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WALTER DAMBROSIO (*)

ABSTRACT - In this paper we are concerned with boundary value problems like

$$\begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) + g(t, u, u') = 0 \\ u(0) = 0 = u(\pi) \end{cases}$$

where $u \mapsto (\phi_p(u'))'$ is the one-dimensional p-laplacian operator. By means of an application of a multi-parameter abstract bifurcation theorem, we prove the existence of global bifurcating sets of solutions to the problem considered.

1. Introduction.

In this paper we present a bifurcation result for a Dirichlet boundary value problem associated to a second order differential equation involving the p-laplacian operator ($p > 1$). The study of bifurcation theory has been widely faced in the past decades, motivated e.g. by many applications to mathematical physics. We refer for instance to the book [4] for a more complete discussion on this topic.

In a first approach to bifurcation problems, the existence of solution pairs $(\lambda, u) \in \boldsymbol{R} \times X$ (X is a Banach space) to an abstract equation of the form

$$u = G(\lambda, u) \tag{1.1}$$

was studied; more precisely, the existence of the *trivial line* of solutions

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$(\lambda, 0)$ is assumed and the problem is to find and characterize the values $\lambda_0 \in \boldsymbol{R}$ (which will be called the «bifurcation points») such that in *any* neiborhood of $(\lambda_0, 0)$ there exists a solution (λ, u) of (1.1) with $u \neq 0$. In particular, for an equation of the form

$$u = \lambda Lu + H(\lambda, u), \tag{1.2}$$

where $L : X \to X$ is a linear compact operator, $H : \boldsymbol{R} \times X \to X$ is compact and $H(\lambda, u) = o(|u|)$ for $u \to 0$, uniformly for λ in bounded sets, the following fundamental result has been proved by P. H. Rabinowitz (see [19,20]):

THEOREM 1.1 [20], Th. 1.10. *If L and H satisfy the above assumptions, then for every λ_0, eigenvalue of odd multiplicity of the linearized equation*

$$u = \lambda Lu, \tag{1.3}$$

the point $(\lambda_0, 0)$ is a bifurcation point for (1.2). *Moreover, from $(\lambda_0, 0)$ bifurcates a connected branch of solutions to* (1.2) *which either is unbounded or contains a point of the form $(\mu_0, 0)$ where $\mu_0 \neq \lambda_0$ is an eigenvalue of* (1.3).

An important application of Theorem 1. 1, given in [11], is the study of solutions to the boundary value problem

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_p(u) + g(t, u, u') = 0, & \lambda \in \boldsymbol{R} \\ u(0) = 0 = u(\pi), \end{cases} \tag{1.4}$$

where, for every $s \in \boldsymbol{R}$ and $p > 1$, $\phi_p(s) = |s|^{p-2}s$ (hence $-(\phi_p(u'))'$ is the one-dimensional p-laplacian operator) and $g : [0, \pi] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is a continuous function such that

$$g(t, \xi, \eta) = o((\phi_p(\xi) + \phi_p(\eta))^{1/p}) \text{ near } (\xi, \eta) = (0, 0) \tag{1.5}$$

$$\text{uniformly in } t \in [0, \pi].$$

Let us denote by $\lambda_{k,p}$ the eigenvalues of the operator $-(\phi_p(u'))'$ with boundary conditions $u(0) = 0 = u(\pi)$. The following result has been proved ([11]):

THEOREM 1.2 [11], Th. 2. *If g satisfies* (1.5), *then for every $k \in \boldsymbol{N}$ there exists an unbounded connected set $\mathcal{C}_k$ of solutions (λ, u) to* (1.4), *with u having exactly k zeros in $[0, \pi)$, bifurcating from the point $(\lambda_{k,p}, 0)$.*

We recall that in the case $p = 2$ Theorem 1 has been proved in the celebrated paper [20] by P. H. Rabinowitz.

We also observe that, in order to obtain Theorem 1.2 from Theorem 1.1, P. Drábek first developed an eigenvalue theory for p-laplacian operators [12], on the lines of the linear theory for the classical case $p = 2$.

Starting from Theorem 1.2, in particular for the case $p = 2$, a lot of authors studied the existence of solutions to various boundary value problems by means of bifurcation techniques; among others we quote the interesting papers by M. J. Esteban [14] and by A. Ambrosetti, J. Garcia-Azorero and I. Peral [2].

As it was done e.g. in [14], Theorem 1.1 applies to problems associated to an equation like $(\phi_p(u'))' + f(t, u, u') = 0$ when f possesses a linearization near $u = 0$, $u' = 0$, i.e. f is differentiable in $(t, 0, 0)$ for every $t \in [0, \pi]$. This remark motivates the result we obtain in this paper: indeed, let us denote $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. If we simply assume $f(t, u, u') = \mu\phi_p(u^+) - \nu\phi_p(u^-) + g(t, u, u')$, with g satisfying (1.6), then the BVP

$$(1.6) \qquad \begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) + g(t, u, u') = 0 \\ u(0) = 0 = u(\pi) \end{cases}$$

cannot be studied through the previous theorem. Nevertheless, a bifurcating result for (1.6) still holds (see Theorem 2.7 in Section 2). Its proof is based on the fact that we can replace the eigenvalues $\lambda_{k,p}$ in Theorem 1.2 by suitable values $(\mu, \nu) \in \boldsymbol{R}^2$ belonging to the set $\mathcal{E}$ usually called *Fučik spectrum* (see [16]): more precisely, $\mathcal{E}$ is the set of those pairs $(\mu, \nu) \in \boldsymbol{R}^2$ such that the autonomous BVP

$$(1.7) \qquad \begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) = 0 \\ u(0) = 0 = u(\pi) \end{cases}$$

admits a nontrivial solution. On the lines of Theorem 1.2 in the differentiable case, once some «eigenvalues» have been introduced, a bifurcation result is required. In the new situation, we need an abstract theorem relative to a multi-dimensional bifurcation; this kind of questions has been intensively studied (see e.g. [15, 18] and the book [4]) and a global result (see Theorem 2.2 in Section 2) for an equation as (1.1) (with $\lambda \in \boldsymbol{R}^n$) has been obtained.

