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Asymptotic Behavior of Solutions of Schrödinger Inequalities on Unbounded Domains of Nilpotent Lie Groups.

FRANCESCO UGUZZONI (*)

1. Introduction.

The aim of this note is to present a technique which allows to find asymptotic behavior at infinity, for solutions of a wide class of equations.

Let $\mathcal{G} = \bigoplus_{j=1}^m \mathcal{G}_j$ be a stratified nilpotent Lie algebra of vector fields and $H = (\mathbb{R}^N, \circ)$ be its associated homogeneous Lie group. Let $\{X_1, \dots, X_n\}$ be a basis of \mathcal{G}_1 and \mathcal{L} be the differential operator

$$\mathcal{L} = \sum_{j=1}^n X_j^2.$$

Moreover we denote by S_{loc} the \mathcal{L} -natural local Sobolev space. We shall assume that H has homogeneous dimension $Q \geq 3$. We work in this setting since we need the existence of a fundamental solution of $-\mathcal{L}$ of the type $\Gamma \sim cd^{2-Q}$, where d is the natural «distance» on H . We refer to section 2 for more precise definitions and additional notation.

The simplest example of this kind of operators is the classical Laplacian $\mathcal{L} = \Delta$ on $H = (\mathbb{R}^N, +)$, for $N = Q \geq 3$. The simplest non-abelian example is the Kohn Laplacian $\mathcal{L} = \Delta_{\mathbb{H}^k}$ on the Heisenberg group $H = \mathbb{H}^k = (\mathbb{R}^{2k+1}, \circ)$, with homogeneous dimension $Q = 2k + 2$.

We consider a nonnegative weak solution $u \in S_{\text{loc}}(\Omega)$ of the Schrödinger-type inequality

$$(1.1) \quad -\mathcal{L}u \leq Vu$$

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in an unbounded domain Ω and we obtain an asymptotic behavior of u at infinity, starting from its L^p properties. The potential V will be supposed to belong to the space $L^q(\Omega)$ for all q in a neighborhood of $Q/2$, i.e. to the set

$$(1.2) \quad L^{]Q/2[}(\Omega) := \left\{ v \in L^{Q/2}(\Omega) \mid \exists q_1, q_2 : q_1 < \frac{Q}{2} < q_2, \quad v \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega) \right\}.$$

Our technique relies on the use of some remarkable representation formulas on the d -balls and is inspired to a work of Simader [S] and one of Citti-Garofalo-Lanconelli [CGL].

We first consider inequality (1.1) in an exterior domain Ω (i.e. $\Omega = H \setminus F$ with F a compact subset of H). The following theorem is the main result of this note.

THEOREM 1.1. *Let Ω be an exterior domain of H and let $V \in L^{]Q/2[}(\Omega)$. If $u \in S_{\text{loc}}(\Omega)$ is a nonnegative weak solution of*

$$-\mathcal{L}u \leq Vu \quad \text{in } \Omega$$

such that $u \in L^p(\Omega)$ for a $p \in [Q/(Q-2), +\infty[$, then

$$u(\xi) = O\left(\frac{1}{d(\xi)^{Q/p}}\right), \quad \text{as } d(\xi) \rightarrow +\infty.$$

If moreover there exists $p_1 \in [1, Q/(Q-2)[$ such that $u \in L^{p_1}(\Omega) \cap L^{Q/(Q-2)}(\Omega)$, then

$$u(\xi) = O\left(\frac{1}{d(\xi)^{Q/p_1}}\right), \quad \text{as } d(\xi) \rightarrow +\infty.$$

REMARK 1.2. Theorem 1.1 holds true also removing the hypothesis on the nonnegativity of u if we replace the inequality $-\mathcal{L}u \leq Vu$ with $|\mathcal{L}u| \leq |Vu|$. In particular the Theorem holds for the equations $-\mathcal{L}u = Vu$, with u that may change sign.

We emphasize that in Theorem 1.1 no assumption is made about the boundary values of u . We also remark that $1/d^{Q/p} \in L^p_{\text{weak}}(H)$; hence our result can be considered optimal.

