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On the Largest Conjugacy Class Size in a Finite Group.

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We set

$$\mathrm{lcs}(G) = \max\{\, |G : C_G(g)| : g \in G\},$$

the largest conjugacy class size of G. Denoting by $\sigma(G)$ the set of prime divisors of $|G|$ and by G_p a Sylow p-subgroup of G, we will prove the following theorem.

THEOREM. *Let G be an abelian-by-nilpotent finite group. Then*

$$(\alpha) \qquad \mathrm{lcs}(G) \geqslant \prod_{p \in \sigma(G)} \mathrm{lcs}(G_p).$$

Our theorem fails for soluble groups in general: for a given $\varepsilon > 0$ we will show how to construct a group G of derived length 3 for which

$$\mathrm{lcs}(G) < \varepsilon\Big(\prod_{p \in \sigma(G)} \mathrm{lcs}(G_p)\Big).$$

We begin by stating and proving three elementary lemmas for use in the proof of the Theorem.

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LEMMA 1. *Let* $H = AB$ *with* $A \trianglelefteq H$ *and* $A \cap B = 1$. *Let* $x \in B$. *Then*

$$C_H(x) = C_A(x)\, C_B(x).$$

PROOF. Let $h \in C_H(x)$, and let

$$h = ab$$

be the unique decomposition with $a \in A$ and $b \in B$. Then

$$ab = (ab)^x = a^x b^x,$$

and since $a^x \in A$ and $b^x \in B$, it follows that $a = a^x$ and $b = b^x$. Thus $a \in C_A(x)$ and $b \in C_B(x)$, and the result is clear.

LEMMA 2. *Let* G *be a group of* π*-length one for some set* π *of primes. If* b *is an element of a Hall* π*-subgroup* B *of* G, *then* $C_B(b)$ *is a Hall* π*-subgroup of* $C_G(b)$.

PROOF. If $T = O^{\pi'}(G)$, the π'-residual of G, then $C_T(b)$ clearly contains a Hall π-subgroup of $C_G(b)$. Thus we can assume that $T = G$ and hence by hypothesis that $G = AB$, where $A(= O_{\pi'}(G))$ is the normal Hall π'-subgroup of G. But then by Lemma 1 we have

$$C_G(b) = C_A(b)\, C_B(b),$$

and the desired conclusion follows.

LEMMA 3. *Let* $A \trianglelefteq G = AB$ *with* $A \cap B = 1$. *If* $g \in G$ *and* $g = ab$ *with* $a \in A$ *and* $b \in B$, *then*

$$(\beta) \qquad |G : C_G(g)| \leqslant |A|\,|B : C_B(b)|.$$

In particular,

$$\mathrm{lcs}(G) \leqslant \mathrm{lcs}(B)\,|A|.$$

PROOF. Let $h = vu$ be an element of G with $u \in A$ and $v \in B$. Then $g^h = a^h b^{vu} = a^h[u, b^{-v}]b^v$. Since $a^h[u, b^{-v}] \in A$, every conjugate of g can be written as a B-conjugate of b times an element of A. The inequality labelled (β) now follows and the rest is clear.

THE PROOF OF THE THEOREM. We argue by induction on the number of primes in $\sigma(G)$. If $|\sigma(G)| = 1$, then G is a p-group and it is clear that

(α) holds. Therefore suppose that $|\sigma(G)| \geqslant 2$, and let

$$\sigma(G) = \pi_1 \dot{\cup} \pi_2$$

be a non-trivial partition of $\sigma(G)$.

Let R denote the nilpotent residual of G and note that, since R is abelian by hypothesis, a system normalizer D of G is a complement to R in G (cf. Doerk and Hawkes [1], Theorem IV, 5.18). Since D is nilpotent, we can write

$$D = D_1 \times D_2,$$

with $D_i \in \mathrm{Hall}_{\pi_i}(D)$; also

$$R = R_1 \times R_2,$$

with $R_i \in \mathrm{Hall}_{\pi_i}(R)$. For $i = 1, 2$ we set

$$H_i = R_i D_i$$

and observe that $H_i \in \mathrm{Hall}_{\pi_i}(G)$. Let x_i be an element of H_i belonging to a conjugacy class of largest size in H_i (thus $|H_i : C_{H_i}(x_i)| = \mathrm{lcs}(H_i)$ for $i = 1, 2$), and write

(γ) $$x_i = r_i d_i$$

with $r_i \in R_i$ and $d_i \in D_i$. Let $\{i, j\} = \{1, 2\}$, and consider the action of D_i on R_j. Since $(o(d_i), |R_j|) = 1$ and R_j is abelian, by Proposition A, 12.5 of Doerk and Hawkes [1] we have