Motivated by the previous remarks, in Section 2 we prove a global bifurcation result (Theorem 2.7) for (1.6); the theorem of Rabinowitz quoted above is a particular case of our Theorem 2.7. To the best of our knowledge, only few results are concerned with bifurcation from the Fučik spectrum; the most significant, in our opinion, has been proved by P. Drábek and M. Kučera in [13] (see also the survey lecture note [12]). Indeed, in [13] it is proved that every point of the Fučik spectrum is a «bifurcation point» for an equation of the form $(\phi_p(u'))' + \mu(t)\, \phi_p(u^+) - \nu(t)\, \phi_p(u^-) = 0$, where the coefficients μ and ν depend on the time variable. However, we stress the fact that this is not a bifurcation result for (1.6) and that also for the equation considered in [13] no description of the global behaviour of the bifurcating set is given.

Next, in Section 3 we give an application of the results of Section 2 to a boundary value problem as (1.6) when g does not depend on the derivative u' and satisfies some growth conditions at infinity. We are mainly motivated by previous works ([7, 9]) in which multiplicity results for Dirichlet or Neumann problems have been obtained; the basic assumption is a superlinear asymmetric growth at infinity of the nonlinearity. In this case,a more complete description of the global bifurcating set can be given (see Lemma 3.1 and Proposition 3.2).

In what follows, for any Banach space X, for any linear compact operator $L : X \to X$ and for any subset $\Omega \subset X$ we will denote by $\deg_{LS}(I - L, \Omega, 0)$ the Leray-Schauder degree of $I - L$ (whenever it is defined) and with $\mathrm{Ind}(L)$ the Leray-Schauder index of $I - L$ near the (isolated) zero 0. We denote by $W_0^{1,p}(0, \pi)$ the usual Sobolev space of the functions $u \in W^{1,p}([0, \pi])$ such that $u(0) = 0 = u(\pi)$; similarly, $C_0^1([0, \pi])$ is the space of functions $u \in C^1([0, \pi])$ such that $u(0) = 0 = u(\pi)$. Moreover, the space $C^1([0, \pi])$ of the continuously differentiable real functions u on $[0, \pi]$ will be equipped with the norm

$$\|u\|_1 = \max\left\{\sqrt{|u(t)|^2 + |u'(t)|^2} : t \in [0, \pi]\right\}.$$

Finally, as it was done in this Introduction, we denote by $\lambda_{k,p}$ the eigenvalues of the one-dimensional p-laplacian operator with Dirichlet boundary conditions in $(0, \pi)$; if we set

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{dz}{(1-z^p)(1/p)}$$

(observe that $\pi_2 = \pi$), then, from [10], we have $\lambda_{k,p} = (k\pi_p/\pi)^p$.

2. Bifurcation from the Fučik spectrum.

We start this section by recalling a multiparameter bifurcation theorem, which can be found in different versions in several papers (see for instance [15, 18] and the book [4]). We confine ourselves to the case of a bifurcation parameter λ varying in $\boldsymbol{R}^2$; we refer to the quoted papers for the general case of $\lambda \in \boldsymbol{R}^n$.

We now introduce the notation to be used in the next results.

Let X be a real Banach space with the norm $\|\cdot\|$. For every $\lambda \in \boldsymbol{R}^2$, let $f: \boldsymbol{R}^2 \times X \to X$ be a compact map of the form $f = I_X - F$ for some compact mapping $F: \boldsymbol{R}^2 \times X \to X$; for each $\lambda \in [0, 1]$, we write $f_\lambda = f(\lambda, \cdot)$ and we assume that $f_\lambda(0) = 0$.

We are interested in the study of the solution pairs $(\lambda, u) \in \boldsymbol{R}^2 \times X$ of the equation

$$f(\lambda, u) = 0\,. \tag{2.1}$$

We observe that equation (2.1) is satisfied by all the pairs of the form $(\lambda, 0)$ for every $\lambda \in \boldsymbol{R}^2$; we refer to the solutions of this form as the trivial solutions. We will denote by S^* the closure of the set of nontrivial solution pairs of equation (2.1). We give the well-known definition:

DEFINITION 2.1. *A pair $(\lambda, 0)$ belonging to the set S^* is called a bifurcation point for equation* (2.1) *if in any neighborhood of $(\lambda, 0)$ in $\boldsymbol{R}^2 \times X$ there is at least a nontrivial solution of* (2.1). *The set of bifurcation points of* (2.11) *is denoted by $\mathcal{F}$.*

The following theorem give conditions to obtain the existence of bifurcating sets of nontrivial solutions to (2.1):

THEOREM 2.2 ([15], Th. 2.4). *For every $\lambda \in \boldsymbol{R}^2$, let $f: \boldsymbol{R}^2 \times X \to X$ be a compact map. Let $\Gamma \subset \boldsymbol{R}^2 \times \{0\}$ be described by $\Gamma = h^{-1}(0)$, where $h: \boldsymbol{R}^2 \to \boldsymbol{R}$ is continuously differentiable and has 0 as a regular value. Suppose that $\alpha \in \boldsymbol{R}^2$ and $\beta \in \boldsymbol{R}^2$ lay in the same component of Γ, that* *neither α nor β are bifurcation points of f and that*

$$\operatorname{Ind}(f_\alpha, 0) \neq \operatorname{Ind}(f_\beta, 0). \tag{2.2}$$

Then, there exists a closed connected surface $\mathcal{C} \subset \mathcal{S}^*$ such that $\mathcal{C} \cap \cap [\alpha, \beta] \neq \emptyset$, where $[\alpha, \beta]$ denotes any arc of Γ between α and β. Moreover, one of the following alternatives is satisfied:

(a) *$\mathcal{C}$ is unbounded,*

(b) $\overline{\mathcal{C}} \cap (\{\Gamma \setminus [\alpha, \beta]\}) \neq \emptyset$.

As already observed in [15], in the formulation of Theorem 2.2 we can recognize the global bifurcation result (Theorem 1.1 of the Introduction) of P. H. Rabinowitz (cf. [19, 20]), relative to the one-dimensional case. We stress the fact that in Theorem 2.2 no regularity (as e.g. differentiability) of the functions f_λ in zero is required.

Theorem 2.2 and some generalizations (see for instance the paper by J. Ize [18]) have been applied in various situations: we mention for instance the results on elasticity obtained by S. Stuart Antman [4] (see also the paper by J. C. Alexander and S. Stuart Antman [1]), the Sturm-Liouville bifurcation theory for systems of second order equations developed, among others, by R. S. Cantrell [6] and S. C. Welsh [21] and a bifurcation result for boundary value problems depending on two real parameters due to P. A. Binding and Y. X. Huang [5].