In Theorem 1.1 we deal with exterior domains since we need to write representation formulas on d -balls and allow to go to infinity both the center and the radius of the balls. This limitation can be overcome if we

are able to find an auxiliary function $w \geq u$ in Ω of the type $w = \Gamma * f$, with $f \leq |V|u$ in H . This idea allows to get asymptotic behavior for solutions of Dirichlet problems related to (1.1) on arbitrary unbounded domains. In particular we derive the following theorem.

THEOREM 1.3. *Let Ω be an (arbitrary) unbounded open subset of H and let $V \in L^{1Q/2I}(\Omega)$. If u is a nonnegative classical solution of*

$$\begin{cases} -\mathcal{L}u \leq Vu & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \\ u(\xi) \rightarrow 0 & \text{as } d(\xi) \rightarrow +\infty \end{cases}$$

then

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as } d(\xi) \rightarrow +\infty, \quad \forall s < Q - 2.$$

REMARK 1.4. Theorem 1.3 holds true also removing the hypothesis on the nonnegativity of u if we replace the inequality $-\mathcal{L}u \leq Vu$ with $|\mathcal{L}u| \leq |Vu|$. In particular the Theorem holds for the equations $-\mathcal{L}u = Vu$, with u that may change sign.

We remark that in Theorem 1.3 we do not assume any a priori summability of u . We also remark that the «limit» behavior $O(1/d(\xi)^{Q-2})$ is the one of the fundamental solution of $-\mathcal{L}$.

The results of this note can be applied to semilinear equations of the type $-\mathcal{L}u = u^q$, whenever we know that u belongs to the suitable L^p spaces. Indeed if u is a weak solution in a (global) Sobolev space, say $u \in S_0^{1,2}$ ($\hookrightarrow L^p$ for every $p \in [2, 2Q/(Q-2)]$), both the condition on u and on the potential $V = u^{q-1}$ could be automatically satisfied. For example, if $1 + 4/Q < q < (Q+2)/(Q-2)$, then we immediately obtain that $V = u^{q-1}$ belongs to the class $L^{1Q/2I}$.

Moreover our results can be used as a starting point in order to obtain nonexistence theorems in unbounded domains. An example of application is given in [LU1] where we find asymptotic behavior of nonnegative weak solutions to the critical semilinear Dirichlet problem

$$(1.3) \quad \begin{cases} -\Delta_{\mathbb{H}^k} u = u^{(Q+2)/(Q-2)} & \text{in } \Omega \\ u \in S_0^1(\Omega) \end{cases}$$

and we also prove some related nonexistence results (see also [U1]). We stress that the Sobolev space S_0^1 considered in [LU1]-[U1] is not embedded in L^2 ; hence a solution u of (1.3) belongs a priori to L^p only for $p = 2Q/(Q-2)$. Actually in [LU1] we prove that such a solution belongs to L^p for every $p \in]Q/(Q-2), +\infty[$ (and so $V = u^{(Q+2)/(Q-2)-1} \in L^{1Q/2[}$) but this result is highly nontrivial, in particular for $Q/(Q-2) < p < 2Q/(Q-2)$.

Other examples of application to nonexistence results for semilinear equations on the Heisenberg group will be given in the forthcoming papers [LU2] and [U2]. In this respect we point out that Liouville-type theorems for semilinear $\Delta_{\mathbb{H}^k}$ -inequalities on some unbounded domains have been recently proved by Birindelli-Capuzzo Dolcetta-Cutrì [BCC]. Finally we quote that asymptotic behavior for capacity problems on exterior domains of groups of Heisenberg type has been treated by Danielli-Garofalo [DG].

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2. Notation and definitions.

Let $(\mathcal{G}, [\cdot])$ be a stratified nilpotent real Lie algebra and let (H, \circ) be its simply connected associated Lie group. We can identify H with \mathbb{R}^N , for a suitable $N \in \mathbb{N}$, and $(\mathcal{G}, [\cdot])$ with the Lie algebra of \circ -left-invariant vector fields on \mathbb{R}^N , with the usual Lie bracket law $[X, Y] = XY - YX$. We denote by $\{\delta_\lambda\}_{\lambda>0}$ the group of dilations naturally associated to H and by Q the homogeneous dimension of H . Let $\mathcal{G} = \bigoplus_{j=1}^m \mathcal{G}_j$ be the stratification of \mathcal{G} and let $\{X_1, \dots, X_n\}$ be a basis of \mathcal{G}_1 . We shall deal with the differential operator

$$\mathcal{L} = \sum_{j=1}^n X_j^2.$$

We recall that X_1, \dots, X_n generate the whole \mathcal{G} by the assumption \mathcal{G} stratified. Moreover, if we set

$$\nabla_{\mathcal{L}} = (X_1, \dots, X_n)$$

then $\nabla_{\mathcal{L}}$ and \mathcal{L} are homogeneous, w.r.t. the dilations δ_λ , of degree one and of degree two respectively.

