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L. FATTORUSSO

G. IDONE

Partial Hölder continuity results for solutions of non linear non variational elliptic systems with strictly controlled growth

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Partial Hölder Continuity Results for Solutions of non Linear non Variational Elliptic Systems with Strictly Controlled Growth.

L. FATTORUSSO - G. IDONE (*)

SUNTO - In un aperto limitato Ω si considera il sistema non lineare non variazionale

$$a(x, u, Du, H(u)) = b(x, u, Du) \tag{1}$$

dove $a(x, u, \mu, \xi)$ e $b(x, u, \mu)$ sono vettori di $\mathbb{R}^N$, $N \geq 1$ misurabili in x e continui nelle altre variabili. Si dimostra che se $u \in H^2(\Omega)$ è soluzione in Ω del sistema (1), *se* $b(x, u, \mu)$ *ha andamenti strettamente controllati, se* $a(x, u, \mu, \xi)$ *è di classe* C^1 *in* ξ *e verifica la condizione* (A) *e, unitamente a* $\partial a/\partial \xi$, *certe condizioni di continutà, allora il vettore* Du *è parzialmente holderiano in* Ω *con ogni esponente* $\alpha < 1 - n/p$.

1. Introduction.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 4$ of class C^2 and let $x = (x_1, x_2, \ldots, x_n)$ be a generic point in it.

The symbols $(\cdot | \cdot)_k$ and $\|\cdot\|_k$ denote the scalar product and the norm in $\mathbb{R}^k$, respectively. We shall omit the index k wherever there is no ambiguity.

(*) Indirizzo degli AA.: D.I.M.E.T. Facoltà di Ingegneria, via Graziella, Feo di Vito, 89128 Reggio Calabria.

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In what follows we denote by $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, $\mu_i \in \mathbb{R}^N$, N integer ≥ 1, a generic vector of $\mathbb{R}^{nN}$ and by $\xi = \{\xi_{ij}\}$ $i, j = 1, 2, \ldots, n$, $\xi_{ij} \in \mathbb{R}^N$ a generic element of $\mathbb{R}^{n^2 N}$.

If $u\colon \Omega \to \mathbb{R}^N$ is a vector, we set

$$D_i u = \frac{\partial u}{\partial x_i}\,; \qquad Du = (D_1 u, D_2 u, \ldots, D_n u)\,;$$

$$H(u) = \{D_i D_j u\} = \{D_{ij} u\}\,; \qquad i, j = 1, 2, \ldots, n\,.$$

We consider in Ω the following second order non linear non variational system

$$a(x, u, Du, H(u)) = b(x, u, Du) \tag{1.1}$$

where $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ are vectors in $\mathbb{R}^N$, measurable in x, continuous in (u, μ, ξ) and (u, μ) respectively, satisfying the conditions:

$$a(x, u, \mu, 0) = 0\,. \tag{1.2}$$

(A) *there exist three positive constants* $\bar{\alpha}, \bar{\gamma}$ *and* $\bar{\delta}$ with $\bar{\gamma} + \bar{\delta} < 1$, *such that,* $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2 N}$ and for *almost every* $x \in \Omega$, *we have* (1)

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \bar{\alpha}[a(x, u, \mu, \tau + \eta) - a(x, u, \mu, \eta)] \right\|^2 \leq \bar{\gamma} \|\tau\|^2 + \bar{\delta} \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2$$

(1.3) *there exists a constant c such that,* $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$ *and for almost every* $x \in \Omega$ *we have*

$$\|b(x, u, \mu)\| \leq c(f(x) + \|u\|^\alpha + \|\mu\|^\beta)$$

(1) From condition (A), assuming $\eta = 0$, it follows, $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$, $\forall \tau \in \mathbb{R}^{n^2 N}$ and for almost all $x \in \Omega$

$$\|a(x, u, \mu, \tau)\| \leq c\|\tau\|\,.$$

Moreover one can show that, if the vector $a(x, u, \mu, \xi)$ is of class C^1 with respect to ξ, with derivatives $\partial a / \partial \xi_{ij}$ bounded, then the operator $a(x, u, \mu, \xi)$ is elliptic (see [7]).

Sufficient conditions that ensure the hypothesis (A) are stated in [4] and [6].

with $f \in L^2(\Omega)$ *and with*

$$1 \leqslant \alpha < \frac{n}{n-4}, \qquad 1 \leqslant \beta < \frac{n}{n-2}.$$

We shall denote by $H^{s,p}(\Omega, \mathbb{R}^N)$, $H_0^{s,p}(\Omega, \mathbb{R}^N)$, s integer $\geqslant 0$, $p \in [1, +\infty)$, the usual Sobolev spaces (2)

By a solution to the system (1.1) we mean a vector $u \in H^2(\Omega, \mathbb{R}^N)$ satisfying (1.1) for almost all $x \in \Omega$. In this work we obtain a partial Hölder continuity result for the gradient of these solutions; this result is similar to the one proved in the case of «linear growth» by S. Campanato in [5] for the elliptic systems and by M. Marino - A. Maugeri in [8] in the parabolic case.