Now, we turn our attention to the case when f can be written in the form $f(\lambda, u) = u - L(\lambda)\, u - H(\lambda, u)$, where, for every $\lambda \in \boldsymbol{R}^2$, $L(\lambda)$: $X \to X$ is a *positively homogeneous* compact operator and $H : \boldsymbol{R}^2 \times X \to X$ is a compact perturbation satisfying the condition

$$H(\lambda, u) = o(\|u\|) \quad \text{for } u \to 0, \quad \text{uniformly for } \lambda \text{ in bounded sets}. \tag{2.3}$$

We observe that in this case equation 2.1 becomes

$$u = L(\lambda)\, u + H(\lambda, u). \tag{2.4}$$

Moreover, we notice the fact that we do not assume, as for instance in the one-dimensional case considered in [19, 20], that the term $L(\lambda)$ is linear.

Now, according to the notation introduced in [4], a number $\lambda \in \boldsymbol{R}^2$ is called *eigenvalue* of the equation

$$v = L(\lambda)\, v \tag{2.5}$$

if (2.5) has a nontrivial solution. We denote by $\mathcal{E}$ the set of eigenvalues of (2.5).

The set $\mathcal{E}$ is closely related with the set of bifurcation points for (2.4); indeed, we have the following:

PROPOSITION 2.3. *If* $(\lambda_0, 0)$ *is a bifurcation point for* (2.1), *then* λ_0 *belongs to* $\mathcal{E}$.

The proof of Proposition 2.3 in the case when $L(\lambda)$ is linear can be found e.g. in [4], Th. 4.1; the proof for the positively homogeneous case is similar and then it is omitted.

It is well-known, also in the one-dimensional situation, that the converse of Proposition 2.3 is not true. A sufficient condition for bifurcation is Theorem 2.2.

Now, we will apply Theorem 2.2 to the study of a Dirichlet boundary value problem associated to a strongly nonlinear second order differential equation. In particular, we will see how the differential equation considered can be put into the form (2.4).

To this aim, let us consider the following bvp:

$$\begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) + g(t, u, u') = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{2.6}$$

where $(\mu, \nu) \in \boldsymbol{R}^2$, $p > 1$ and $g : [0, \pi] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is a continuous map such that

$$g(t, \xi, \eta) = o((\phi_p(\xi) + \phi_p(\eta))^{1/p}) \text{ near } (\xi, \eta) = (0, 0), \quad \textit{uniformly in } t \in [0, \pi]. \tag{2.7}$$

In what follows, we will use the notation $\lambda = (\mu, \nu) \in \boldsymbol{R}^2$.

Let $X = W_0^{1,p}([0, \pi])$; we say that $u \in X$ is a weak solution of (2.6) if and only if

$$\begin{aligned} &\int_0^\pi |u'(t)|^{p-2} u'(t)\, v'(t)\, dt + \mu \int_0^\pi |u^+(t)|^{p-2} u^+(t)\, v(t)\, dt + \\ &-\nu \int_0^\pi |u^-(t)|^{p-2} u^-(t)\, v(t)\, dt + \int_0^\pi g(t, u(t), u'(t))\, v(t)\, dt = 0 \end{aligned} \tag{2.8}$$

for every $v \in X$. In [12] it has been proved that whenever u is a weak solution to (2.6) then $\phi_p(u') \in C_0^1([0, \pi])$ and the equation in (2.6) holds for every $t \in (0, \pi)$: therefore a weak solution of (2.6) is indeed a classical solution.

Let us also denote by X^* the dual space of X; we define the operators $J, S, G: X \to X^*$ by

$$
\begin{aligned}
(Ju, v) &= \int_0^\pi |u'(t)|^{p-2} u'(t)\, v'(t)\, dt \\
(Su, v) &= \int_0^\pi |u(t)|^{p-2} u(t)\, v(t)\, dt \qquad (2.9)\\
(G(u), v) &= \int_0^\pi g(t, u(t), u'(t))\, v(t)\, dt
\end{aligned}
$$

for every $u, v \in X$. It can be shown [12] that J is a homeomorphism; moreover, as in [12], u is a weak (and hence a classical) solution to (2.6) if and only if

$$Ju = \mu Su^+ - \nu Su^- + G(u),$$

i.e.

$$v = Ju, \qquad v = L(\lambda)\, v + H(\lambda, v), \qquad (2.10)$$

where

$$
\begin{aligned}
L(\lambda)\, v &= \mu S(J^{-1} v)^+ - \nu S(J^{-1} v)^- \\
H(\lambda, v) &\equiv G(J^{-1} v).
\end{aligned} \qquad (2.11)
$$

Moreover, from [12], we deduce that, for every $\lambda \in \boldsymbol{R}^2$, the operator $L(\lambda): X \to X$ and the map $H: \boldsymbol{R} \times X \to X$ are compact; it is also easy to check that the map $\lambda \mapsto L(\lambda)$ is continuous. Next, from (2.7), we infer that condition (2.3) is fulfilled. Finally, the following lemma proves that for every $\lambda \in \boldsymbol{R}^2$ the operator $L(\lambda)$ is positively homogeneous:

LEMMA 2.4. *The operators J and S defined in* (2.9) *are positively homogeneous of degree $p-1$; the operator J^{-1} is positively homogeneous of degree $1/(p-1)$ and, for every $\lambda \in \boldsymbol{R}^2$, the operator $L(\lambda)$ is positively homogeneous* (*of degree* 1).

PROOF. Let us consider $\xi > 0$; then, for every $u \in X$ and for every $v \in \in X$ we have

$$(J(\xi u), v) = \int_0^\pi |(\xi u)'(t)|^{p-2}(\xi u)'(t)\, v'(t)\, dt = (\xi^{p-1} Ju, v),$$

i.e.

$$J(\xi u) = \xi^{p-1} Ju \ \ \forall \xi > 0, \qquad \forall u \in X. \tag{2.12}$$

Analogously, it can be proved that S is homogeneous of degree $p-1$ as well.