Throughout the paper we will make the assumption

$$Q \geq 3$$

on the homogeneous dimension of H . We remark that, if $Q \leq 3$, then it is necessarily $\mathcal{G} = \mathcal{G}_1$, (H, \circ) is simply $(\mathbb{R}^Q, +)$ and \mathcal{L} is the Laplace operator Δ on \mathbb{R}^Q . If $Q \geq 3$, there exists a homogeneous norm $|\cdot|_{\mathcal{L}}$ on H such that, setting

$$d_\xi(\eta) = |\eta^{-1} \circ \xi|_{\mathcal{L}},$$

a fundamental solution of $-\mathcal{L}$ with pole at ξ is given by

$$(2.1) \quad \Gamma_\xi = \frac{c_Q}{d_\xi^{Q-2}}$$

where c_Q is a suitable positive constant depending only on Q (see [G]). We set

$$d(\xi, \eta) = d_\xi(\eta), \quad \Gamma(\xi, \eta) = \Gamma_\xi(\eta).$$

Moreover we will often write $d(\eta)$ instead of $d_0(\eta)$ and $\Gamma(\eta)$ instead of $\Gamma_0(\eta)$. We recall that a homogeneous norm on H is a function $|\cdot|: H \rightarrow [0, +\infty[$ such that $|\cdot| \in C^\infty(H \setminus \{0\}) \cap C(H)$, $|\xi| = 0$ iff $\xi = 0$, $|\xi| = |\xi^{-1}|$ ($= |-\xi|$) and

$$(2.2) \quad |\delta_\lambda \xi| = \lambda |\xi|.$$

Moreover any homogeneous norm satisfies the following triangle inequality

$$|\xi \circ \eta| \leq c(|\xi| + |\eta|)$$

for a suitable $c \geq 1$. Hence there exists $\beta \geq 1$ such that

$$(2.3) \quad d(\xi, \eta) \leq \beta(d(\xi, \zeta) + d(\zeta, \eta))$$

for every $\xi, \eta, \zeta \in H$. Moreover, since $|\xi|_{\mathcal{L}} = |\xi^{-1}|_{\mathcal{L}}$, it is also

$$d(\xi, \eta) = d(\eta, \xi).$$

We denote the d -balls on H by

$$B_d(\xi, r) = \{\eta \in H \mid d(\xi, \eta) < r\}.$$

Since the Lebesgue measure is a Haar measure on H we have

$$|B_d(\xi, r)| = |B_d(0, r)| = r^Q |B_d(0, 1)|.$$

Moreover on H the following polar coordinates formula holds:

$$\int_H f(d(\xi)) d\xi = Q |B_d(0, 1)| \int_0^{+\infty} f(\varrho) \varrho^{Q-1} d\varrho.$$

In particular it follows that, for every $s \in]0, Q[$,

$$(2.4) \quad \frac{1}{d_\xi^s} \in L^p(B_d(\xi, 1)) \cap L^q(H \setminus B_d(\xi, 1)), \quad \text{for } 1 \leq p < \frac{Q}{s} < q \leq +\infty.$$

We also remark that

$$(2.5) \quad |\nabla_{\mathcal{L}} d_0| \in L^\infty(H)$$

since $\nabla_{\mathcal{L}} d_0$ is homogeneous of degree zero w.r.t. the dilations δ_λ and then

$$\sup_{\xi \neq 0} |(\nabla_{\mathcal{L}} d_0)(\xi)| = \sup_{\xi \neq 0} |(\nabla_{\mathcal{L}} d_0)(\delta_{1/d_0(\xi)} \xi)| = \max_{d_0(\eta)=1} |(\nabla_{\mathcal{L}} d_0)(\eta)|.$$

More details on nilpotent Lie algebras and homogeneous groups can be found, for example, in [FH], [F] and [RS].

