

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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balance model in climatology**

Rendiconti del Seminario Matematico della Università di Padova,
tome 103 (2000), p. 181-192

http://www.numdam.org/item?id=RSMUP_2000__103__181_0

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Periodic Solutions for a Sellers Type Diffusive Energy Balance Model in Climatology.

MAURIZIO BADIO (*)

1. Introduction.

In this paper we consider the mathematical treatment of a time evolution model of the temperature on the Earth surface, obtained by an energy balance model. Climate models were independently introduced in 1969 by Budyko [1] and Sellers [7]. These models have a global character i.e. refer to all Earth and involves a relatively long-time scales with respect to the prediction time.

We want to study the existence of periodic solutions for the nonlinear parabolic problem

$$(P) \quad \begin{cases} u_t - (\varrho(x) |u_x|^{p-2} u_x)_x = R_a(x, t, u) - R_e(x, u), & \text{in } Q := (-1, 1) \times \mathbb{R}_+ \\ (\varrho(x) |u_x|^{p-2} u_x)(\pm 1, t) = 0, & \forall t > 0, \quad p \geq 2 \end{cases}$$

where

$$(1) \quad \varrho(x) := k(1 - x^2), \quad \forall x \in [-1, 1], \quad k > 0;$$

$$(2) \quad \begin{cases} R_a(x, t, u) := Q(x, t) \beta(u), & \text{where } Q(x, t) \geq 0, \quad Q \in C([-1, 1] \times \mathbb{R}_+), \\ Q(x, \cdot) \text{ is 1-periodic } \forall x \in [-1, 1] \text{ and} \\ \beta \text{ is a nonnegative, bounded nondecreasing function for any } u \in \mathbb{R} \end{cases}$$

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Partially supported by G.N.A.F.A. and M.U.R.S.T. 40% Equazioni Differenziali.

$$(3) \quad \begin{cases} R_e \in C([-1, 1] \times \mathbb{R}), & R_e(x, \cdot) \text{ is a strictly increasing odd} \\ \text{function for } x \in [-1, 1], & R_e(x, 0) = 0, \quad R_e(x, s) \geq Bs - A \\ \text{for any } x \in [-1, 1], & \forall u \geq 0 \text{ and } B, A \text{ positive constants} \end{cases}$$

In (2), $Q(x, t)$ describes the incoming solar radiation flux and the assumption $Q(x, t) \geq 0$, allows to consider also the polar night phenomena. Function R_a represents the fraction of the solar energy absorbed by the Earth, clearly it depends on the albedo or reflexivity of the Earth surface.

The albedo function $\alpha(u)$ is usually taken such that $0 < \alpha(u) < 1$, thus the coalbedo function $\beta(u) := 1 - \alpha(u)$, represents the fraction of the absorbed light.

In (3), function R_e represents the emitted energy by the Earth to the outer space. In the balance of energy models, one considers a rapid variation of the coalbedo function near to the critic temperature $u = -10^\circ \text{C}$. In this paper, we want to study the existence of periodic solutions for the Sellers model. For his model, Sellers proposed as coalbedo a function allowing a partially ice-free zone, $u_i < u < u_w$. An example of such function is

$$\beta(u) = \begin{cases} a_w, & \text{if } u_w < u \\ a_i + ((u - u_i)/(u_w - u_i))(a_w - a_i), & \text{if } u_i \leq u \leq u_w \\ a_i, & \text{if } u < u_i \end{cases}$$

where a_i is the «ice» coalbedo (~ 0.38), a_w is the «ice-free» coalbedo (~ 0.71), u_i and u_w are fixed temperatures very close to -10°C and R_e is taken of the form $R_e(x, u) = B|u|^3 u$. Our interest in the periodic forcing term is motivated by the seasonal variation of the incoming solar radiation flux during one year. As usual, $u(x, t)$ represents the mean annual temperature averaging on the latitude circles around the Earth (denoted by $x = \sin \phi$, where ϕ is the latitude).