(δ) $$R_j = [R_j, d_i] \times C_{R_j}(d_i),$$

and because D_j centralizes d_i, the two subgroups $[R_j, d_i]$ and $C_{R_j}(d_i)$ are D_j-invariant and are therefore normal in H_j. We set

$$A_j = [R_j, d_i] \quad \text{and} \quad B_j = C_{R_j}(d_i)\, D_j.$$

[Note for use below that $A_j \trianglelefteq A_j B_j = H_j$, that $A_j \cap B_j = 1$, and that A_j is a normal subgroup of each conjugate of H_j.] According to Equation (δ) we can write $r_j = a_j c_j$ with $a_j \in A_j$ and $c_j \in C_{R_j}(d_i)$, and then we obtain

$$x_j = a_j b_j$$

with $b_j = c_j d_j \in B_j$. Since $[d_i, c_j] = [d_i, d_j] = [c_i, c_j] = 1$, it follows that b_i

commutes with b_j. We aim to show that the element $g = b_i b_j$ satisfies

$$(\varepsilon) \qquad |G : C_G(g)| \geqslant \operatorname{lcs}(H_1)\operatorname{lcs}(H_2).$$

For by induction we have

$$\operatorname{lcs}(H_i) \geqslant \prod_{p \in \sigma(H_i)} \operatorname{lcs}(G_p),$$

and since $\sigma(H_1) \cup \sigma(H_2) = \sigma(G)$, the conclusion of the Theorem will then follow.

As before, let $\{i, j\} = \{1, 2\}$. As our first step in justifying the inequality labelled (ε), we choose a conjugate H of H_i so that $H \cap C_G(b_i b_j)$ is a Hall π_i-subgroup of $C_G(b_i b_j)$. Since b_i is a π_i-element of the centre of $C_G(b_i b_j)$, evidently $b_i \in H$. Because b_i and b_j have relatively prime orders and commute, we have

$$C_H(b_i b_j) = C_H(b_i) \cap C_H(b_j).$$

Now b_j acts fixed-point-freely on $A_i = [R_i, b_j]$, and so $C_H(b_i b_j) \cap A_i \leqslant$ $\leqslant C_H(b_j) \cap A_i = 1$. Hence

$$(\zeta) \qquad |C_H(b_i b_j)| = |C_H(b_i b_j)A_i|/|A_i| \leqslant |C_H(b_i)A_i|/|A_i|.$$

[Here we have used the fact that H normalizes A_i.] Next we observe that

$$|C_H(b_i)A_i| = |C_H(b_i)|\,|A_i|/|C_H(b_i) \cap A_i|.$$

Since metanilpotent groups have π-length one for all sets π of primes, we can twice apply Lemma 2 (with H and then H_i in place of B) to conclude that

$$(\eta) \qquad |C_H(b_i)| = |C_{H_i}(b_i)|.$$

Since $b_i \in B_i$ and $H_i = A_i B_i$ is a semidirect product of A_i by B_i, it follows from Lemma 1 that

$$(\theta) \qquad |C_{H_i}(b_i)| = |C_{A_i}(b_i)|\,|C_{B_i}(b_i)|.$$

Hence from (ζ) we obtain

$$(\iota) \qquad \begin{aligned} |C_H(b_i b_j| &\leqslant |C_H(b_i)|/|C_{A_i}(b_i)| \\ &= |C_{H_i}(b_i)|/|C_{A_i}(b_i)| \\ &= |C_{B_i}(b_i)| \qquad \text{(by } (\theta)). \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
|H : C_H(b_i b_j)| &= |H|/|C_H(b_i b_j)| \\
&\geqslant |H|/|C_{B_i}(b_i)| \qquad \text{(by } (\iota)\text{)} \\
&= |A_i||B_i : C_{B_i}(b_i)| \\
&\geqslant |H_i : C_{H_i}(x_i)| \qquad \text{(by Lemma 3)} \\
&= \operatorname{lcs}(H_i).
\end{aligned}
$$

If $\widetilde{H}$ is a conjugate of H_j with the property that $\widetilde{H} \cap C_G(b_i b_j)$ is a Hall π_j-subgroup of $C_G(b_i b_j)$, we similarly obtain

$$|\widetilde{H} : C_{\widetilde{H}}(b_i b_j)| \geqslant \operatorname{lcs}(H_j).$$

Thus, finally, we can deduce that

$$
\begin{aligned}
|C : C_G(b_i b_j)| &= |H : C_H(b_i b_j)|\,|\widetilde{H} : C_{\widetilde{H}}(b_i b_j)| \\
&\geqslant \operatorname{lcs}(H_i)\operatorname{lcs}(H_j).
\end{aligned}
$$

We have now justified the inequality labelled (ε) and the Theorem is proved.