If Ω is an open subset of H , we denote by $S(\Omega)$ the Sobolev space of the functions $u \in L^2(\Omega)$ such that $\nabla_{\mathcal{L}} u \in L^2(\Omega)$. The norm in $S(\Omega)$ is given by

$$\|u\|_{S(\Omega)} = \left(\int_{\Omega} |\nabla_{\mathcal{L}} u|^2 + u^2 \right)^{1/2}.$$

Moreover we denote by $S_{\text{loc}}(\Omega)$ the set of those $u \in L^2_{\text{loc}}(\Omega)$ such that $\varphi u \in S(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. Let $V \in L^{1Q/2l}(\Omega)$; a function $u \in S_{\text{loc}}(\Omega)$ is called a weak solution of

$$-\mathcal{L}u \leq Vu \quad \text{in } \Omega$$

if

$$\int_{\Omega} \langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} \varphi \rangle \leq \int_{\Omega} Vu \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0.$$

We remark that in the definition above $Vu \in L^1_{loc}$ since $u \in S_{loc} \subseteq L^{Q/(Q-2)}_{loc}$ and $V \in L^{Q/2}$. We also remark that every classical solution of $-\mathcal{L}u \leq Vu$ is a weak solution in our definition, since $X_j^* = -X_j$ for $j = 1, \dots, n$.

3. Proof of the main theorems.

The following representation formula plays a basic role in the proof of Theorem 1.1. Let Ω be an open subset of H and $u \in S_{loc}(\Omega)$; then for a.e. $\xi \in \Omega$ and every $r > 0$ such that $\overline{B_d(\xi, r)} \subseteq \Omega$, we have

$$(3.1) \quad u(\xi) = (M_r u)(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(\xi, \varrho)} \langle \nabla_{\mathcal{L}} \Gamma_{\xi}, \nabla_{\mathcal{L}} u \rangle \right) d\varrho$$

where M_r is the mean value operator defined by

$$(3.2) \quad (M_r u)(\xi) = \frac{c}{r^Q} \int_{B_d(\xi, r)} |\nabla_{\mathcal{L}} d_{\xi}|^2 u$$

and c is a suitable positive constant. This formula has been proved in [CGL], Proposition 2.3. We have only replaced the \mathcal{L} -balls $\Omega_r(\xi) = \{ \Gamma_{\xi} > 1/r \}$ of [CGL] with our d -balls and used the «explicit» expression of Γ (2.1). We remark that the integral in the right-hand side of (3.1) is finite since (using (2.5))

$$|\langle \nabla_{\mathcal{L}} \Gamma_{\xi}, \nabla_{\mathcal{L}} u \rangle(\eta)| \leq |\nabla_{\mathcal{L}} \Gamma_{\xi}| |\nabla_{\mathcal{L}} u|(\eta) \leq cd(\xi, \eta)^{1-Q} |\nabla_{\mathcal{L}} u(\eta)| \in L^1_{loc, \eta}$$

for a.e. $\xi \in \Omega$ by Tonelli's Theorem, since $d(\xi, \eta)^{1-Q} \in L^1_{loc, \xi}$ (see (2.4)) and $|\nabla_{\mathcal{L}} u(\eta)| \in L^1_{loc, \eta}$.

We define $r: H \rightarrow [0, +\infty[$,

$$(3.3) \quad r(\xi) = \frac{d(\xi)}{4\beta^3}$$

where β has been introduced in (2.3).

LEMMA 3.1. *Let Ω be an exterior domain of H , let $V \in L^{1Q/2l}(\Omega)$ and let $u \in S_{loc}(\Omega)$ be a nonnegative weak solution of*

$$-\mathcal{L}u \leq Vu \quad \text{in } \Omega$$

such that $u \in L^p(\Omega)$ for a $p \in [1, +\infty[$. If there exist $s \in]0, Q/p]$ and two

positive constant R, M such that

$$(3.4) \quad \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u \leq \frac{M}{r(\xi)^s}, \quad \text{for } d(\xi) > R,$$

then

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \quad \text{as } d(\xi) \rightarrow +\infty.$$

PROOF. For every exponent $t \in]1, +\infty[$ we shall denote by $t' = t/(t-1)$ the conjugate exponent of t . Since $V \in L^{1Q/2}(\Omega)$ there exist q_1, q_2 such that $1 < q_1 < Q/2 < q_2 < +\infty$ and

$$(3.5) \quad V \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega).$$

Moreover $q_2' < (Q/2)' = Q/(Q-2) < q_1'$ and then

$$(3.6) \quad \Gamma \in L^{q_2'}(B_d(0, 1)) \cap L^{q_1'}(H \setminus B_d(0, 1))$$

by means of (2.1) and (2.4).

