

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

V. DE FILIPPIS

O. M. DI VINCENZO

**Prime rings with hypercommuting derivations
on a Lie ideal**

Rendiconti del Seminario Matematico della Università di Padova,
tome 102 (1999), p. 305-317

http://www.numdam.org/item?id=RSMUP_1999__102__305_0

© Rendiconti del Seminario Matematico della Università di Padova, 1999, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Prime Rings with Hypercommuting Derivations on a Lie Ideal.

V. DE FILIPPIS(*) - O. M. DI VINCENZO(**)(***)

ABSTRACT - Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R , L a non-central Lie ideal of R . If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R is an order in a simple algebra of dimension at most 4 over its center.

1. Introduction.

The classical hypercenter theorem, proved by I. N. Herstein [10], asserts that for a ring R not containing non-zero nil two-sided ideals, an inner derivation d_a , induced by $a \in R$, satisfying $d_a(x^m) = 0$, $m = m(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R , i.e. $a \in Z(R)$.

Moreover in [4, theorem 4] Chuang and Lin proved that for a ring R not containing non-zero nil right ideals, the inner derivation d_a , induced by $a \in R$, satisfying $[d_a(x^m), x^m]_k = 0$, $m = m(x) \geq 1$, $k = k(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R .

Later Chuang generalized the results above to arbitrary derivation d , defined in a prime ring R not containing non-zero nil right ideals. He proved in [3, corollary 2] that if $[d(x^m), x^m]_k = 0$, $m = m(x) \geq 1$,

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Messina, C/da Papardo salita Sperone 31, 98166 Messina.

(**) Dipartimento di Matematica, Università della Basilicata, Via N. Sauro 85, 85100 Potenza.

(***) Research supported by a grant from M.U.R.S.T.

$k = k(x) \geq 1$, for all $x \in R$, then either d vanishes identically on the whole ring R or R is commutative.

The starting point of this paper is the following result, obtained in [5]:

Let R be a prime ring, with no non-zero nil right ideals, L a non-central Lie ideal of R , d_a an inner derivation induced by $a \in R$. If $[d_a(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then either d_a vanishes identically on the whole ring R , that is $a \in Z(R)$, or R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree 4.

The last conclusion is well known to be equivalent, by Posner's theorem, to saying that R is an order in a simple algebra at most 4-dimensional over its center.

Moreover we remark that if $R = M_2(C)$, the ring of all 2×2 matrices over a commutative ring C , then for any $u \in [R, R]$ one has $u^2 \in Z(R)$ (see example (3) page 12 in [12]), and so our condition $[d(u^m), u^m]_k = 0$ holds for $m = 2$ and $k = 1$.

The purpose of this note is to generalize the result obtained in [5] to arbitrary derivation d . We will prove the following:

THEOREM. *Let R be a prime ring with no non-zero nil right ideal, L a non-central Lie ideal of R , d a non-zero derivation of R satisfying $[d(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. Then R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree 4.*

2. Preliminaries.

The proof of our theorem is based upon two results obtained respectively in [3] and [5].

The first one is a result of Chuang [3, proposition 2] concerning a careful analysis of derivations satisfying a particular property on semiprime rings.

The related objects we need to mention are the left Utumi quotient ring U , and also the two-sided Utumi quotient ring Q of a ring R (sometimes, as in [1], U and Q are called the maximal left ring of quotients and the symmetric ring of quotients respectively).

The definitions, the axiomatic formulations and the properties of these quotient rings U , Q can be found in [15], [8], [1].

For instance U , the left Utumi quotient ring of R , exists if and only if R is right faithful, that is for any $a \in R$, $Ra = 0$ implies $a = 0$.

In the same way we can define Q if R is both right and left faithful.

In any case, when R is a prime ring, all that we need here about these objects is that

$$1) R \subseteq Q \subseteq U.$$

$$2) U \text{ and } Q \text{ are prime rings [8, page 74].}$$

3) For all $q \in Q$ there exists a dense left ideal M of R such that $Mq \subseteq R$, moreover if $Mq = 0$, for some dense left ideal M of R , then $q = 0$.

$$4) \text{ The center of } U, \text{ denoted by } C, \text{ coincides with the center of } Q.$$

C is a field which is called the extended centroid of R [1, pages 68-70].