Moreover, from (2.12), we deduce that

$$\xi u = J^{-1}(\xi^{p-1} Ju) \ \ \forall \xi > 0, \qquad \forall u \in X;$$

therefore, setting $Ju = p$ and $\varsigma = \xi^{p-1}$, we obtain

$$J^{-1}(\varsigma p) = \varsigma^{1/p-1} J^{-1}(p) \ \ \forall \varsigma > 0, \qquad \forall p \in X^*$$

and this proves the statement for J^{-1}. Finally, let us observe that, for every $\xi > 0$ and for every $u \in X$, we have $(\xi u)^+ = \xi u^+$ and $(\xi u)^- = \xi u^-$; from this, we can immediately infer that, for every $\lambda = (\mu, \nu) \in \boldsymbol{R}^2$, for every $\xi > 0$ and for every $u \in X$ we have:

$$\begin{aligned} L(\lambda)(\xi u) &= \mu S(J^{-1}(\xi u))^+ - \nu S(J^{-1}(\xi u))^- = \\ &= \mu S(\xi^{1/p-1} J^{-1}(u))^+ - \nu S(\xi^{1/p-1} J^{-1}(u))^- = \\ &= \mu \xi S(J^{-1}(u))^+ - \nu \xi S(J^{-1}(u))^- = \\ &= \xi L(\lambda)\, u\,. \end{aligned}$$

The lemma is proved. ■

Now, in order to apply Theorem 2.2, we have to study the set $\mathcal{E}$ of the eigenvalues of the «linearized» equation (2.5); to do this, we observe that, in the present case, (λ, v) is a solution of (2.5) if and only if (λ, v) satisfies the equation

$$(\phi_p(v'))' + \mu \phi_p(v^+) - \nu \phi_p(v^-) = 0 \tag{2.13}$$

with boundary condition

$$v(0) = 0 = v(\pi). \tag{2.14}$$

Now, following the ideas of [8, 16], in [12] it has been proved that problem (2.27)-(2.28) has a nontrivial solution $v \not\equiv 0$ if and only if λ belongs to the set $\mathcal{E}$ defined by

$$\mathcal{E} = \{(\mu, \nu) \in \boldsymbol{R}^2 : \mu = 1 \ \text{ or } \ \nu = 1\} \cup \big(\bigcup_{k \in \boldsymbol{N}} \gamma_k\big) \cup \big(\bigcup_{k \in \boldsymbol{N}} \gamma'_k\big) \cup \big(\bigcup_{k \in \boldsymbol{N}} \gamma''_k\big),$$

where, for every $k \in \boldsymbol{N}$,

$$\gamma_k = \left\{(\mu, \nu) \in \boldsymbol{R}^2 : \frac{k}{\sqrt[p]{\mu}} + \frac{k}{\sqrt[p]{\nu}} = \frac{1}{\sqrt[p]{\lambda_{1,p}}}\right\},$$

$$\gamma'_k = \left\{(\mu, \nu) \in \boldsymbol{R}^2 : \frac{k}{\sqrt[p]{\mu}} + \frac{k+1}{\sqrt[p]{\nu}} = \frac{1}{\sqrt[p]{\lambda_{1,p}}}\right\}$$

and

$$\gamma''_k = \left\{(\mu, \nu) \in \boldsymbol{R}^2 : \frac{k+1}{\sqrt[p]{\mu}} + \frac{k}{\sqrt[p]{\nu}} = \frac{1}{\sqrt[p]{\lambda_{1,p}}}\right\}.$$

The set $\mathcal{E}$ defined above is usually known as the *Fučik spectrum.*

Let us introduce the following regions of $\boldsymbol{R}^2$:

$$Q = \{(\mu, \nu) \in \boldsymbol{R}^2 : \mu > 0, \ \nu > 0\},$$

$$A_1 = \bigcup_{k \in \boldsymbol{N}} A_k^+, \quad A_{-1} = \bigcup_{k \in \boldsymbol{N}} A_k^-$$

and

$$A_0 = Q \setminus (\mathcal{E} \cup A_1 \cup A_{-1}),$$

where

$$A_k^+ = \left\{(\mu, \nu) \in Q : \frac{k}{\sqrt[p]{\mu}} + \frac{k+1}{\sqrt[p]{\nu}} < \frac{1}{\sqrt[p]{\lambda_{1,p}}}, \right.$$
$$\left. \frac{k}{\sqrt[p]{\mu}} + \frac{k}{\sqrt[p]{\nu}} < \frac{1}{\sqrt[p]{\lambda_{1,p}}} < \frac{k+1}{\sqrt[p]{\mu}} + \frac{k}{\sqrt[p]{\nu}}\right\}$$

and

$$A_k^- = \left\{(\mu, \nu) \in Q : \frac{k+1}{\sqrt[p]{\mu}} + \frac{k}{\sqrt[p]{\nu}} < \frac{1}{\sqrt[p]{\lambda_{1,p}}}, \right.$$
$$\left. \frac{k}{\sqrt[p]{\mu}} + \frac{k+1}{\sqrt[p]{\nu}} < \frac{1}{\sqrt[p]{\lambda_{1,p}}} < \frac{k+1}{\sqrt[p]{\mu}} + \frac{k+1}{\sqrt[p]{\nu}}\right\}.$$

The following proposition, whose proof can be found in [12], will be crucial to prove our main result:

PROPOSITION 2.5. *Let us assume that* $\lambda \in Q \setminus \mathcal{E}$; *then the Leray-Schauder index* $\operatorname{Ind}(I - L(\lambda), 0)$ *is well defined and*

$$\operatorname{Ind}(I - L(\lambda), 0) = \begin{cases} 1 & \text{if } \lambda \in A_1 \\ -1 & \text{if } \lambda \in A_{-1} \\ 0 & \text{if } \lambda \in A_0 . \end{cases}$$

The following lemma will be the key in the verification of the assumptions of Theorem 2.2:

LEMMA 2.6. *Let* $\lambda \in Q \setminus \mathcal{E}$. *Then*

$$\operatorname{Ind}(I - L(\lambda) - H(\lambda, \cdot), 0) = \operatorname{Ind}(I - L(\lambda), 0). \tag{2.15}$$

PROOF. Let us start by observing that the thesis first means that the Leray-Schauder index $\operatorname{Ind}(I - L(\lambda) - H(\lambda, \cdot), 0)$ is well defined and then that the equality in (2.15) holds.