For every $\xi \in H$ and $r > 0$ such that $\overline{B_d(\xi, r)} \subseteq \Omega$, we define (see (3.2))

$$M_r(\xi) = (M_r u)(\xi),$$

$$N_r(\xi) = \int_{B_d(\xi, r)} \Gamma_\xi |V|$$

and

$$I_r(\xi) = \int_{B_d(\xi, r)} \Gamma_\xi |V| u = \int_{B_d(\xi, r)} \Gamma(\xi, \eta) |V|(\eta) u(\eta) d\eta.$$

We remark that $I_r(\xi) < +\infty$ a.e. by Tonelli's Theorem, since $\Gamma(\xi, \eta) \in L^1_{\text{loc}, \xi}$ and $(Vu)(\eta) \in L^1_{\text{loc}, \eta}$ because $u \in S_{\text{loc}} \subseteq L^{Q/(Q-2)}_{\text{loc}}$ and $V \in L^{Q/2}$. Moreover

$$(3.7) \quad \begin{aligned} N_r(\xi) &\leq \|\Gamma_\xi\|_{L^{q_2}(B_d(\xi, 1))} \|V\|_{L^{q_2}(B_d(\xi, r))} + \|\Gamma_\xi\|_{L^{q_1}(H \setminus B_d(\xi, 1))} \|V\|_{L^{q_1}(B_d(\xi, r))} \\ &\leq c(\|V\|_{L^{q_2}(B_d(\xi, r))} + \|V\|_{L^{q_1}(B_d(\xi, r))}) \end{aligned}$$

by (3.6). In particular (3.5) and (3.7) give

$$(3.8) \quad \sup_{\{\xi, r\} \in \mathcal{B}_d(\bar{\xi}, \bar{r}) \subseteq \Omega} N_r(\xi) \leq c.$$

By (2.5), (3.2) and the assumption $u \in L^p(\Omega)$ we can estimate also M_r and get

$$(3.9) \quad M_r(\xi) \leq \frac{c}{r^Q} \int_{B_d(\xi, r)} u \leq \frac{c}{r^Q} \|u\|_p \|1\|_{L^{p'}(B_d(\xi, r))} = \frac{c}{r^Q} r^{Q/p'} = \frac{c}{r^{Q/p}}.$$

Let us now recall the representation formula (3.1) for u . Since $-\mathcal{L}u \leq Vu$ weakly in Ω , it is not difficult to see that

$$\int_{B_d(\xi, \varrho)} \langle \nabla_{\mathcal{L}} \Gamma_{\xi}, \nabla_{\mathcal{L}} u \rangle \leq \int_{B_d(\xi, \varrho)} \left(\Gamma_{\xi} - \frac{c_Q}{\varrho^{Q-2}} \right) |V| u$$

(one only needs to approximate $(\Gamma_{\xi} - c_Q/\varrho^{Q-2})$ with a suitable sequence of functions $C_0^\infty(\Omega)$). Then we get

$$(3.10) \quad \begin{aligned} u(\xi) &\leq M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(\xi, \varrho)} \left(\Gamma_{\xi} - \frac{c_Q}{\varrho^{Q-2}} \right) |V| u \right) d\varrho \\ &\leq M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(\xi, r)} \Gamma_{\xi} |V| u \right) d\varrho \\ &= M_r(\xi) + I_r(\xi) \leq \frac{c}{r^{Q/p}} + I_r(\xi) \end{aligned}$$

(by (3.9)). Hence, if $\overline{B_d(\xi, 2\beta r)} \subseteq \Omega$ (see (2.3)), we have

$$(3.11) \quad \begin{aligned} I_r(\xi) &\leq \int_{B_d(\xi, r)} \Gamma_{\xi}(\eta) |V|(\eta) \left(\frac{c}{r^{Q/p}} + I_r(\eta) \right) d\eta \\ &\leq \frac{c}{r^{Q/p}} N_r(\xi) + \int_{B_d(\xi, r)} \Gamma_{\xi} |V| I_r \leq \frac{c}{r^{Q/p}} + \int_{B_d(\xi, r)} \Gamma_{\xi} |V| I_r \end{aligned}$$