The diffusion coefficient ρ in (P), degenerates at $x = \pm 1$ and for $p > 2$ the equation in (P) degenerates also on the set of points where $u_x = 0$.

To prove the existence of periodic solutions for (P), we consider an initial-boundary problem associated to (P)

$$(P_1) \quad \begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x = R_a(x, t, u) - R_e(x, u), & \text{in } [-1, 1] \times (0, T) \\ (\rho(x)|u_x|^{p-2}u_x)(\pm 1, t) = 0, & t \in (0, T) \\ u(x, 0) = u_0(x), & x \in (-1, 1) \end{cases}$$

with $T \geq 1$ arbitrary and

$$(4) \quad u_0 \in L^\infty(-1, 1).$$

The problem (P_1) is a model used in climatology to describe the climate energy balance models. Since (P_1) degenerates at $x = \pm 1$ and where $u_x = 0$, we cannot expect that (P_1) has classical solutions (see [3] for $\varrho = 1$ and $R_a = 0$), thus we shall deal with a weak solution for (P_1) .

It was proved in [2] that if $u_0 \in L^\infty(-1, 1)$ there exists at least one bounded weak solution for (P_1) .

The assumption

$$(5) \quad \begin{cases} \text{There exists a constant } L > 0 \text{ such that} \\ s \rightarrow R_a(x, t, s) - R_e(x, s) - Ls \text{ is nonincreasing} \end{cases}$$

shall be utilized to prove the uniqueness of the bounded weak solution for (P_1) . Because of the degenerate diffusion coefficient $\varrho(x)$, the natural energy space associated to (P_1) , is the one defined by

$$V := \{w \in L^2(-1, 1) : w_x \in L^p(-1, 1; \varrho)\}$$

where

$$L^p(-1, 1; \varrho) := \left\{ v : \|v\|_{L^p(-1, 1; \varrho)} := \left(\int_{-1}^1 \varrho(x) |v(x)|^p dx \right)^{1/p} < +\infty \right\}.$$

V is a separable and reflexive Banach space with the norm

$$\|v\|_v := \|v\|_{L^2(-1, 1)} + \|v_x\|_{L^p(-1, 1; \varrho)}.$$

To prove the existence of periodic solutions of the problem (P) , we construct a subsolution $\underline{v}(x)$ and a supersolution $\overline{u}(x)$ of (P_1) .

Then, we consider the Poincaré map F associated to (P_1) i.e. the operator assigning to every initial data of the ordered interval $[\underline{v}(x), \overline{u}(x)]$ the solution of (P_1) after 1-period. One proves that F is continuous, compact and pointwise increasing. By the Schauder fixed point theorem, there exists at least a fixed point for F .

This fixed point is a periodic solution for the problem (P) .

Finally, one shows that (P) has a smallest and a greatest periodic solution.

The existence of periodic solutions for (P) both on a Riemannian manifold without boundary and for the Budyko type mode, (β is

a bounded maximal monotone graph of \mathbb{R}^2 and $R_e(x, u) = Bu + A$, $B > 0$, $A > 0$), shall be the argument of a forthcoming paper.

In the nondegenerate case i.e. $p = 2$, the study of the periodic case for the climate energy balance models has been carried out by [4-5].

2. Existence and uniqueness of the solution.

DEFINITION 1. For a bounded weak solution to (P_1) we mean a function $u \in C([0, T]; L^2(-1, 1)) \cap L^\infty(Q_T) \cap L^p(0, T; V)$, $(Q_T := (-1, 1) \times (0, T))$ such that

$$\begin{aligned} & \int_{-1}^1 u(x, T) v(x, T) dx - \int_0^T \int_{-1}^1 u(x, t) v_t(x, t) dx dt + \\ & + \int_0^T \int_{-1}^1 \varrho(x) |u_x(x, t)|^{p-2} u_x(x, t) v_x(x, t) dx dt = \\ & = \int_0^T \int_{-1}^1 (Q(x, t) \beta(u(x, t)) - R_e(x, u(x, t)) v(x, t) dx dt + \\ & + \int_{-1}^1 u_0(x) v(x, 0) dx \end{aligned}$$

$\forall v \in L^p(0, T; V) \cap L^\infty(Q_T)$ such that $v_t \in L^{p'}(0, T; V')$.