Before we move to the promised construction of examples, we prove another elementary lemma.

LEMMA 4. *Let A be an abelian normal subgroup of prime index p in a group G. If x and y are elements of G not in A, then*

$$|x^G| = |y^G|.$$

PROOF. Let $C = C_G(x)$. Since $x \in C$ and $\langle x, A\rangle = G$, we have $G = CA$ and therefore

$$|x^G| = |CA : C| = |A : A \cap C| = |A : C_A(x)|.$$

If $y \in G \backslash A$, then $y = x^i a$ for some $a \in A$ and $i \in \{1, \ldots, p-1\}$. Then we have

$$C_A(y) = C_A(x^i) = C_A(\langle x^i \rangle) = C_A(\langle x \rangle) = C_A(x),$$

and the conclusion of the lemma follows.

A FAMILY OF EXAMPLES. Let p be a prime, $p \geqslant 5$, and let q be a prime dividing $p-1$. Let E be a non-abelian group of order pq. Thus $E = PQ$, where $Z_p \cong P = \boldsymbol{O}_p(E) = \boldsymbol{F}(E)$, the Fitting subgroup of E, and $Z_q \cong Q \in \in \mathrm{Syl}_q(E)$; moreover, the non-trivial elements of P fall into $(p-1)/q$ orbits of length q under the action by conjugation of Q. We now define two abelian groups A and B on which E acts as an operator group.

(A) If $q = 2$, let A be a cyclic group of order 2^n $(n \geqslant 3)$, and let E act on A with P as the kernel of the action so that the elements of the non-identity coset of P in E act on A by inversion, sending each $a \in A$ to its inverse.

If $q > 2$, let U be the trivial simple P-module over the field $\mathbb{F}_q$ of q-elements and let $A = U^E$; thus A is isomorphic with the regular $\mathbb{F}_q(E/P)$-module, and, in particular, $A_Q \cong \mathbb{F}_q Q$, the regular Q-module.

(B) Next let V be the trivial simple Q-module over $\mathbb{F}_p$ and let $B = V^E$. By easy applications of Mackey's theorem for induced representations we have:

(i) $B_P \cong \mathbb{F}_p P$ and, in particular, $|C_B(x)| = p$ for all $1 \neq x \in P$;

(ii) $B_Q \cong V \oplus r\mathbb{F}_p Q$, where $r = (p-1)/q$.

Let G be the semidirect product

$$G = [A \oplus B]\, E\,,$$

where the action of E as a group of operators on $A \oplus B$ is determined by the action of E on A and B described above. In what follows we will use multiplicative notation for $A \oplus B$ when it is regarded as a subgroup AB of G. Evidently BP is a Sylow P-subgroup of G and AQ is a Sylow q-subgroup of G. Set

$$M = \begin{cases} \min\{2^{n-2}, p^{r-1}\} & \text{if } q = 2, \ \textit{and} \\ \min\{q^{q-2}, p^{r-1}\} & \text{if } q > 2 \end{cases}$$

Since

$$r = (p-1)/q$$

and $p \geqslant 5$, it follows that $r \geqslant 2$ and hence that $M \geqslant 2$. In fact, it is easy to see that by judicious choice of p and q we can make M arbitrarily large. We will show that

$$(\kappa) \qquad \mathrm{lcs}\,(BP)\, \mathrm{lcs}\,(AQ) \geqslant M\, \mathrm{lcs}\,(G)\,.$$

Step 1: We assert that

$$(\lambda) \qquad \operatorname{lcs}(BP) = p^{p-1}.$$

Since B_P is a regular $\mathbb{F}_p P$-module, the group BP is isomorphic with $Z_p \wr_{\text{reg}} Z_p$ and $|C_B(P)| = p$. The conjugacy classes of BP contained in B obviously have lengths 1 or p, while elements x in $BP\backslash B$ belong to classes of length $|B : C_B(P)| = p^{p-1}$ by Lemma 4. Thus Assertion (λ) is justified.