To achieve the partial Hölder regularity of the gradient we need to preliminary obtain the local L^q −regularity of the derivatives $D_{ij}u$. Section n° 2 is devoted to the above study, which in itself is of interest.

2. Local L^q −regularity of the matrix $H(u)$.

Let $u \in H^2(\Omega, \mathbb{R}^N)$ be a solution in Ω to the non variational system

$$a(x, u, Du, H(u)) = b(x, u, Du) \tag{2.1}$$

being $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ vectors of $\mathbb{R}^N$ satisfying assumptions (1.2), (1.3) and (A).

Proceeding with the same technique used in [5] to obtain the estimate

(2) If s, j are integers $\geqslant 0$ and $p \in [1, +\infty)$,

$$|u|_{j,p,\Omega} = \left[\int_\Omega \left(\sum_{|\alpha|=j} \|D^\alpha u\|^2\right)^{p/2} dx\right]^{1/p};$$

$$\|u\|_{s,p,\Omega} = \left\{\sum_{j=0}^{s} |u|^p_{j,p,\Omega}\right\}^{1/p}$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}, \ldots, D_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, α_j integer $\geqslant 0$.

Particularly $|u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \|u\|_{L^p(\Omega, \mathbb{R}^N)}$.

(3.2) of lemma (3.1), we have:

$$(2.2)\qquad \int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx \leqslant c\sigma^{-2}\int_{B(x^0,\,2\sigma)} \|D(u)-(Du)_{2\sigma}\|^2\,dx + \\ + c\int_{B(x^0,\,2\sigma)} \|b(x,\,u,\,Du)\|^2\,dx\,(^3)(^4)(^5).$$

To prove the L^q – local regularity result of the matrix $H(u)$, we suppose that, in the assumption (1.3), we have

$$(2.3)\qquad f\in L^{q_0}(\Omega) \quad \text{with } 2<q_0\leqslant 2^*.$$

Moreover by the theorem of Poincaré, we have

$$(2.4)\qquad \int_{B(x^0,\,2\sigma)} \|Du-(Du)_{2\sigma}\|^2\,dx \leqslant c\left(\int_{B(x^0,\,2\sigma)} \|H(u)\|^{2n/(n+2)}\,dx\right)^{(n+2)/n}$$

where c does not depend on σ.

Now from (2.3) and from the assumption (1.3) we will also have $b(x,\,u,\,Du)\in L^{\bar q}_{loc}(\Omega)$ with

$$2<\bar q\leqslant q_0\,\frac{2n}{\alpha(n-4)}\wedge\frac{2^*}{\beta}\,.$$

If we set

$$F(x)=\|H(u)\|^{2n/(n+2)},\qquad G(x)=\|b(x,\,u,\,Du)\|^{2n/(n+2)}.$$

The estimate (2.2) by means of the (2.4) can be written in the following manner:

$$\fint_{B(x^0,\,\sigma)} F^{(n+2)/n}\,dx \leqslant \left(\fint_{B(x^0,\,2\sigma)} F\,dx\right)^{(n+2)/n} + c\left(\fint_{B(x^0,\,2\sigma)} G^{(n+2)/n}\,dx\right).$$

From this and by a well known lemma of Gehring - Giaquinta - G. Modica (see lemma 10.1, page 100 of [2]) written for $r=(n+2)/n$,

(3) $B(x^0,\,\sigma)=\{x\in\mathbb{R}^n\colon \|x-x^0\|<\sigma\}$.

(4) If $E\subset\mathbb{R}^k$ is a measurable set with positive measure and $f\in L^1(E,\,\mathbb{R}^h)$, we set

$$f_E=\fint_E f\,dx=\frac{\int_E f\,dx}{mis\,E}$$

(5) $u_\sigma=u_{B(x^0,\,\sigma)},\,(Du)_\sigma=(Du)_{B(x^0,\,\sigma)},\;(H(u))_\sigma=(H(u))_{B(x^0,\,\sigma)}$.

$s = (\bar{q}(n+2))/2n$, we deduce that there exists $\varepsilon \in (0, s-r]$ such that

$$F \in L^t_{loc}(\Omega) \qquad \forall t \in \left[\frac{n+2}{n}, \frac{n+2}{n} + \varepsilon\right)$$

and, $\forall B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < 1$, one has

$$\left(\fint_{B(x^0, \sigma)} F^t\,dx\right)^{1/t} \le K\left\{\left(\int_{B(x^0, 2\sigma)} F^r\,dx\right)^{1/r} + \left(\fint_{B(x^0, 2\sigma)} G^t\,dx\right)^{1/t}\right\}. \tag{2.5}$$

From (2.5) written for $t = (q(n+2))/2n$ with $q \in [2, \bar{q})$ we achieve the following

THEOREM 2.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system* (2.1) *and if the assumptions* (1.2), (1.3) *with $f \in L^{q_0}(\Omega)$, $q_0 \le 2^*$, and* (A) *hold, then there exists $\bar{q}$*

$$2 < \bar{q} \le q_0 \wedge \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \bar{q})$, $u \in H^{2,q}_{loc}(\Omega, \mathbb{R}^N)$ and $\forall B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < 1$, one has:

$$\left(\fint_{B(x^0, \sigma)} \|H(u)\|^q\,dx\right)^{1/q} \le \tag{2.6}$$

$$\le K\left\{\left(\fint_{B(x^0, 2\sigma)} \|H(u)\|^2\,dx\right)^{1/2} + \left(\fint_{B(x^0, 2\sigma)} \|b\|^q\,dx\right)^{1/q}\right\}$$

where K does not depend on σ.