DEFINITION 2. For an 1-periodic bounded weak solution to (P) , we mean a function $u \in C(\mathbb{R}_+; L^2(-1, 1)) \cap L^\infty(Q)$ such that $u \in L^p_{loc}(\mathbb{R}_+; V)$, $u(x, t + 1) = u(x, t)$, $u_t \in L^{p'}_{loc}(\mathbb{R}_+; V')$ and satisfies $\forall I := [t_0, t_1]$ the following equality

$$\begin{aligned} & \int_I \langle u_t, z \rangle dt + \int_{I-1}^1 \int_{-1}^1 (\varrho(x) |u_x|^{p-2} u_x) z_x dx dt - \\ & - \int_{I-1}^1 \int_{-1}^1 (R_a(x, t, u) - R_e(x, u)) z(x, t) dx dt = 0 \end{aligned}$$

$\forall z \in L^p(I; V) \cap L^\infty((-1, 1) \times I)$.

In [2] has been proved, by means of a regularization argument, the existence of solutions to (P_1) . This method consists to replace $\varrho(x)$ by

$$(6) \quad \varrho_\varepsilon(x) = \varrho(x) + \varepsilon.$$

In order to approximate u by classical solutions of a related problem to (P_1) , we replace the data u_0, β, Q and R_e by C^∞ functions $u_{0,m}, \beta_\varepsilon, Q_n, R_{e,k}$ such that $u_0(\pm 1) = 0, \|u_{0,m}\|_{L^\infty(-1,1)} \leq \|u_0\|_{L^\infty(-1,1)}$ and $u_{0,m} \rightarrow u_0$ in $L^2(-1, 1)$ as $m \rightarrow \infty, Q_n \rightarrow Q$ in $C(\bar{Q}_T), Q_n$ 1-periodic in t .

$R_{e,k}$ satisfies (3), $R_{e,k}(\cdot, u) \rightarrow R_e(\cdot, u)$ in $C([-1, 1])$ for any fixed $u \in \mathbb{R}$.

Then, given ε, m, n and k positive constants, we consider the approximating problem for $T \geq 1$

$$(P_\varepsilon) \quad \begin{cases} u_t - (\varrho_\varepsilon(x)|u_x|^{p-2}u_x)_x - \varepsilon u_{xx} = Q_n(x, t)\beta_\varepsilon(u) - R_{e,k}(x, u), & \text{in } Q_T \\ \varrho_\varepsilon(x)(|u_x|^{p-2}u_x + \varepsilon u_x)(\pm 1, t) = 0, & \text{in } (0, T) \\ u(x, 0) = u_{0,m}(x), & \text{in } (-1, 1). \end{cases}$$

The problem (P_ε) is now uniformly parabolic and by well-known results (see [6]) has a unique classic solution $u_{\varepsilon, m, n, k}$.

Moreover, it has been proved in [2] that

$$(7) \quad \|u_{\varepsilon, m, n, k}\|_{L^\infty(Q_T)} \leq C$$

$$(8) \quad \|\varrho_\varepsilon(u_{\varepsilon, m, n, k})\|_{L^p(0, T; L^p(-1, 1))} \leq C$$

where C is a positive constant, independent of ε, m, n, k .