(by (3.8)). Moreover, if $\overline{B_d(\xi, 3\beta^2 r)} \subseteq \Omega$, then

$$\begin{aligned}
 (3.12) \quad \int_{B_d(\xi, r)} \Gamma_\xi |V| I_r &= \int_{B_d(\xi, r)} \Gamma(\xi, \eta) |V|(\eta) \left(\int_{B_d(\eta, r)} \Gamma(\eta, \zeta) |V|(\zeta) u(\zeta) d\zeta \right) d\eta \\
 &= \int_{B_d(\xi, 2\beta r)} |V|(\zeta) u(\zeta) \left(\int_{B_d(\xi, r) \cap B_d(\zeta, r)} \Gamma(\xi, \eta) \Gamma(\eta, \zeta) |V|(\eta) d\eta \right) d\zeta \\
 &\leq c \sup_{\zeta \in B_d(\xi, 2\beta r)} N_r(\zeta) \int_{B_d(\xi, 2\beta r)} \Gamma_\xi |V| u
 \end{aligned}$$

since, setting $A = B_d(\xi, r) \cap B_d(\zeta, r)$, $A_1 = \{\eta \in A \mid d(\eta, \zeta) \leq d(\xi, \zeta)/(2\beta)\}$ and $A_2 = A \setminus A_1$, we have (note that $d(\xi, \eta) \geq d(\xi, \zeta)/(2\beta)$ for every $\eta \in A_1$)

$$\begin{aligned}
 \int_A \Gamma_\xi \Gamma_\zeta |V| &= \int_{A_1} \Gamma_\xi \Gamma_\zeta |V| + \int_{A_2} \Gamma_\xi \Gamma_\zeta |V| \\
 &\leq c \Gamma(\xi, \zeta) \left(\int_{A_1} \Gamma_\zeta |V| + \int_{A_2} \Gamma_\xi |V| \right) \\
 &\leq c \Gamma(\xi, \zeta) (N_r(\zeta) + N_r(\xi)).
 \end{aligned}$$

From now on we will take $r = r(\xi)$ (see (3.3)); in this way $\overline{B_d(\xi, 3\beta^2 r)} \subseteq H \setminus B_d(0, r) \subseteq \Omega$, for large $d(\xi)$. Using this fact we obtain

$$\sup_{\zeta \in B_d(\xi, 2\beta r(\xi))} N_{r(\xi)}(\zeta) \leq c (\|V\|_{L^{q_2}(H \setminus B_d(0, r(\xi)))} + \|V\|_{L^{q_1}(H \setminus B_d(0, r(\xi)))}) \xrightarrow{d(\xi) \rightarrow +\infty} 0$$

by means of (3.7) and (3.5). Hence there exists $R_0 > R$ such that, for $d(\xi) > R_0$, we have (see (3.12))

$$\begin{aligned}
 (3.13) \quad \int_{B_d(\xi, r(\xi))} \Gamma_\xi |V| I_{r(\xi)} &\leq \frac{1}{2} \int_{B_d(\xi, 2\beta r(\xi))} \Gamma_\xi |V| u \\
 &= \frac{1}{2} I_{r(\xi)}(\xi) + \frac{1}{2} \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u \\
 &\leq \frac{1}{2} I_{r(\xi)}(\xi) + \frac{M}{2r(\xi)^s}
 \end{aligned}$$

by assumption (3.4). From (3.11) and (3.13) we finally get, for $d(\xi) > R_0$,

$$\frac{1}{2} I_{r(\xi)}(\xi) \leq \frac{c}{r(\xi)^{Q/p}} + \frac{c}{r(\xi)^s}.$$

This estimate and (3.10) allow us to conclude that

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \quad \text{as } d(\xi) \rightarrow +\infty$$

since $r(\xi) = d(\xi)/(4\beta^3)$ and $s \leq Q/p$. ■

PROOF OF THEOREM 1.1. To prove the first part of the statement we only need to obtain (3.4) for $s = Q/p$ and then use Lemma 3.1. If $p = Q/(Q-2)$ we have (for large $d(\xi)$)

$$\begin{aligned} \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V|u &= \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \frac{c_Q}{d_\xi^{Q-2}} |V|u \leq \frac{c}{r(\xi)^{Q-2}} \int_H |V|u \\ &\leq \frac{c}{r(\xi)^s} \|V\|_{Q/2} \|u\|_{Q/(Q-2)} \end{aligned}$$

and then (3.4) holds.

