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ALIREZA ABDOLLAHI

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## Finitely Generated Soluble Groups with an Engel Condition on Infinite Subsets.

ALIREZA ABDOLLAHI (\*)

ABSTRACT - In this note, we prove that, in every finitely generated soluble group  $G$ ,  $G/Z_2(G)$  is finite if and only if in every infinite subset  $X$  of  $G$  there exist different  $x, y$  such that  $[x, y, y] = 1$ .

B. H. Neumann proved in [9] that a group  $G$  is centre-by-finite if and only if every infinite subset  $X$  of  $G$  contains two different commuting elements. This answered a question posed by Paul Erdős. Extensions of problems of this type are studied in [1], [4], [5], [8] and [11].

We denote by  $E(\infty)$  (respectively,  $N(\infty)$ ) the class of groups  $G$  such that, every infinite subset  $X$  of  $G$ , contains different elements  $x$  and  $y \in X$  such that  $[x, {}_k y] = 1$  (respectively,  $\langle x, y \rangle$  is nilpotent of class at most  $k$ ) for some  $k = k(x, y) \geq 1$ . If the integer  $k$  is the same for all infinite subsets of  $G$ , we say that  $G$  is in the class  $E_k(\infty)$  (respectively,  $N_k(\infty)$ ).

It is easy to see that the above classes are closed with respect to forming subgroups and homomorphic images.

In [6] J. C. Lennox and J. Wiegold studied the class  $N(\infty)$  and proved that a finitely generated soluble group is in  $N(\infty)$  if and only if it is finite-by-nilpotent.

Also, in [7] P. Longobardi and M. Maj studied the class  $E(\infty)$  and proved that a finitely generated soluble group is in  $E(\infty)$  if and only if it is finite-by-nilpotent. Moreover, they proved that a finitely generated soluble group  $G$  is in  $E_2(\infty)$  if and only if  $G/R(G)$  is finite, where  $R(G)$  is

(\*) Indirizzo dell'A.: Department of Mathematics, University of Isfahan, Isfahan-Iran.

the characteristic subgroup of  $G$  consisting of all right 2-Engel elements of  $G$ .

In [2] and [3] C. Delizia proved that, a finitely generated soluble (or residually finite) group  $G$  is in  $N_2(\infty)$  if and only if  $G/Z_2(G)$  is finite.

Here we prove the following:

**THEOREM.** *Let  $G$  be a finitely generated soluble group. Then  $G \in E_2(\infty)$  if and only if  $G/Z_2(G)$  is finite.*

**PROOF.** Let  $G$  be a finitely generated soluble  $E_2(\infty)$ -group. By Theorem 1 of [7],  $G$  contains a finite normal subgroup  $N$  such that  $G/N$  is torsion-free nilpotent. Now by Theorem 2 of [7],  $R(G)$  has finite index in  $G$ , where  $R(G) = \{a \in G \mid [a, x, x] = 1 \text{ for all } x \in G\}$ , thus  $R(G)N/N$  has finite index in  $G/N$ . So  $R(G)N/N$  is a torsion-free 2-Engel group, therefore by Theorem 7.14 in [10],  $R(G)N/N$  is nilpotent group of class at most 2. Since  $G/N$  is torsion-free nilpotent and  $R(G)N/N$  is of finite index in  $G/N$ , thus  $G/N$  is nilpotent group of class at most 2. We note that  $G$  is residually finite since it is a finitely generated nilpotent-by-finite group. Thus it contains a normal subgroup  $L$  of finite index such that  $L \cap N = 1$ . Now  $[L, G, G] \leq N \cap L = 1$ . Then  $L \leq Z_2(G)$  as required to be shown.

Conversely, if  $G/Z_2(G)$  is finite and  $\{x_i : i \in I\}$  is an infinite set of elements of  $G$ , there exist  $i, j \in I$  with  $i \neq j$  such that  $x_i \equiv x_j \pmod{Z_2(G)}$ . Therefore  $x_i x_j^{-1} = z \in Z_2(G)$ , so  $\langle x_i, x_j \rangle = \langle z, x_j \rangle$  is nilpotent of class at most 2. Hence  $G \in N_2(\infty) \subset E_2(\infty)$ .

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