Using the a priori estimates, we can pass to the limit as ε goes to zero and $m, n, k \rightarrow \infty$ and we get

THEOREM 1 ([2]). *With assumptions (1)-(3) for any $u_0 \in L^\infty(-1, 1)$, there exists at least a bounded weak solution to (P_1) .*

The uniqueness of the bounded weak solution for (P_1) , is obtained using the assumption (5). In fact

THEOREM 2. *If (1)-(3) and (5) hold, for any $u_0 \in L^\infty(-1, 1)$ there exists a unique bounded weak solution for the problem (P_1) .*

PROOF. If by contradiction there exist two solutions u_1 and u_2 for (P_1) , multiplying by $(u_1 - u_2)^+ \in L^p(0, T; V)$

$$\begin{aligned} u_{1t} - u_{2t} - (\varrho(x) |u_{1x}|^{p-2} u_{1x})_x + (\varrho(x) |u_{2x}|^{p-2} u_{2x})_x = \\ = R_a(x, t, u_1) - R_e(x, u_1) - R_a(x, t, u_2) + R_e(x, u_2) \end{aligned}$$

and integrating on $(-1, 1)$, since $(u_1 - u_2)_t \in L^{p'}(0, T; V')$ (see [2]), one has

$$\begin{aligned} (9) \quad (1/2) \frac{d}{dt} \int_{-1}^1 (u_1 - u_2)^+{}^2 dx = \\ = - \int_{-1}^1 (\varrho(x) |u_{1x}|^{p-2} u_{1x} - \varrho(x) |u_{2x}|^{p-2} u_{2x})(u_1 - u_2)_x^+ dx + \\ + \int_{-1}^1 (R_a(x, t, u_1) - R_e(x, u_1) - R_a(x, t, u_2) + R_e(x, u_2))(u_1 - u_2)^+ dx. \end{aligned}$$

Since the operator $A(u_x) := \varrho(x) |u_x|^{p-2} u_x$ is nondecreasing, by (5) and integrating on $(0, t)$ we have

$$\begin{aligned} (10) \quad \int_{-1}^1 (u_1(x, t) - u_2(x, t))^+{}^2 dx \leq \int_{-1}^1 (u_{01}(x) - u_{02}(x))^+{}^2 dx + \\ + 2L \int_0^t \int_{-1}^1 (u_1(x, s) - u_2(x, s))^+{}^2 dx ds. \end{aligned}$$

By the Gronwall lemma, it follows the uniqueness of the solution.

3. Subsolutions-supersolutions.

We assume that

$$(11) \quad Q_1(x) \leq Q(x, t) \leq Q_2(x), \quad \text{with } Q_1, Q_2 \in C([-1, 1]), Q_1 \geq 0 \text{ and } Q_2 > 0.$$

We consider the stationary problems

$$(PS)_1 \quad \begin{cases} -(\varrho(x) |v_x|^{p-2} v_x)_x = Q_1(x) \beta(v) - R_e(x, v), & \text{in } (-1, 1) \\ (\varrho(x) |v_x|^{p-2} v_x)(\pm 1) = 0 \end{cases}$$

$$(PS)_2 \quad \begin{cases} -(\varrho(x) |u_x|^{p-2} u_x)_x = Q_2(x) \beta(u) - R_e(x, u), & \text{in } (-1, 1) \\ (\varrho(x) |u_x|^{p-2} u_x)_x(\pm 1) = 0 \end{cases}$$

A subsolution for $(PS)_1$ is given by the function

$$\underline{v}(x) = -10 - a|x|^{p^*} - b, \quad \forall x \in [-1, 1]$$

with $a < 0, b > 0, 10 < a + b$, suitable constants to be chosen later with $1/p + 1/p^* = 1$.