We now suppose $p > Q/(Q-2)$. For every exponent $t \in]1, +\infty[$ we shall denote by $t' = t/(t-1)$ the conjugate exponent of t . Since $V \in L^{1Q/2}(\Omega)$ there exists $q < Q/2$ such that $Q/(Q-2) = (Q/2)' < q' < p$ and $V \in L^q(\Omega)$. Moreover $Q-2-s = Q-2-(Q/p) > 0$. Hence

$$\frac{Q-2-s}{Q} q' > \frac{Q-2-Q/p}{Q-2} = 1 - \frac{Q/(Q-2)}{p} > 1 - \frac{q'}{p} = \frac{1}{(p/q)'}.$$

and then (see (2.4))

$$\frac{1}{d_0^{(Q-2-s)q'}} \in L^{(p/q)'}(H \setminus B_d(0, 1)).$$

We can now obtain (3.4). For large $d(\xi)$ we have

$$\begin{aligned}
 \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u &\leq \frac{c}{r(\xi)^s} \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} |V| \frac{u}{d_\xi^{Q-2-s}} \\
 &\leq \frac{c}{r(\xi)^s} \|V\|_q (\|u\|_{p/q}^{q'} \|d_\xi^{-(Q-2-s)q'}\|_{L^{(p/q)'}(H \setminus B_d(\xi, 1))})^{1/q'} \\
 &\leq \frac{c}{r(\xi)^s} \|d_0^{-(Q-2-s)q'}\|_{L^{(p/q)'}(H \setminus B_d(0, 1))}^{1/q'} \leq \frac{c}{r(\xi)^s}.
 \end{aligned}$$

Let us now prove the second part of the theorem. We know that $u \in \epsilon L^{p_1} \cap L^{Q/(Q-2)}$ for a $p_1 \in [1, Q/(Q-2)]$. By means of Lemma 3.1 we only need to prove that (3.4) holds for $s = Q/p_1$. For every $t \in]0, Q/p_1[$ we set

$$\sigma(t) = Q - 2 + t - \frac{p_1 t(Q-2)}{Q}$$

and we claim that (3.4) holds for $s = \sigma(t)$ if (3.4) holds for $s = t$. Indeed, by Lemma 3.1, if (3.4) holds for $s = t$ then

$$(3.14) \quad u(\xi) = O\left(\frac{1}{d(\xi)^t}\right), \quad \text{as } d(\xi) \rightarrow +\infty;$$

therefore, for large $d(\xi)$,

$$\begin{aligned}
 \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u &\leq \frac{c}{r(\xi)^{Q-2}} \|V\|_{Q/2} \left(\int_{B_d(\xi, 2\beta r(\xi))} u^{Q(Q-2)-p_1 u^{p_1}} \right)^{(Q-2)/Q} \\
 &\leq \frac{c}{r(\xi)^{Q-2}} \|u\|_{L^\infty(H \setminus B_d(0, r(\xi)))}^{1-(p_1(Q-2))/Q} \|u\|_{p_1}^{p_1(Q-2)/Q} \leq \frac{c}{r(\xi)^{\sigma(t)}}
 \end{aligned}$$

(by means of (3.14) and (3.3)).