Indeed, let us consider $t \in [0, 1]$ and define

$$Z_t(u) = u - L(\lambda)\, u - tH(\lambda, u) \qquad \forall u \in X .$$

We will show that there exist $R > 0$ such that for every $t \in [0, 1]$ and for every $u \in X$ with $\|u\| = R$ we have $Z_t(u) \neq 0$. Indeed, suppose that for every $n \in N$ there exist $t_n \in [0, 1]$ and $u_n \in X$ with $\|u_n\| = 1/n$ such that

$$u_n = L(\lambda)\, u_n + t_n H(\lambda, u_n). \tag{2.16}$$

Dividing (2.16) by $\|u_n\|$, setting $w_n = u_n / \|u_n\|$ ($\|w_n\| = 1$) and using the positive homogeneity of $L(\lambda)$, we obtain

$$w_n = L(\lambda)\, w_n + t_n \frac{H(\lambda, u_n)}{\|u_n\|}. \tag{2.17}$$

Letting $n \to +\infty$, from (2.3) we deduce that the last term in (2.17) is infinitesimal; moreover, by the compactness of $L(\lambda)$ we can find a subsequence w_{n_j} such that $L(\lambda)\, w_{n_j}$ converges. Therefore, also w_{n_j} must converge to some $w \in X$, $w \neq 0$ ($\|w\| = 1$); from (2.17), w must satisfy

$$w = L(\lambda)\, w$$

and so $\lambda \in \mathcal{E}$: absurd.

Then, the Leray-Schauder index $\mathrm{Ind}\,(I - L(\lambda) - H(\lambda, \cdot), 0)$ is well defined and the homotopy invariance of the degree permits to conclude. ∎

Now, we recall that the set $\mathcal{E}$ is the union of the curves γ_k, γ'_k and γ''_k, k varying in $\boldsymbol{N}$. Moreover, let $\mathcal{S}^+$ denote the set of functions χ with $\chi'(0) > 0$ and S_l^+ denote the set of the functions $\varphi \in X$ with exactly l simple zeros in $[0, \pi)$, with $\varphi'(0) > 0$ and such that all the zeros of φ in $[0, \pi]$ are simple. Let $\mathcal{S}^- = -\mathcal{S}^+$, $S_l^- = -S_l^+$ and $S_l = S_l^+ \cup S_l^-$; we observe that the sets $S_l^\pm$ are open in S_l.
We are ready to prove our bifurcation result:

THEOREM 2.7. *Let $g : [0, \pi] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ be a continuous map satisfying* (2.7) *and let $\lambda_0 = (\mu_0, \nu_0) \in \mathcal{E}$. Moreover, consider $h : \boldsymbol{R}^2 \to \boldsymbol{R}$ defined by $h(x, y) = \nu_0 x - \mu_0 y$ and $\Gamma = h^{-1}(0)$.*

If $(\mu_0, \nu_0) \in \gamma_k$ (for some $k \in \boldsymbol{N}$), then there exist two closed connected unbounded *surfaces $\mathcal{C}^\pm \subset ((\boldsymbol{R} \times S_m^\pm) \cup (\mathcal{E} \times \{0\})) \cap \mathcal{S}^*$ bifurcating from (μ_0, ν_0).*

If $(\mu_0, \nu_0) \in \gamma'_k$ (resp. $(\mu_0, \nu_0) \in \gamma''_k$), then there exists a closed connected unbounded *surface $\mathcal{C}^- \subset ((\boldsymbol{R} \times S_m^-) \cup (\mathcal{E} \times \{0\})) \cap \mathcal{S}^*$ (resp. $\mathcal{C}^+ \subset$ $\subset ((\boldsymbol{R} \times S_m^+) \cup (\mathcal{E} \times \{0\})) \cap \mathcal{S}^*$) bifurcating from (μ_0, ν_0).*

REMARK 2.8. 1) *From the statement of Theorem* 2.7 *we deduce that every point of the Fučik spectrum for* (2.13)-(2.14) *is a bifurcation point. Moreover, the set of solutions which bifurcates from any point of the Fučik spectrum is a unbounded closed connected surface. To the best of our knowledge, the only result which is concerned with a bifurcating behaviour from the points of the Fučik spectrum has been obtained by P. Drábek and M. Kučera: in the paper* [13], *they show that every point of the set $\mathcal{E}$ is a bifurcation point not for* (2.6), *but for the boundary value problem with variable coefficients*

$$\begin{cases} (\phi_p(u'))' + \mu(t)\, \phi_p(u^+) - \nu(t)\, \phi_p(u^-) = 0 \\ u(0) = 0 = u(\pi)\,. \end{cases} \tag{2.18}$$

Moreover, we stress the fact that in [13] (*see also* [12]), *no* (*local or global*) *description of the bifurcating set is given.*

2) *The proof of Theorem* 2.7 *will follow from an application of Theorem* 2.2. *As in the one-dimensional case considered by Rabinowitz* (*see* [20] *and Theorem* 1.1 *in the Introduction*), *the abstract theorem cont ains an alternative for the behaviour of the bifurcating set; in the application to differential equations, it is shown that one of the alternatives never occurs.*

Before proving Theorem 2.7, we need some preliminary results. First of all, we denote by v_{2k}^+ any eigenfunction corresponding to $\lambda \in \gamma_k$ with $(v_{2k}^+)'(0) > 0$ and by v_{2k}^- any eigenfunction corresponding to $\lambda \in \gamma_k$ with $(v_{2k}^-)'(0) < 0$. Analogously, let v_{2k+1}^+ be any eigenfunction corresponding to $\lambda \in \gamma_k''$ (observe that $(v_{2k+1}^+)'(0) > 0$) and v_{2k+1}^- be any eigenfunction corresponding to $\lambda \in \gamma_k'$ (observe that $(v_{2k+1}^-)'(0) < 0$). It is easy to check, following the construction and the description of the set $\mathcal{S}$ given e.g. in [17], that, for every $k \in \boldsymbol{N}$, each v_{2k} has exactly $2k$ zeros in $[0, \pi)$ and each v_{2k+1} has exactly $2k+1$ zeros in $[0, \pi)$; therefore, $v_l^\pm \in S_l^\pm$ and the normalization $\|v_l^\pm\| = 1$ makes v_l unique.

This remark on the number of zeros of each v_l $(l \in \boldsymbol{N})$ in $[0, \pi)$ will be crucial for the proof of Theorem 2.7.

Now, we give a lemma on the uniqueness of the solution to a Cauchy problem associated to the equation in (2.6); the proof of this result is quite classical and therefore it will be omitted.