In fact

$$|\underline{v}_x(x)|^{p-2} \underline{v}_x(x) = (|a|p^*)^{p-1} |x| \operatorname{sgn} x, \quad ((p^* - 1)(p - 1) = 1).$$

Hence,

$$-(k(1 - x^2) |\underline{v}_x|^{p-2} \underline{v}_x)_x = -k(|a|p^*)^{p-1} (1 - 3x^2).$$

We want that

$$-k(|a|p^*)^{p-1} (1 - 3x^2) \leq Q_1(x) \beta(\underline{v}) - R_e(x, \underline{v}).$$

Since, $\underline{v}(x) \leq -10 - b - a$, we have by (3) that

$$\begin{aligned} Q_1(x) \beta(\underline{v}) - R_e(x, \underline{v}) &\geq -R_e(x, -10 - b - a) \geq \\ &\geq R_e(x, 10 + b + a) \geq B(10 + b + a) - A. \end{aligned}$$

Moreover, $-k(|a|p^*)^{p-1} (1 - 3x^2) \leq 2k(|a|p^*)^{p-1}$, therefore we choose a, b such that

$$2k(|a|p^*)^{p-1} \leq (10 + b + a) B - A, \quad \text{with } (10 + b + a) B > A.$$

A supersolution for $(PS)_2$ is given by the function

$$\bar{u}(x) = -10 + a|x|^{p^*} + b, \quad \forall x \in [-1, 1]$$

with a, b suitable constants, with $a < 0, b > 0, 10 < a + b$ as before, $1/p + 1/p^* = 1$.

In fact

$$|\bar{u}_x(x)|^{p-2} \bar{u}_x(x) = -(|a|p^*)^{p-1} |x|^{(p^*-1)(p-1)} \operatorname{sgn} x = -(|a|p^*)^{p-1} |x| \operatorname{sgn} x.$$

Hence,

$$-(k(1 - x^2) |\bar{u}_x|^{p-2} \bar{u}_x)_x = k(|a|p^*)^{p-1} (1 - 3x^2).$$

We require that

$$k(|a|p^*)^{p-1}(1-3x^2) \geq Q_2(x)\beta(\bar{u}) - R_e(x, \bar{u}).$$

Since,

$$\bar{u}(x) \geq -10 + a + b,$$

we have

$$Q_2(x)\beta(\bar{u}) - R_e(x, \bar{u}) \leq$$

$$\leq \tilde{Q}_2 M - R_e(x, -10 + a + b) \leq \tilde{Q}_2 M - (-10 + a + b) B + A,$$

because of (3), where $\tilde{Q}_2 := \max\{Q_2(x), x \in [-1, 1]\}$ and M is such that $\beta(u) \leq M$, for any $u \in \mathbb{R}$.

Moreover, $k(|a|p^*)^{p-1}(1-3x^2) \geq -2k(|a|p^*)^{p-1}$, therefore we want that a, b verify

$$2k(|a|p^*)^{p-1} \leq (-10 + a + b) B - (A + \tilde{Q}_2 M),$$

$$\text{with } (-10 + a + b) B > A + \tilde{Q}_2 M.$$

Now, it is possible to prove the following result

THEOREM 3. *If (1)-(5) and (11), hold the solution u of (P_1) with $u_0 \in [\underline{v}(x), \bar{u}(x)]$ verifies*

$$\underline{v}(x) \leq u(x, t) \leq \bar{u}(x), \quad \forall (x, t) \in Q_T$$

PROOF. Multiplying by $(u - \bar{u})^+$ and integrating on $(-1, 1)$, we obtain

$$\begin{aligned} (12) \quad & (1/2) \frac{d}{dt} \int_{-1}^1 (u - \bar{u})^+{}^2 dx \leq \\ & \leq - \int_{-1}^1 (\varrho(x) |u_x|^{p-2} u_x - \varrho(x) |\bar{u}_x|^{p-2} \bar{u}_x) (u - \bar{u})_x^+ dx + \\ & + \int_{-1}^1 (R_a(x, t, u) - R_e(x, u) - R_a(x, \bar{u}) + R_e(x, \bar{u})) (u - \bar{u})^+ dx, \end{aligned}$$

where $R_a(x, \bar{u}) = Q_2(x)\beta(\bar{u})$.