Since $u \in L^{Q/(Q-2)}(\Omega)$, (3.4) holds for $s = Q - 2$ (see the beginning of the proof); moreover if we set

$$\begin{cases} t_1 = Q - 2 \\ t_{k+1} = \sigma(t_k) \end{cases}$$

it is easy to see that $t_k \nearrow (Q/p_1)$. Henceforth (3.4) holds for every $s \in$

$\in]0, Q/p_1[$. In particular if we choose $q < Q/2$ such that $V \in L^q(\Omega)$ and we set

$$\tau = \frac{Q/p_1 - Q + 2}{1 - p_1/q'}$$

we have $0 < \tau < Q/p_1$ and then (3.4) holds for $s = \tau$. Hence, by Lemma 3.1,

$$(3.15) \quad u(\xi) = O\left(\frac{1}{d(\xi)^\tau}\right), \quad \text{as } d(\xi) \rightarrow +\infty.$$

We are now able to prove (3.4) for $s = Q/p_1$. In fact we have (for large $d(\xi)$)

$$\begin{aligned} \int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u &\leq \frac{c}{r(\xi)^{Q-2}} \|V\|_q \left(\int_{B_d(\xi, 2\beta r(\xi))} u^{q' - p_1} u^{p_1} \right)^{1/q'} \\ &\leq \frac{c}{r(\xi)^{Q-2}} \|u\|_{L^\infty(H \setminus B_d(0, r(\xi)))}^{1 - p_1/q'} \|u\|_{p_1}^{p_1/q'} \\ &\leq \frac{c}{r(\xi)^{Q-2 + \tau(1 - p_1/q')}} = \frac{c}{r(\xi)^{Q/p_1}} \end{aligned}$$

(by means of (3.15) and (3.3)). ■

PROOF OF THEOREM 1.3. We set

$$f = \begin{cases} |V|u & \text{in } \Omega \\ 0 & \text{in } H \setminus \Omega \end{cases}$$

and we define $w: H \rightarrow \mathbb{R}$,

$$w(\xi) = (\Gamma * f)(\xi) = \int_H \Gamma(\xi^{-1} \circ \eta) f(\eta) d\eta.$$

Then $w \geq 0$ and $-\mathcal{L}w = f$ weakly in H . In particular $u \leq w$ in $\partial\Omega \cup \{\infty\}$ and $-\mathcal{L}u \leq |V|u = -\mathcal{L}w$ in Ω . Hence

$$(3.16) \quad 0 \leq u \leq w \quad \text{in } \Omega$$

by the weak maximum principle for \mathcal{L} . Moreover if $t \in]1, Q/2[$ and $f \in$

$\in L^t(H)$ then

$$(3.17) \quad w \in L^{(1/t-2/Q)^{-1}}(H)$$

and

$$(3.18) \quad \nabla_{\varepsilon} w \in L^{(1/t-1/Q)^{-1}}(H)$$

(see [RS], Proposition B, p. 264).

Since $V \in L^{]Q/2[}(\Omega)$ there exists $q_1 \in]1, Q/2[$ such that $V \in L^{q_1}(\Omega) \cap L^{Q/2}(\Omega)$. We now fix $q \in]q_1, Q/2[$ and we set $q' = q/(q-1)$ and

$$\varepsilon = \frac{1}{q} - \frac{2}{Q}.$$

If $p \in]q', +\infty]$ and $u \in L^p(\Omega)$ then $f \in L^t(H)$ for $t = (1/q + 1/p)^{-1}$, since

$$\int_{\Omega} |Vu|^t \leq \| |V|^t \|_{q/t} \|u^t\|_{p/t} = \|V\|_q^t \|u\|_p^t.$$

Moreover $t \in]1, q] \subseteq]1, Q/2[$ and then we get $w \in L^{(1/t-2/Q)^{-1}}(H) = L^{(1/p+\varepsilon)^{-1}}(H)$ (see (3.17)) and also $u \in L^{(1/p+\varepsilon)^{-1}}(\Omega)$ (see (3.16)). We know a priori that $u \in L^{\infty}(\Omega)$; hence we can iterate the process above and get $w \in L^{q'}(H)$. Since $q \in]q_1, Q/2[$ is arbitrary, we finally obtain

$$(3.19) \quad w \in L^p(H) \quad \forall p \in \left] \frac{Q}{Q-2}, +\infty \right[.$$

Moreover, using (3.18) one can easily see that $w \in S_{\text{loc}}(H)$. Hence w is a nonnegative weak solution of

$$\begin{cases} -\mathcal{L}w \leq |V|w & \text{in } H \\ w \in S_{\text{loc}}(H) \end{cases}$$

(see (3.16)). It follows that

$$w(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as } d(\xi) \rightarrow +\infty, \quad \forall s < Q-2,$$

by means of (3.19) and Theorem 1.1. Again by (3.16) we finally get our statement. ■

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