Step 2: We now assert that

$$(\mu) \qquad \operatorname{lcs}(AQ) = \begin{cases} 2^{n-1} & \text{if } q = 2, \text{ and} \\ q^{q-1} & \text{if } q > 2. \end{cases}$$

The conjugacy classes of AQ contained in the abelian normal subgroup A obviously have length 1 or q. In the case $q = 2$, as well as in the case $q > 2$, it is easy to see from the action of Q on A that $|C_A(Q)| = q$. Hence, if $x \in AQ\backslash A$, it follows from Lemma 4 that

$$(\nu) \qquad |x^{AQ}| = |A : C_A(Q)| = \begin{cases} 2^{n-1} & \text{if } q = 2, \text{ and} \\ q^{q-1} & \text{if } q > 2. \end{cases}$$

Assertion (μ) is now clear.

Step 3: Our next assertion is that

$$(\xi) \qquad \operatorname{lcs}(G) = \begin{cases} \max\{2p^{p-1}, 2^{n-1}p^{p-r}\} & \text{when } q = 2, \text{ and} \\ \max\{qp^{p-1}, q^{q-1}p^{p-r}\} & \text{when } q > 2. \end{cases}$$

Let $x \in G$, and let y be a generator of Q. We consider three cases.

Case 1: We have $x \notin ABP$. Since $G/AB(\cong E)$ is a Frobenius group, $AB\langle x\rangle$ is conjugate to ABQ, and therefore in calculating $|x^G|$, we can suppose without loss of generality that $x \in ABQ\backslash AB$. Since x, like y, acts fixed-point-freely on P, we have $C_G(x) \leq AB\langle x\rangle = ABQ$, and so $|x^G| = |P|\,|x^{ABQ}|$. By Lemma 4 we have

$$|x^{ABQ}| = |y^{ABQ}| = |AB : C_{AB}(Q)| = |A : C_A(Q)|\,|B : C_B(Q)|.$$

Since the restriction B_Q of B to Q is the sum of a trivial module and r regular modules, we have $|C_B(Q)| = p^{r+1}$; hence from (ν) we conclude that

$$(\pi) \qquad |x^G| = \begin{cases} 2^{n-1} p^{p-r} & \text{if } q = 2, \text{ and} \\ q^{q-1} p^{p-r} & \text{if } q > 2 . \end{cases}$$

We note that pq divides $|x^G|$ in this case because $p - r > (q-1)r \geqslant r \geqslant \geqslant 2$ and by assumption $n \geqslant 3$.

Case 2: We have $x \in ABP \backslash AB$. Since ABP/AB is self-centralizing in G/AB, it follows that $C_G(x) \leqslant ABP$ and hence from Lemma 4 that $|x^G| = |Q|\,|x^{ABP}| = |AB : C_{AB}(P)|$. Now P centralizes A and B_P is a regular module, and therefore

$$|AB : C_{AB}(P)| = p^{p-1}.$$

Hence

$$(\varrho) \qquad |x^G| = qp^{p-1}$$

in this case, and again $|x^G|$ is divisible by pq.

Case 3: We have $x \in AB$. Since AB is abelian, we have $|x^G| = = |E : C_E(x)|$, which is a divisor of pq. In this case $|x^G|$ is smaller than the values obtained for it is Cases 1 and 2.

Assertion (ξ) now follows from (π) and (ϱ).

To justify the inequality labelled (κ), we deduce from (λ), (μ), and (ξ) that for $q = 2$

$$\begin{aligned} \frac{\operatorname{lcs}(BP)\operatorname{lcs}(QA)}{\operatorname{lcs}(G)} &= \frac{2^{n-1}p^{p-1}}{\max\{2p^{p-1}, 2^{n-1}p^{p-r}\}} \\ &\geqslant \min\left\{\frac{2^{n-1}p^{p-1}}{2p^{p-1}}, \frac{2^{n-1}p^{p-1}}{2^{n-1}p^{p-r}}\right\} \\ &= \min\{2^{n-2}, p^{r-1}\} \\ &= M . \end{aligned}$$

Similarly, for $q > 2$, we obtain

$$\frac{\operatorname{lcs}(BP)\operatorname{lcs}(QA)}{\operatorname{lcs}(G)} \geqslant \min\left\{ \frac{q^{q-1}p^{p-1}}{qp^{p-1}}, \frac{q^{q-1}p^{p-1}}{q^{q-1}p^{p-r}} \right\}$$

$$= \{q^{q-2}, p^{r-1}\}$$

$$= M .$$

Thus we have shown that (κ) holds for all values of q. Given $\varepsilon > 0$, it is easy to find primes p and q so that $1/M < \varepsilon$. Thus, as promised at the outset, we have shown the existence of $\{p, q\}$-groups of derived length 3 satisfying

$$\operatorname{lcs}(G) < \varepsilon \Big(\prod_{p \in \sigma(G)} \operatorname{lcs}(G_p) \Big).$$

REFERENCES

[1] K. Doerk - T. O. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

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