(u, B_{x_1}(d)) \\ &\leq (1-\eta)^m 2K. \end{aligned}$$

From (48) we get

$$-m-1 < \log_4 \varrho - \log_4 d,$$

that is

$$m > -1 - \log_4 \frac{\varrho}{d}.$$

Hence, recalling (49), we have

$$\operatorname{osc}(u, B_{x_1}(\varrho)) \leq (1-\eta)^{-1 - \log_4 \frac{\varrho}{d}} 2K \leq \left(\frac{4\varrho}{d}\right)^{-\log_4(1-\eta)} 2K.$$

Put  $a = -\log_4(1-\eta)$ , where  $\eta$  is the constant defined in Lemma 5 from (45), we obtain

$$\operatorname{osc}(u, B_{x_1}(\varrho)) \leq 2K \left(\frac{4\varrho}{d}\right)^a,$$

from which it follows (47). ■

We can conclude that the differential quotients

$$u(x) = u(x, t, e) = \frac{w(x+te) - w(x)}{t}$$

are Hölder-continuous in every ball  $B_y\left(\frac{d}{2}\right) \subset \Omega$  with  $0 < 2d < \operatorname{dist}(y, (\mathbb{R}^n - \Omega))$ , for all  $e \in S^{n-1}$  and  $t \in \mathbb{R} - \{0\}$  sufficiently small. Moreover the Hölder constant and exponent are independent of  $t$  and  $e$ . Therefore from (47), letting  $t \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow 0} |u(x_1, t, e) - u(x_2, t, e)| = |D_e u(x_1) - D_e u(x_2)| \leq 2K \left(\frac{4|x_1 - x_2|}{d}\right)^a,$$

for all  $x_1, x_2 \in B_y\left(\frac{d}{2}\right)$  and for all  $e \in S^{n-1}$ .

This completes the proof of Theorem 1.

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Manoscritto pervenuto in redazione l'8 aprile 1999