Since $A(u_x) := -\varrho(x) |u_x|^{p-2} u_x$ is a nondecreasing operator, we get

$$(13) \quad (1/2) \frac{d}{dt} \int_{-1}^1 (u - \bar{u})^{+2} dx \leq \int_{-1}^1 (R_a(x, t, u) - R_e(x, u) - R_a(x, t, \bar{u}) + R_e(x, \bar{u}) + R_a(x, t, \bar{u}) - R_a(x, \bar{u})) (u - \bar{u})^+ dx .$$

Now, (5) gives us

$$(14) \quad (1/2) \frac{d}{dt} \int_{-1}^1 (u - \bar{u})^{+2} dx \leq L \int_{-1}^1 (u - \bar{u})^{+2} dx + \int_{-1}^1 (Q(x, t) - Q_2(x)) \beta(\bar{u}) (u - \bar{u})^+ dx \leq L \int_{-1}^1 (u - \bar{u})^{+2} dx .$$

Integrating on $(0, t)$ and by the Gronwall lemma, one has

$$u(x, t) \leq \bar{u}(x), \quad \forall (x, t) \in Q_T$$

In a similar way one proves that

$$u(x, t) \geq \underline{v}(x), \quad \forall (x, t) \in Q_T.$$

If we denote with F the Poincaré map defined by

$$F(u_0(x)) = u(x, 1)$$

(u is the solution of (P_1)), to apply the Schauder fixed point theorem in the space $L^\infty(-1, 1)$, we need of a closed and convex set $K \subset L^\infty(-1, 1)$ and to show that

- i) $F(K) \subset K$;
- ii) $F|_K$ is continuous;
- iii) $F(K)$ is relatively compact in $L^\infty(-1, 1)$.

Define

$$K := \{w \in L^\infty(-1, 1) : \underline{v}(x) \leq w(x) \leq \bar{u}(x)\}$$

it is easy to prove that K is a closed, convex and nonempty set.

Now, i) it follows from the Theorem 3 because we have showed that $F[\underline{v}(x), \bar{u}(x)] \subset [\underline{v}(x), \bar{u}(x)]$.

LEMMA 4. *With the assumptions of the Theorem 3, let $u_{0n}, u_0 \in K$ be such that $u_{0n} \rightarrow u_0$ in $L^\infty(-1, 1)$ as $n \rightarrow \infty$. Then, if u_n (respectively) u are the solutions of (P_1) with initial data u_{0n} and u_0 respectively, we have that $u_n(x, t) \rightarrow u(x, t)$ in $L^\infty(-1, 1)$ as $n \rightarrow \infty$, $\forall t \in [0, T]$.*

PROOF. Subtracting member to member and multiplying by $\text{sgn}^+(u_n - u) \in V$, after an integration on Q_t we have

$$(15) \quad \int_0^t \int_{-1}^1 (u_n - u)_s \text{sgn}^+(u_n - u) \, dx \, ds - \\ - \int_0^t \int_{-1}^1 [(\varrho(x) |u_{nx}|^{p-2} u_{nx})_x - (\varrho(x) |u_x|^{p-2} u_x)_x] \text{sgn}^+(u_n - u) \, dx \, ds = \\ = \int_0^t \int_{-1}^1 (R_a(x, s, u_n) - R_e(x, u_n) - R_a(x, s, u) + R_e(x, u)) \text{sgn}^+(u_n - u) \, dx \, ds$$

by which

$$(16) \quad \int_{-1}^1 (u_n(t) - u(t))^+ \, dx - \\ - \int_{-1}^1 (u_{0n}(x) - u_0(x))^+ \, dx \leq L \int_0^t \int_{-1}^1 (u_n(s) - u(s))^+ \, dx \, ds .$$

The Gronwall lemma gives us

$$(17) \quad \int_{-1}^1 (u_n(t) - u(t))^+ \, dx \leq \exp(LT) \int_{-1}^1 |u_{0n}(x) - u_0(x)| \, dx .$$

Changing $u(t)$ with $u_n(t)$, one has

$$(18) \quad \int_{-1}^1 |u_n(t) - u(t)| \, dx \leq \exp(LT) \int_{-1}^1 |u_{0n}(x) - u_0(x)| \, dx .$$

Since u_{0n} converges in $L^\infty(-1, 1)$ to u_0 as $n \rightarrow \infty$ we have that $u_n(t)$ converges in $L^1(-1, 1)$ and a.e. to $u(t)$ as n goes to infinity.