LEMMA 2.9. *If* (λ, u) *is a solution of* (2.6) *and there exists* $\tau \in [0, \pi]$ *such that* $u(\tau) = u'(\tau) = 0$, *then* $u \equiv 0$ *in* $[0, \pi]$.

The following lemma is required to prove Theorem 2.7 (cf. [20]):

LEMMA 2.10. *For each* $j \in \boldsymbol{N}$ *and for each* $\lambda_j \in \gamma_{[j-1/2]} \cup \gamma'_{[j-1/2]} \cup \cup \gamma''_{[j-1/2]}$ *there exists a neighborhood* $\mathcal{N}_j^\pm$ *of* $(\lambda_j, 0)$ *such that if* $(\lambda, u) \in \in \mathcal{N}_j^\pm \cap \mathcal{S}^\pm$ *and* $u \not\equiv 0$ *then* $u \in S_j^\pm$.

PROOF. We use an argument similar to the one already developed in the proof of Lemma 2.6. We prove the result only for the sets S_j^+, the other case being similar.

Suppose by contradiction that there exists a sequence $(\lambda_n, u_n) \in \mathcal{S}^+$ such that $u_n \not\equiv 0$, $u_n \notin S_j^\pm$ and $(\lambda_n, u_n) \to (\lambda_j, 0)$. Let $w_n = u_n/\|u_n\|$ and let us divide (2.10) by $\|u_n\|$: using the positive homogeneity of $L(\lambda)$ for every

$\lambda \in \boldsymbol{R}^2$, we obtain

$$w_n = L(\lambda_n)\, w_n + H(u_n)/\|u_n\| , \tag{2.19}$$

i.e.

$$w_n = \{L(\lambda_n) - L(\lambda_j)\}\, w_n + L(\lambda_j)\, w_n + H(u_n)/\|u_n\| . \tag{2.20}$$

Now, as already observed, by (2.7), $H(u_n)/\|u_n\| \to 0$ when $n \to +\infty$; moreover, by (2.11),

$$\|\{L(\lambda_n) - L(\lambda_j)\}\, w_n\| \leq \|L(\lambda_n) - L(\lambda_j)\|\|w_n\| \leq \|L(\lambda_n) - L(\lambda_j)\| ;$$

the continuity of $\lambda \mapsto L(\lambda)$ implies that $\|\{L(\lambda_n) - L(\lambda_j)\} w_n\| \to 0$ and therefore the first term on the right of (2.20) tends to 0 as $n \to +\infty$. Finally, by the compactness of $L(\lambda_j)$, there exists a subsequence of $L(\lambda_j) w_n$ which is convergent. Therefore, from these remarks and from (2.20), it follows that there exists a subsequence w_{n_i} of w_n converging to some function w ($w \in S^+$) such that $\|w\| = 1$ and

$$w = L(\lambda_j)\, w .$$

Then, $w \in S_j^+$; since this set is open, for n_i large enough we deduce that $w_{n_i} \in S_j^+$, contrary to our assumption. ∎

Proof of Theorem 2.7. *Step* 1. We apply Theorem 2.2. We only have to take two suitable points α and β on Γ; we give the details for the choice of α and β for the case when $(\mu_0, \nu_0) \in \gamma_k$ (for some $k \in \boldsymbol{N}$), the other cases being similar. Indeed, if $(\mu_0, \nu_0) \in \gamma_k$, it is sufficient to take $\alpha \in \Gamma \cap A_k^+$ and $\beta \in \Gamma \cap A_k^-$.

Then, neither α nor β belong to $\mathcal{E}$: Proposition 2.3 implies that α and β are not bifurcation points for (2.4). Moreover, from Proposition 2.5 and Lemma 2.6, we have

$$\operatorname{Ind}(I - L(\alpha) - H(\alpha, \cdot), 0) \neq \operatorname{Ind}(I - L(\beta) - H(\beta, \cdot), 0) .$$

Therefore, all the assumptions of Theorem 2.2 are satisfied and the existence of a bifurcating set is proved.

Step 2. In the sequel, we denote by $\mathcal{C}(\lambda_0)$ the set bifurcating from λ_0, whose existence has been proved in the previous Step.

Let us suppose that $\lambda_0 \in \gamma_k$; since $S_j \cap S_l = \emptyset$ if $j \neq l$, we only have to show that $\mathcal{C}(\lambda_0)$ is contained in $(\boldsymbol{R}^2 \times S_{2k}) \cup (\gamma_k \times \{0\})$. Indeed, if this is true, then alternative (b) of Theorem 2.2 cannot hold; suppose in fact that there exists $\lambda^* \in \Gamma$, $\lambda^* \neq \lambda_0$, such that $(\lambda^*, 0) \in \mathcal{C}(\lambda_0)$. Then necessarily $\lambda^* \in \gamma_k$ (since $0 \notin S_{2k}$) and this is absurd since $\gamma_k \cap \Gamma = \{\lambda_0\}$.

Hence, suppose that $\mathcal{C}(\lambda_0)$ is not contained in $(\boldsymbol{R}^2 \times S_{2k}) \cup (\gamma_k \times \times \{0\})$. Then there exists $(\lambda, u) \in \mathcal{C}(\lambda_0) \cap (\boldsymbol{R}^2 \times \partial S_{2k})$ with $(\lambda, u) \notin (\gamma_k \times \times \{0\})$ and $(\lambda, u) = \lim (\lambda_n, u_n)$ with $u_n \in S_{2k}$. Now, if $u \in \partial S_{2k}$ then, by Lemma 2.9 $u \equiv 0$: hence $\lambda \in \mathcal{E} \backslash \gamma_k$. Therefore, for some $j \neq 2k$, we can find a neighbourhood $\mathcal{N}_j$ of $(\lambda, 0)$ satisfying the properties stated in Lemma 2.10; now, we observe that, for n large enough, $(\lambda_n, u_n) \in \mathcal{N}_j \cap \cap (\boldsymbol{R}^2 \times S_{2k})$. But, by Lemma 2.10, we also have $u_n \in S_j$ and this is absurd.

In the case $\lambda_0 \in \gamma_k'$ (resp. $\lambda_0 \in \gamma_k''$) we can prove in the same way that $\mathcal{C}(\lambda_0)$ is contained in $(\boldsymbol{R}^2 \times S_{2k+1}^-) \cup (\gamma_k' \times \{0\})$ (resp. $(\boldsymbol{R}^2 \times S_{2k+1}^+) \cup \cup (\gamma_k'' \times \{0\})$) and from this the thesis easily follows. ■

3. Applications to superlinear asymmetric boundary value problems.

In this section, we consider an application of the results proved in Section 2 to the study of the solution set of a boundary value problem as (2.6) when some conditions on the behaviour of g at infinity are assumed. We are mainly motivated by previous works [7, 9], where nonlinearities satisfying a superlinear asymmetric growth at infinity are considered.