Now we can show the following

LEMMA 2.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system* (2.1) *and if the assumptions* (1.2), (1.3) *with $f \in L^p(\Omega)$, $p > n$, and* (A) *hold, then there exists $\bar{q}$ with*

$$2 < \bar{q} \le \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \bar{q})$, $\forall B(x^0, 2\sigma) \subset\subset \Omega$, *with* $\sigma < 1$, *it results*:

$$\left(\int_{B(x^0,\sigma)} \|H(u)\|^q \, dx\right)^{1/q} \leqslant c\sigma^{n(1/q-1/2)}[\phi(u, x^0, 2\sigma)]^{1/2} \tag{2.7}$$

where c does not depend on σ *and*

$$\phi(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0,\sigma)} \|u\|^{2n/(n-4)} \, dx + \int_{B(x^0,\sigma)} \|D(u)\|^{2n/(n+2)} \, dx + \int_{B(x^0,\sigma)} \|H(u)\|^2 \, dx \tag{2.8}$$

with

$$\xi = n\left(1 - \frac{2}{p}\right).$$

PROOF. From (2.6) we have $\forall B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < 1$

$$\left(\int_{B(x^0,\sigma)} \|H(u)\|^q dx\right)^{1/q} \leqslant K\sigma^{n(1/q-1/2)}[\phi(u, x^0, 2\sigma)]^{1/2} + \left(\int_{B(x^0,2\sigma)} \|b\|^q \, dx\right)^{1/q}. \tag{2.9}$$

Now we can evaluate the last term of the right hand side of (2.9).

From assumption (1.3) it follows:

$$\|b(x, u, Du)\|^q \leqslant c\{|f(x)|^q + \|u\|^{\alpha q} + \|Du\|^{\beta q}\}. \tag{2.10}$$

On the other hand, one gets

$$\left(\int_{B(x^0,2\sigma)} |f|^q \, dx\right)^{1/q} \leqslant c\left(\int_{B(x^0,2\sigma)} |f|^p \, dx\right)^{1/p} \sigma^{n(1-q/p)(1/q)} \leqslant c\|f\|_{L^p(\Omega, \mathbb{R}^N)} \sigma^{n(1/q-1/2)}[\phi(u, x^0, 2\sigma)]^{1/2}; \tag{2.11}$$

$$(2.12)\qquad \left(\int_{B(x^0,\,2\sigma)} \|u\|^{\alpha q}\,dx\right)^{1/q} \leqslant$$

$$\leqslant c\left(\int_{B(x^0,2\sigma)} \|u\|^{2n/(n-4)}dx\right)^{\alpha(n-4)/2n} \cdot \sigma^{n(1-\alpha q(n-4)/2n)(1/q)} =$$

$$= c\sigma^{n/q}\left(\frac{\int_{B(x^0,\,2\sigma)} \|u\|^{2n/(n-4)}\,dx}{\sigma^n}\right)^{\alpha(n-4)/2n} \leqslant$$

$$\leqslant \sigma^{n/q}\left(1+\frac{\int_{B(x^0,2\sigma)}\|u\|^{2n/(n-4)}dx}{\sigma^n}\right)^{1/2} \leqslant c\sigma^{n(1/q-1/2)}\left(\sigma^{\xi}+\int_{B(x^0,2\sigma)} \|u\|^{2n/(n-4)}dx\right)^{1/2} \leqslant$$

$$\leqslant c\sigma^{n(1/q-1/2)}[\phi(u,\,x^0,\,2\sigma)]^{1/2};$$

$$(2.13)\qquad \left(\int_{B(x^0,\,2\sigma)} \|Du\|^{\beta q}\,dx\right)^{1/q} \leqslant$$

$$\leqslant c\left(\int_{B(x^0,\,2\sigma)} \|Du\|^{2n/(n-2)}\,dx\right)^{\beta(n-2)/2n} \cdot \sigma^{n(1-\beta q(n-2)/2n)(1/q)} \leqslant$$

$$\leqslant c\sigma^{n/q}\left(1+\int_{B(x^0,\,2\sigma)} \|Du\|^{2n/(n-2)}\,dx\right)^{1/2} \leqslant c\sigma^{n(1/q-1/2)}[\phi(u,\,x^0,\,2\sigma)]^{1/2}.$$

Then, from the estimates (2.10), (2.11), (2.12), (2.13), we deduce

$$(2.14)\qquad \left(\int_{B(x^0,\,2\sigma)} \|b\|^{q}\,dx\right)^{1/q} \leqslant K\sigma^{n(1/q-1/2)}[\phi(u,\,x^0,\,2\sigma)]^{1/2}.$$

The estimate (2.7) easily follows from (2.9) and (2.14). ■

Moreover we can deduce the following

LEMMA 2.2. *If* $u \in H^2(B(x^0,\sigma),\mathbb{R}^N)$ *with* $\sigma \in (0,1)$ *is a solution to the system* (2.1) *and if the assumptions* (1.2), (1.3) *with* $f \in L^p(\Omega)$, $p > n$

and (A) *hold, then there exists* $\overline{q}$

$$2 < \overline{q} \leqslant \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \overline{q})$, $\forall \tau \in (0, 1)$, *one has*

$$\left(\int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} + \|Du\|^{2n/(n-2)}\right) dx \leqslant \tag{2.15}$$

$$\leqslant c\tau^n \left(\int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx + \int_{B(x^0, \sigma)} \|Du\|^{2n/(n-2)} dx\right) +$$

$$+ c\sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx$$

where c *does not depend on* σ *and* τ.