As $u_n(x, t) \in K$, by the Lebesgue theorem, one has that $u_n(\cdot, t)$ strongly converges to $u(\cdot, t)$ in $L^p(-1, 1)$, $\forall 1 \leq p \leq +\infty$. This proves ii).

The proof that $F(u)$ is relatively compact follows by a result of [2], where it is showed that $V \subset L^\infty(-1, 1)$, with compact embedding, for $p > 2$.

Now, $F(K)$ is bounded in V and by the quoted result, it follows that $F(K)$ is relatively compact in $L^\infty(-1, 1)$. Then, by the Schauder fixed point theorem, there exists a fixed point for the Poincaré map F . This fixed point is a periodic solution for (P).

If together to (P_1) , we consider the problems

$$\begin{aligned}
 \text{(P)} \quad & \begin{cases} z_t - (\varrho(x) |z_x|^{p-2} z_x)_x = R_a(x, t, z) - R_e(x, z), & \text{in } Q_T \\ (\varrho(x) |z_x|^{p-2} z_x)(\pm 1, t) = 0, & \text{in } (0, T) \\ z(x, 0) = \underline{v}(x) & \text{in } (-1, 1) \end{cases} \\
 \text{(\bar{P})} \quad & \begin{cases} w_t - (\varrho(x) |w_x|^{p-2} w_x)_x = R_a(x, t, w) - R_e(x, w), & \text{in } Q_T \\ (\varrho(x) |w_x|^{p-2} w_x)(\pm 1, t) = 0, & \text{in } (0, T) \quad (\bar{P}) \\ w(x, 0) = \bar{u}(x), & \text{in } (-1, 1) \end{cases}
 \end{aligned}$$

as it was proved in the Theorem 3, for the solution of (P) we have $\underline{v} \leq F(\underline{v})$, while for the solution of (P̄) one has $F(\bar{u}) \leq \bar{u}$.

If we define by recurrence the sequences

$$z_1 = F(\underline{v}), \dots, z_n = F(z_{n-1}), \dots$$

and

$$w_1 = F(\bar{u}), \dots, w_n = F(w_{n-1}), \dots$$

the Picar iterates $\{z_n\}$ and $\{w_n\}$ makes two sequences, the first one is nondecreasing, the second one nonincreasing regard to the pointwise ordering,

$$\underline{v} \leq z_1 \leq \dots \leq z_n \leq w_n \leq \dots \leq w_1 \leq \bar{u},$$

with

$$\|z_n(1)\|_{L^\infty(-1, 1)} \leq C \quad \text{and} \quad \|w_n(1)\|_{L^\infty(-1, 1)} \leq C.$$

There exist the following pointwise limits

$$(19) \quad \lim_n z_n(x, 1) = \underline{z}(x, 1)$$

$$(20) \quad \lim_n w_n(x, 1) = \bar{w}(x, 1).$$

By the Lebesgue theorem, the convergence in (19) and (20) is uniform.

Since F is a continuous map, $\underline{z}(x, 1) = \lim_n z_n(x, 1) = \lim_n F(z_{n-1}) = F(\underline{z}(x, 1))$ and $\bar{w}(x, 1) = F(\bar{w}(x, 1))$.

Thus, $\underline{z}(x, 1)$ and $\bar{w}(x, 1)$ are the smallest respectively greatest periodic solutions of (P) in the ordered interval $[\underline{v}(x), \bar{u}(x)]$ of $L^\infty(-1, 1)$.

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Manoscritto pervenuto in redazione l'8 settembre 1998