We recall the problem we are dealing with:

$$\begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) + g(t, u) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{3.31}$$

where $(\mu, \nu) \in \boldsymbol{R}^2$, $p > 1$ and $g: [0, \pi] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is a continuous map such that

(H_0) $\quad g(t, u) = o(\phi_p(u))$ for $u \to 0$, *uniformly in* $t \in [0, \pi]$.

As before, we will use the notation $\lambda = (\mu, \nu) \in \boldsymbol{R}^2$.

Now, let us assume that there exists a continuous function $\beta : [0, \pi] \to \boldsymbol{R}^+$ such that

$$(H_\infty) \quad \begin{cases} \lim\limits_{u \to +\infty} \dfrac{g(t, u)}{\phi_p(u)} = +\infty & \text{uniformly in } t \in [0, \pi] \\ \lim\limits_{u \to -\infty} \dfrac{g(t, u)}{\phi_p(u)} = \beta(t) & \text{uniformly in } t \in [0, \pi]. \end{cases}$$

Under the only assumption (H_0), from Theorem 2.7 we deduce that from every point $\lambda_0 = (\mu, \nu) \in \gamma_k$ (for some $k \in \boldsymbol{N}$) of the Fučik spectrum $\mathcal{E}$ emanate (bifurcate) two unbounded continua (i.e. closed and connected branches) $\mathcal{C}^\pm(\lambda_0)$ of nontrivial solutions (λ, u) of (3.1); moreover, these continua are contained in $(\boldsymbol{R} \times S_m^\pm) \cup (\mathcal{E} \times \{0\})$ (for $m = [k/2]$). Analogously, from every point $\lambda_0 = (\mu, \nu) \in \gamma_k'$ (resp. $\lambda_0 = (\mu, \nu) \in \gamma_k''$) bifurcates an unbounded continuum $\mathcal{C}^-(\lambda_0) \subset (\boldsymbol{R} \times S_m^-) \cup (\mathcal{E} \times \{0\})$ (resp. $\mathcal{C}^+(\lambda_0) \subset (\boldsymbol{R} \times S_m^+) \cup (\mathcal{E} \times \{0\})$) of nontrivial solutions (λ, u) of (3.1) (for $m = [k/2]$).

If, in addition, we assume condition (H_∞), then some global results for $\mathcal{C}(\lambda_0)$ can be obtained. More precisely, the following variant of Lemma 7 in [14] (where the case $p = 2$ and $\mu = \nu$ is considered) can be proved:

LEMMA 3.1 ([14], Lemma 7). *Under assumptions (H_0) and (H_∞), for every $k \in \boldsymbol{N}$ there exists a positive constant ξ_k such that if (μ, ν, u) is a solution of (3.1) and u has exactly k zeros in $[0, \pi)$, then $\mu \leq \xi_k$ and $\nu \leq \xi_k$.*

PROOF. First of all, we observe that conditions (H_0) and (H_∞) imply that there exists a constant $C_1 > 0$ such that

$$g(t, u)/\phi_p(u) \geq -C_1 \tag{3.2}$$

for every $t \in [0, \pi]$ and for every $u \in \boldsymbol{R}$.

Now, let (μ, ν, u) be a solution of (3.1) and suppose that u has exactly k zeros in $[0, \pi)$; it is easy to check that (μ, ν, u) satisfies

$$(\phi_p(u'))' + \nu\phi_p(u) + (\mu - \nu)\, \phi_p(u^+) + g(t, u) = 0\,; \tag{3.3}$$

setting $p(t, u) = (\mu - \nu)\, \phi_p(u^-) + g(t, u)$, from (3.2) we deduce that $p(t, u)/\phi_p(u) \geq -C := \min(-C_1, \ -C_1 + \mu - \nu)$ for every $t \in [0, \pi]$, $u \in \boldsymbol{R}$ and $(\mu, \nu) \in \boldsymbol{R}^2$.

Hence, by (3.3), ν is the kth eigenvalue of $-(\phi_p(u'))' - p(t, u)$ with Dirichlet boundary conditions. By standard comparison theorems (see e.g. [3, 17]), ν is smaller than the k-th eigenvalue of the operator $-(\phi_p(u'))' + C\phi_p(u)$; this implies that $\nu \leqslant \xi_k := \lambda_{k,p} + C_1$ if $\mu \geqslant \nu$ and $\mu \leqslant \xi_k$ if $\mu < \nu$. By swapping the role of μ and ν we infer that $\mu \leqslant \xi_k$ if $\mu \geqslant \nu$ and $\nu \leqslant \xi_k$ if $\mu < \nu$. This proves the result. ∎

In terms of the continuum $\mathcal{C}(\lambda_0)$, Lemma 3.1 means that it is contained in the region of space of equation $\{(\mu, \nu): \mu \leqslant \xi_0, \nu \leqslant \xi_0\}$, for some ξ_0 depending on λ_0. We stress the fact that this further information is a consequence of assumption (H_∞).

The following result specifies the global behaviour of the bifurcating branches. It is a characterization of the values (μ, ν) where a branch $\mathcal{C}(\lambda_0)$ can become unbounded:

PROPOSITION 3.2. *Assume conditions* (H_0) *and* (H_∞). *For* $k \in \boldsymbol{N}$, *let* $m = [k/2]$ *and let* (μ_m, ν_m) *be a point in* $\gamma_m \cup \gamma''_m$; *moreover, let* $\beta_0 = \min\{\beta(t): t \in [0, \pi]\}$ *and* $\beta_1 = \max\{\beta(t): t \in [0, \pi]\}$. *We denote by* $\mathcal{C}_k^+$ *the branch (contained in* $\boldsymbol{R}^2 \times S_k^+ \cup \mathcal{E} \times \{0\}$*) bifurcating from* (μ_m, ν_m).