PROOF. Let $P = (P_1, P_2, \ldots, P_N)$ be the polynomial vector of degree $\leqslant 1$ such that

$$\int_{B(x^0, \sigma)} D^\alpha (u - P) = 0 \qquad \forall \alpha, \ |\alpha| \leqslant 1 . \tag{2.16}$$

We obtain that

$$\int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} dx \leqslant c(\tau\sigma)^n \|P\|^{2n/(n-4)} + \int_{B(x^0, \tau\sigma)} \|u - P\|^{2n/(n-4)} dx \tag{2.17}$$

and hence taking into account (2.16) we get

$$(\tau\sigma)^n \|P\|^{2n/(n-4)} \leqslant \tau^n (\|u\|_{L^{2n/(n-4)}(B(x^0, \sigma))})^{2n/(n-4)} + c(u)\, \tau^n \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx . \tag{2.18}$$

Now, by the theorem of Poincaré, we have

$$\left(\int_{B(x^0, \tau\sigma)} \|u - P\|^{2n/(n-4)} dx\right)^{(n-4)/2n} \leqslant c \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx\right)^{1/2} \tag{2.19}$$

and then from the estimates (2.17), (2.18), (2.19) we obtain

$$(2.20)\qquad \int_{B(x^0,\,\tau\sigma)} \|u\|^{2n/(n-4)}\,dx \leqslant c\tau^n\Big(\int_{B(x^0,\,\sigma)} \|u\|^{2n/(n-4)}\,dx\Big)+$$

$$+c(u)\,\tau^n\int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx + c\Big(\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\Big)^{n/(n-4)};$$

Since

$$(2.21)\qquad \Big(\int_{B(x^0,\sigma)} \|H(u)\|^2 dx\Big)^{n/(n-4)} \leqslant \Big(\int_{B(x^0,\sigma)} \|H(u)\|^2 dx\Big)\,\sigma^{n(1-2/q)4/(n-4)}.$$

$$\cdot\Big[\int_{B(x^0,\,d_{x^0})} \|H(u)\|^q\,dx\Big]^{8/q(n-4)} \leqslant c(u,\,q)\,\sigma^{2(1-2/q)}\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\,,$$

from (2.20) and (2.21), we have

$$(2.22)\qquad \int_{B(x^0,\,\tau\sigma)} \|u\|^{2n/(n-4)}\,dx \leqslant c\tau^n\Big(\int_{B(x^0,\,\sigma)} \|u\|^{2n/(n-4)}\,dx\Big)+$$

$$+c(u)\,\tau^n\int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx + c(u,\,q)\,\sigma^{2(1-2/q)}\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\,.$$

Moreover being

$$(2.23)\qquad \int_{B(x^0,\,\tau\sigma)} \|Du\|^{2^*}\,dx \leqslant$$

$$\leqslant c\Big(\int_{B(x^0,\,\tau\sigma)} \|Du-(Du)_\sigma\|^{2^*}\,dx + \int_{B(x^0,\,\tau\sigma)} \|(Du)_\sigma\|^{2^*}\,dx\Big),$$

and taking into account that

$$(2.24)\qquad \int_{B(x^0,\,\tau\sigma)} \|(Du)_\sigma\|^{2^*}\,dx \leqslant \tau^n\,\|Du\|^{2^*}_{L^{2^*}(B(x^0,\,\sigma))}$$

and that, in virtue of (3.19) of [2],

$$(2.25)\qquad \int_{B(x^0,\tau\sigma)} \|Du-(Du)_\sigma\|^{2^*}dx \leqslant c\left(\int_{B(x^0,\sigma)} \|H(u)\|^2 dx\right)^{2^*/2} \leqslant$$

$$\leqslant c\left(\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\right)\sigma^{n(1-2/q)(2/(n-2))}\left[\int_{B(x^0,\,d_{x^0})} \|H(u)\|^q\,dx\right]^{4/q(n-2)} \leqslant$$

$$\leqslant c(u,\,q)\,\sigma^{2(1-2/q)}\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\,,$$

we obtain

$$(2.26)\qquad \int_{B(x^0,\,\tau\sigma)} \|Du\|^{2^*}\,dx \leqslant c\tau^n\int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx\,+$$

$$+\,c(u,\,q)\,\sigma^{2(1-2/q)}\int_{B(x^0,\,\sigma)} \|H(u)\|^2\,dx\,.$$

Therefore from (2.22) and (2.26) we deduce (2.15). ■

3. Partial Hölder continuity of the vector Du.

Let $u\in H^2(\Omega,\,\mathbb{R}^N)$ be a solution to the non varational system

$$(3.1)\qquad a(x,\,u,\,Du,\,H(u)) = b(x,\,u,\,Du)$$

where $a(x,\,u,\,\mu,\,\xi)$ and $b(x,\,u,\,\mu)$ are vectors of $\mathbb{R}^N$ with the following properties:

(3.2) $b(x,\,u,\,\mu)$ is measurable in x, continuous in $(u,\,\mu)$ and such that $\forall u\in\mathbb{R}^N$, $\forall\mu\in\mathbb{R}^{nN}$ and for x a.e. in Ω

$$\|b(x,\,u,\,\mu)\| \leqslant c\{f(x)+\|u\|^\alpha+\|\mu\|^\beta\}\,,$$

with $f\in L^p(\Omega,\,\mathbb{R}^n)$, $p>n$ and with $1\leqslant\alpha<n/(n-4)$, $1\leqslant\beta<$ $<n/(n-2)$;

(3.3) $a(x, u, \mu, \xi)$ is continuous in (x, u, μ), of class C^1 in ξ, with derivatives $\partial a/\partial \xi_{ij}$ (6) uniformly continuous and bounded in $\Omega \times \mathbb{R}^N \times \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2 N}$ and such that

$$a(x, u, \mu, 0) = 0\,,$$

(A) there exist three positive constants $\overline{\alpha}, \overline{\gamma}$ and $\overline{\delta}$ with $\overline{\gamma} + \overline{\delta} < 1$, such that, $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2 N}$ and for almost every $x \in \Omega$

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \overline{\alpha}[a(x, u, \mu, \tau + \eta) - a(x, u, \mu, \eta)] \right\|^2 \leqslant$$

$$\leqslant \overline{\gamma} \|\tau\|^2 + \overline{\delta} \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2,$$

(B) there exists a non negative function $\omega(t)$, defined for $t \geqslant 0$, continuous, bounded, concave, non decreasing with $\omega(0) = 0$ such that $\forall x, y \in \Omega$, $\forall u, v \in \mathbb{R}^N$, $\forall \mu, \overline{\mu} \in \mathbb{R}^{nN}$ and $\forall \xi, \tau \in \mathbb{R}^{n^2 N}$:

$$\|a(x, u, \mu, \xi) - a(y, v, \overline{\mu}, \xi)\| \leqslant \omega(d^2(x, y) + \|u - v\|^2 + \|\mu - \overline{\mu}\|^2) \cdot \|\xi\|,$$

$$\left\| \frac{\partial a(x, u, \mu, \xi)}{\partial \xi} - \frac{\partial a(x, u, \mu, \tau)}{\partial \xi} \right\| \leqslant \omega(\|\xi - \tau\|^2)\,(7).$$

Let us start by showing the following

LEMMA 3.1. *If* $u \in H^2(\Omega, \mathbb{R}^N)$ *is a solution to the system* (3.1) *and if the assumptions* (3.2) *and* (3.3) *hold, then* $\forall B(x^0, \sigma) \subset\subset \Omega$, *with* $\sigma < 2$, $\forall \tau \in (0, 1)$ *and* $\forall \varepsilon \in (0, n - 2 - 2(1/p + 1/q)]$, *where* $q \in (2, \overline{q})$ (8), *it results*:

$$(3.4) \qquad \phi(u, x^0, \tau\sigma) \leqslant A\phi(u, x^0, \sigma)\{\tau^\lambda + \sigma^{2(1-2/q)} +$$

$$+ [\omega(c\sigma^{2-n}\phi(u, x^0, \sigma))]^{1-2/q} + \left[\omega \fint_{B(x^0, \sigma)} \|H(u) - (H(u))_\sigma\|^2 d\sigma\right]^{1-2/q},$$

(6) $\dfrac{\partial a(x, u, \mu, \xi)}{\partial \xi_{ij}} = \left\{ \dfrac{\partial a^h(x, u, \mu, \xi)}{\partial \xi_{ij}^k} \right\}$, $h, k = 1, 2, \ldots, N.$

(7) $\dfrac{\partial a(x, u, \mu, \eta)}{\partial \xi} = \left\{ \dfrac{\partial a(x, u, \mu, \eta)}{\partial \xi_{ij}} \right\}$, $i, j = 1, 2, \ldots, n.$

(8) $\overline{q}$ is the constant (> 2) that appears in theorem 2.1.

where

$$\lambda = n\left(1 - \frac{2}{p}\right) - \varepsilon ,$$

and

$$\phi(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0, \sigma)} [\|u\|^{2n/(n-4)} + \|D(u)\|^{2^*} + \|H(u)\|^2]\, dx \tag{3.5}$$

with

$$\xi = n\left(1 - \frac{2}{p}\right).$$

PROOF. Let us fix the ball $B(x^0, \sigma)$, with $\sigma < 1$, such that $B(x^0, 2\sigma) \subset\subset \Omega$ and let us set:

$$\frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi_{ij}} = \left\{ \int_0^1 \frac{\partial a^h(x, u, \mu, t\eta)}{\partial \xi_{ij}^k} dt \right\} \qquad h, k = 1, 2, \ldots, N$$

$$\frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi} = \left\{ \frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi_{ij}} \right\} \qquad i, j = 1, 2, \ldots, n .$$

In $B(x^0, \sigma)$ the system (3.1) can also be written in the following form:

$$\begin{aligned} &a(x^0, u_\sigma, (Du)_\sigma, H(u)) = \\ &= [-a(x, u, Du, H(u)) + a(x^0, u_\sigma, (Du)_\sigma, H(u))] + b(x, u, Du) = \\ &\qquad = B_1 + b(x, u, Du). \end{aligned} \tag{3.6}$$

On the other hand, denoting by $a^h(x^0, u_\sigma, (Du)_\sigma, \eta)$, $h = 1, 2, \ldots, N$, the $h-th$ component of the vector $a(x^0, u_\sigma, (Du)_\sigma, \eta)$ one gets:

$$a^h(x^0, u_\sigma, (Du)_\sigma, \eta) = a^h(x^0, u_\sigma, (Du)_\sigma, \eta) - a^h(x^0, u_\sigma, (Du)_\sigma, 0) =$$

$$= \sum_{i,j=1}^{n} \sum_{k=1}^{N} \int_0^1 \frac{\partial a^h(x^0, u_\sigma, (Du)_\sigma, t\eta)}{\partial \xi_{ij}^k} dt \eta_{ij}^k \qquad h = 1, 2, \ldots, N$$

from which

$$a(x^0, u_\sigma, (Du)_\sigma, \eta) = \sum_{i,j=1}^{n} \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, \eta)}{\partial \xi_{ij}} \eta_{ij}.$$

Hence, from (3.6) the system (3.1) can be written in the following form:

$$\sum_{i,j=1}^{n} \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, H(u))}{\partial \xi_{ij}} D_{ij} u = B_1 + b(x, u, Du)$$

or, equivalently

$$(3.7) \qquad \sum_{i,j=1}^{n} \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} u =$$

$$= \sum_{i,j=1}^{n} \left(- \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, H(u)_\sigma)}{\partial \xi_{ij}} + \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} \right) D_{ij} u +$$

$$+ B_1 + b(x, u, Du) = B_2 + B_1 + b(x, u, Du).$$

Letting w to be solution in $B(x^0, \sigma)$ to the elliptic Dirichlet problem

$$(3.8) \qquad \begin{cases} w \in H_0^2(B(x^0, \sigma), \mathbb{R}^N) \\ \displaystyle\sum_{i,j=1}^{n} \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} w = B_2 + B_1 \end{cases}$$

it results, in $B(x^0, \sigma)$, $u = w + v$ where $v \in H^2(B(x^0, \sigma), \mathbb{R}^N)$ is solution to the linear system

$$(3.9) \qquad \sum_{i,j=1}^{n} \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} v = b(x, u, Du).$$

It is known (see [1]) that for v we have the following estimate

$$\int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\tau^n \int_{B(x^0, \sigma)} \|H(v)\|^2 dx +$$

$$+ c \int_{B(x^0, \sigma)} \|b(x, u, Du)\|^2 dx, \qquad \forall \tau \in (0, 1)$$

from which and in virtue of assumption (3.2), it follows

$$\int_{B(x^0,\,\tau\sigma)} \|H(v)\|^2\,dx \leqslant c\tau^n \int_{B(x^0,\,\sigma)} \|H(v)\|^2\,dx + \tag{3.10}$$

$$+ c \int_{B(x^0,\,\sigma)} |f|^2\,dx + \int_{B(x^0,\,\sigma)} \|u\|^{2\alpha}\,dx + \int_{B(x^0,\,\sigma)} \|Du\|^{2\beta}\,dx$$

where c does not depend on x^0, τ and σ.

Moreover, setting

$$F(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0,\,\sigma)} \|u\|^{2n/(n-4)}\,dx + \int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx\,,$$

being

$$\int_{B(x^0,\,\sigma)} |f|^2\,dx \leqslant c\sigma^\xi \|f\|^2_{0,\,p,\,\Omega} \leqslant \bar{c}\sigma^\xi \tag{3.11}$$

$$\int_{B(x^0,\,\sigma)} \|u\|^{2\alpha}\,dx \leqslant \sigma^{n-(n-4)\alpha} \left(\int_{B(x^0,\,\sigma)} \|u\|^{2n/(n-4)}\,dx \right)^{(n-4)(\alpha/n)} \leqslant \tag{3.12}$$

$$\leqslant \sigma^\xi + \int_{B(x^0,\,\sigma)} \|u\|^{2n/(n-4)}\,d\sigma$$

$$\int_{B(x^0,\,\sigma)} \|Du\|^{2\beta}\,dx \leqslant \sigma^{n-\beta(n-2)} \left(\int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx \right)^{2\beta/2^*} \leqslant \tag{3.13}$$

$$\leqslant \sigma^\xi + \int_{B(x^0,\,\sigma)} \|Du\|^{2^*}\,dx\,,$$

the estimate (3.10) can be written in the following form

$$\int_{B(x^0,\,\tau\sigma)} \|H(v)\|^2\,dx \leqslant c\tau^\xi \int_{B(x^0,\,\sigma)} \|H(v)\|^2\,dx + cF(u, x^0, \sigma)\,. \tag{3.14}$$