Then, either $\mathcal{C}_k^+$ *intersects the set* $O = \{(\mu, \nu, u): \mu = 0\} \cup \cup \{(\mu, \nu, u): \nu = 0\}$ *or it is unbounded in a neighborhood of a value* ν *such that*

$$\left(\frac{\pi_p}{\pi} m\right)^p - \beta_1 \leqslant \nu < \left(\frac{\pi_p}{\pi}(m+1)\right)^p - \beta_0. \tag{3.4}$$

An analogous statement holds for $\mathcal{C}_k^-$, *with* $m = [(k+1)/2]$, *when* $(\mu_m, \nu_m) \in \gamma_m \cup \gamma'_m$.

The proof of Proposition 3.2 will be an easy consequence of some lemmas already proved in [7, 9]. First, let l be the largest integer such that

$$\lambda_{l,p} < \nu + \beta_0 \tag{3.5}$$

and l' be the smallest integer such that

$$\nu + \beta_1 < \lambda_{l'+1,p}. \tag{3.6}$$

Moreover, let us denote by $\boldsymbol{n}(u)$ the number of simple zeros in $[0, \pi)$ of a solution u of (3.1); it is well defined because of Lemma 2.9.

According to [7, 9], we have:

LEMMA 3.3. *There exists* $M_l > 0$ *such that if* u *is a solution of* (3.35) *then*

$$|u(0)| > M_l \Rightarrow \boldsymbol{n}(u) > 2l + 1 .$$

Moreover, for every $k \in N$ *there exists* $M_{l'}^* > 0$ *such that if* u *is a solution of* (3.1) *with* $u(0) > M_{l'}^*$, *then*

$$\boldsymbol{n}(u) > k \quad or \quad \begin{cases} \boldsymbol{n}(u) < 2l' + 1 & if\ \boldsymbol{n}(u)\ is\ even \\ \boldsymbol{n}(u) < 2l' + 2 & if\ \boldsymbol{n}(u)\ is\ odd . \end{cases}$$

Analogously, if u *is a solution of* (3.1) *with* $u(0) < -M_{l'}^*$, *then*

$$\boldsymbol{n}(u) > k \quad or \quad \begin{cases} \boldsymbol{n}(u) < 2l' + 1 & if\ \boldsymbol{n}(u)\ is\ even \\ \boldsymbol{n}(u) < 2l' & if\ \boldsymbol{n}(u)\ is\ odd . \end{cases}$$

Finally, we can state a classical lemma, which gives an *a priori* bound for solutions to (3.1) with a prescribed number of zeros:

LEMMA 3.4 ([9], Corollary 3.11). *For every* $k \in N$ *and for every* $M^* > 0$ *there exists* $M > 0$ *such that if* u *is a solution of* (3.1) *with* $\boldsymbol{n}(u) = k$ *and* $|u(0)| \leq M^*$, *then* $\|u\|_1 \leq M$.

PROOF OF PROPOSITION. 3.2. Let us fix $k \in \boldsymbol{N}$; if $\mathcal{C}_k^+$ does not meets O, then, by Lemma 3.1, it is contained in $\{(\mu, \nu, u) : 0 < \mu \leq \xi_k, 0 < \nu \leq \leq \xi_k\}$. Therefore, it must be unbounded in u. We will show that this happens for the values ν described in (3.4).

We consider the definitions of l and l' given in (3.5) and (3.6), respectively. According to the different values of ν, we distinguish the following cases:

Case A. $l = [\pi/\pi_p \sqrt[p]{\nu + \beta_0}]$ and $l' = [\pi/\pi_p \sqrt[p]{\nu + \beta_1}]$.
Case B. $l = [\pi/\pi_p \sqrt[p]{\nu + \beta_0}]$ and $l' = [\pi/\pi_p \sqrt[p]{\nu + \beta_1}] + 1$.
Case C. $l = [\pi/\pi_p \sqrt[p]{\nu + \beta_0}] - 1$ and $l' = [\pi/\pi_p \sqrt[p]{\nu + \beta_1}]$.
Case D. $l = [\pi/\pi_p \sqrt[p]{\nu + \beta_0}] - 1$ and $l' = [\pi/\pi_p \sqrt[p]{\nu + \beta_1}] + 1$.

We continue our discussion in the case k is even, $k = 2m$; the other case is left to the reader.

Case A. Let $M^* = \max(M_l, M_{l'}^*)$ and let us assume $u(0) > M^*$: then, by Lemma 3,

$$2l + 1 < 2m < 2l' + 1 ,$$

i.e.

$$l + 1 \leqslant m \leqslant l' .$$

By the definition of l and l' we infer that

$$\left[\frac{\pi}{\pi_p} \sqrt[p]{\nu + \beta_0}\right] + 1 \leqslant m \leqslant \left[\frac{\pi}{\pi_p} \sqrt[p]{\nu + \beta_1}\right]$$

and

$$\left(\frac{\pi_p}{\pi} m\right)^p - \beta_1 \leqslant \nu < \left(\frac{\pi_p}{\pi} m\right)^p - \beta_0 . \tag{3.7}$$

Therefore, if ν does not satisfy (3.7), $0 < u(0) < M^*$ and, by Lemma 3,4 there exists a constant M such that $\|u\|_1 \leqslant M$: actually, this is the thesis.

Case B. As in the previous case, if $u(0) > M^*$, we get

$$l + 1 \leqslant m \leqslant l' .$$

This implies that

$$\left[\frac{\pi}{\pi_p} \sqrt[p]{\nu + \beta_0}\right] + 1 \leqslant m \leqslant \left[\frac{\pi}{\pi_p} \sqrt[p]{\nu + \beta_1}\right] + 1$$

and so

$$\left(\frac{\pi_p}{\pi} m\right)^p - \beta_1 \leqslant \nu < \left(\frac{\pi_p}{\pi} (m + 1)\right)^p - \beta_0 . \tag{3.8}$$

Case C. It is immediate to check that in this case we obtain again condition (3.8).

Case D. In this case, from

$$l + 1 \leqslant m \leqslant l'$$

we deduce that

$$\left[\frac{\pi}{\pi_p}\sqrt[p]{\nu+\beta_0}\right]-1+1\leqslant m\leqslant\left[\frac{\pi}{\pi_p}\sqrt[p]{\nu+\beta_1}\right]+1\,.$$

By a simple computation we obtain

$$\left(\frac{\pi_p}{\pi}m\right)^p-\beta_1\leqslant\nu<\left(\frac{\pi_p}{\pi}m\right)^p-\beta_0.$$

In any case, condition (3.39) must be fulfilled and the result is proved. ■

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