On the other hand from (2.15) we obtain, $\forall \tau \in (0, 1)$:

$$F(u, x^0, \tau\sigma) \leqslant \tau^\xi F(u, x^0, \sigma) + c(u, q)\,\sigma^{2(1-2/q)}\,\phi(u, x^0, \sigma)\,. \tag{3.15}$$

In virtue of the estimates (3.14) and (3.15), using the lemma 1, II of

Chap. I of [2], we get

$$\forall \varepsilon \in \left(0, (n-2) - 2\left(\frac{n}{p} - \frac{2}{q}\right)\right], \qquad \forall \tau \in (0, 1), \forall \sigma \in (0, 1)$$

$$\int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\left\{ \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + F(u, x^0, \sigma)\right\} \cdot$$

$$\cdot \tau^{n(1-2/p)-\varepsilon} + K(\tau\sigma)^{2(1-2/q)} \phi(u, x^0, \sigma),$$

from which we obtain $\forall \lambda \in [2(1-2/q), \xi)$, $\forall \tau \in (0, 1)$ and $\forall \sigma \in (0, 1)$

$$\text{(3.16)} \qquad \int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\tau^\lambda \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + c(\tau^\lambda + \sigma^{2(1-2/q)}\phi(u, x^0, \sigma).$$

The function w satisfies the following estimate:

$$\text{(3.17)} \qquad \int_{B(x^0, \sigma)} \|H(w)\|^2 dx \leq c\left(\int_{B(x^0, \sigma)} \|\mathcal{B}_1\|^2 dx + \int_{B(x^0, \sigma)} \|\mathcal{B}_2\|^2 dx\right).$$

Now let us estimate the integrals in the right hand side of (3.17).

From assumption (3.2), (3.3) and (2.6) we have

$$\text{(3.18)} \qquad \int_{B(x^0,\sigma)} \|\mathcal{B}_1\|^2 dx \leq$$

$$\leq \int_{B(x^0, \sigma)} \omega^2(\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) \cdot \|H(u)\|^2 dx \leq$$

$$\leq c\sigma^n \left(\fint_{B(x^0, \sigma)} \|H(u)\|^q dx\right)^{2/q} \cdot$$

$$\cdot \left(\fint_{B(x^0, \sigma)} \omega(\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) dx\right)^{1-2/q} \leq$$

$$\leq \left[\omega\left(\fint_{B(x^0, \sigma)} (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) dx\right)\right]^{1-2/q} \cdot$$

$$\cdot \left[c\left(\sigma^n \fint_{B(x^0, 2\sigma)} \|H(u)\|^2 + \sigma^n \left(\fint_{B(x^0, 2\sigma)} \|b\|^q dx\right)^{2/q}\right)\right].$$

Now we observe that:

$$(3.19)\qquad \fint_{B(x^0,\,\sigma)} \|u - u_\sigma\|^2\, dx \leqslant$$

$$\leqslant \sigma^4 \left(\frac{\fint_{B(x^0,\,\sigma)} \|Du\|^{2^*}\, dx}{\sigma^{2^*}} \right)^{2/2^*} \leqslant \sigma^2 \left(1 + \fint_{B(x^0,\,\sigma)} \|Du\|^{2^*}\, dx\right)$$

$$(3.20)\qquad \fint_{B(x^0,\,\sigma)} \|Du - (Du)_\sigma\|^2\, dx \leqslant c\sigma^2 \fint_{B(x^0,\,\sigma)} \|H(u)\|^2\, dx\,.$$

Hence

$$(3.21)\qquad \omega\left[\fint_{B(x^0,\sigma)} (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2)\, dx\right] \leqslant \omega(c\sigma^{2-n}\phi(u, x^0, \sigma))\,.$$

Moreover, from (2.13), we have

$$(3.22)\qquad \sigma^n \left(\fint_{B(x^0,\,2\sigma)} \|b\|^q\, dx\right)^{2/q} \leqslant c(u,\, f)\, \phi(u,\, x^0,\, 2\sigma)\,.$$

Therefore, from (3.18)-(3.22) we deduce

$$(3.23)\qquad \int_{B(x^0,\,\sigma)} \|\mathcal{B}_1\|^2\, dx \leqslant \left[\omega(c\sigma^{2-n}\phi(u,\, x^0,\, 2\sigma))\right]^{1-2/q} \cdot c\phi(u,\, x^0,\, 2\sigma)\,.$$

Similarly we obtain

$$(3.24)\qquad \int_{B(x^0,\,\sigma)} \|\mathcal{B}_2\|^2\, dx \leqslant$$

$$\leqslant c\left[\omega\left(\int_{B(x^0,\,2\sigma)} (\|H(u) - (H(u))_{2\sigma}\|^2\, dx)\right)\right]^{1-2/q} \phi(u,\, x^0,\, 2\sigma)\,.$$

Finally, from (3.17) (3.23) and (3.24) we get

$$(3.25)\qquad \int_{B(x^0,\sigma)} \|H(w)\|^2 dx \leq c\phi(u,x^0,2\sigma)\cdot$$

$$\cdot\left\{\left[\omega(c\sigma^{2-n}\phi(u,x^0,2\sigma))\right]^{1-2/q} + \left[\omega\left(\fint_{B(x^0,2\sigma)} \|H(u)-(H(u))_{2\sigma}\|^2 dx\right)\right]^{1-2/q}\right\}.$$

Since $u = v + w$ it follows, from (3.16) and (3.25) that $\forall \tau \in (0, 1)$ and $\forall \lambda \in [2(1-2/q), \xi)$ we have

$$(3.26)\qquad \int_{B(x^0,\tau\sigma)} \|H(u)\|^2 dx \leq c\phi(u, x^0, 2\sigma)\cdot$$

$$\cdot\left\{\tau^\lambda + \sigma^{2(1-2/q)} + \left[\omega(c\sigma^{2-n}\phi(u, x^0, 2\sigma))\right]^{1-2/q} + \right.$$

$$\left. + \left[\omega\left(\fint_{B(x^0,2\sigma)} \|H(u)-(H(u))_{2\sigma}\|^2 dx\right)\right]^{1-2/q}\right\}.$$

From the estimates (3.15) and (3.26) we achieve $\forall \tau \in (0, 1)$, $\forall \sigma \in \in (0, 1)$ and $\forall \lambda \in [2(1-2/q), \xi)$

$$(3.27)\qquad \phi(u, x^0, \tau\sigma) \leq c\phi(u, x^0, 2\sigma)\cdot$$

$$\cdot\left\{\tau^\lambda + \sigma^{2(1-2/q)} + \left[\omega(c\sigma^{2-n}\phi(u, x^0, 2\sigma)\right]^{1-2/q} + \right.$$

$$\left. + \left[\omega\left(\fint_{B(x^0,2\sigma)} \|H(u)-(H(u))_{2\sigma}\|^2 dx\right)\right]^{1-2/q}\right\}.$$

This estimate is easily true for $\tau \in [1, 2)$ and thus the lemma is proved. ■

Let us set:

$$\mathcal{B}_1 = \left\{x \in \Omega : \lim_{\sigma\to 0}{}'' \fint_{B(x,\sigma)} \|H(u)-(H(u))_\sigma\|^2 dy > 0\right\}$$

$$\mathcal{B}_2 = \left\{x \in \Omega : \lim_{\sigma\to 0}{}' \ \sigma^{2-n}\phi(u, x, \sigma) > 0\right\}.$$

By a well known property of the Lebesgue integral we have

$$mis\ \mathcal{B}_1 = 0$$

and taking into account a well known theorem of Giusti [2], we obtain

$$\mathcal{H}_{n-2}(\mathcal{B}_2) = 0$$

where $\mathcal{H}_\gamma$ is the γ – dimensional Hausdorff measure.

Hence the set $\mathcal{B}_1 \cup \mathcal{B}_2$ has measure zero.

Now, reasoning exactly as in theorem 5.1 of [3], it is easy to prove

LEMMA 3.2. *If* $u \in H^2(\Omega, \mathbb{R}^N)$ *is a solution to the system* (3.1) *if the assumptions* (3.2) *and* (3.3) *hold, then, for every fixed* $\varepsilon \in (0, 1 - n/p)$, *it is possible to associate to every* $x^0 \in \Omega \setminus \mathcal{B}_1 \cup \mathcal{B}_2$ *a ball* $B(x^0, R_{x^0}) \subset \subset \Omega \setminus \mathcal{B}_2$ *and a positive number* σ_ε *such that,* $\forall t \in (0, 1)$ *and* $\forall y \in \in B(x^0, R_{x^0})$

$$\phi(u, y, t\sigma_\varepsilon) \leqslant (1 + A) t^{n(1-2/p)-2\varepsilon} \phi(u, y, \sigma_\varepsilon)$$

and hence (see [2])

$$H(u) \in L^{2, n(1-2/p)-2\varepsilon}(B(x^0, R_{x^0}), \mathbb{R}^{n^2 N}))$$

$$D(u) \in \mathcal{L}^{2, n(1-2/p)-2\varepsilon+2}(B(x^0, R_{x^0}), \mathbb{R}^{nN}) .$$

From lemma (3.2) the following result of partial Hölder continuity for Du easily follows.

THEOREM 3.1. *If* $u \in H^2(\Omega, \mathbb{R}^N)$ *is a solution to the system* (3.1) *and if the hypotheses* (3.2) (3.3) *are fulfilled, then there exists a set* $\mathcal{B}_0$, *closed in* $\mathcal{B}$ (9) *with*

$$\mathcal{B}_2 \subset \mathcal{B}_0 \subset \mathcal{B}_1 \cup \mathcal{B}_2$$

such that

$$Du \in C^{0, \alpha}(\Omega \setminus \mathcal{B}_0, \mathbb{R}^{nN}), \qquad \forall \alpha < 1 - \frac{n}{p} .$$

(9) In particular $m(\mathcal{B}_0) = 0$.

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