Moreover if R is a prime P.I. ring then, by Posner's theorem [9, theorem 1.4.3 page 40], C is the quotient field of $Z(R)$ and

$$RZ^{-1} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\} = RC$$

is a simple algebra finite dimensional over its center.

In this case it is easy to see that $RC = Q = U$.

Finally we recall that a map $d: R \rightarrow R$ is a derivation if, for any $x, y \in R$, $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$. Each derivation of a prime ring R can be uniquely extended to a derivation of its Utumi quotient ring U and thus all derivations of R will be implicitly assumed to be defined on the whole U (see [15, page 101] or [16, lemma 2]).

Now we are ready to state the result of Chuang ([3, proposition 1, page 46]) for prime rings. In this case as we said above U and Q are prime too and so any central idempotent is trivial.

Hence, for any $a \in U - \{0\}$, the norm $\|a\|$ of a , defined in [3, page 39, (7)], is always 1 and of course $\|0\| = 0$.

PROPOSITION 2.1. *Let R be a prime ring with extended centroid C . Let d a derivation of U , the left Utumi quotient of R , satisfying $d(a)(1 + a)^{-1} \in C$, for any $a \in R$, with $a^2 = 0$. Let Q be the two-sided Utumi quotient ring of R . Then*

either the ring Q is reduced, that is Q does not have any non-zero nilpotent element

or $Q = U = M_2(C)$, the ring of all two by two matrices over C

or the derivation d is the inner derivation induced by a square-zero

element $c \in U$, satisfying the property that, for any $x, y \in Q$ with $xy = 0$, we also have $xc = cy = 0$.

The second result concerns the generalized hypercentralizer of a non-central Lie ideal of R .

More precisely the generalized hypercentralizer of an arbitrary subset S of R is the following subring of R :

$$H_R(S) = \{a \in R: \text{ for all } s \in S \text{ there exist } n = n(a, s) \geq 1,$$

$$k = k(a, s) \geq 1 \text{ such that } [a, s^n]_k = 0\}.$$

In [5] we proved that if L is a non-central Lie ideal of a prime ring R with no non-zero nil right ideal, then either $H_R(L) = Z(R)$ or R satisfies $S_4(x_1, x_2, x_3, x_4)$.

Finally we remark that an important tool in our proof will be the theory of differential identities initiated by Kharchenko [13].

3. Proof of the theorem.

Throughout this paper we will use the following notation:

R will always be an associative prime ring, with no non-zero nil right ideal, L will be a non-central Lie ideal of R and d will be a derivation of R satisfying $[d(u^m), u^m]_k = 0$, for any $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. U will be the left Utumi quotient ring of R , and Q will be the two-sided Utumi quotient ring of R .

We start with an easy remark:

REMARK 1. *If R has characteristic $p \neq 0$ then for all $u \in L$ there exists $n = n(u) \geq 1$ such that $d(u^n) = 0$.*

PROOF. Let $u \in L$ be arbitrarily given. There exist $m = m(u) \geq 1$, $k = k(u) \geq 1$ such that $[d(u^m), u^m]_k = 0$. Pick an integer $t \geq 1$ such that $p^t \geq k$. Then

$$0 = [d(u^m), u^m]_{p^t - k} = [d(u^m), u^m]_{p^t}.$$

Since R is of characteristic $p > 0$, we have

$$0 = [d(u^m), u^m]_{p^t} = [d(u^m), u^{mp^t}]$$

and this implies immediately $[d(u^{mp^t}), u^{mp^t}] = 0$. Let $a = u^{mp^t}$. We obtain $[d(a), a] = 0$.

Using the commutativity we have $d(a^p) = pa^{p-1}d(a) = 0$, that is

$$d(u^{mp^{t+1}}) = 0$$

and so we have shown that for all $u \in L$ there exists $l = l(u) \geq 1$ such that $d(u^l) = 0$. ■

Notice that in this case, if $p \neq 2$, then our result follows immediately by theorem 2 of [7].

Now we make some other reductions.

It is well known that if L is a non-central Lie ideal of a prime ring R then either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or there exists a non-zero two-sided ideal I of R such that $[I, R] \subseteq L$ and $[I, R] \not\subseteq Z(R)$.

Therefore we will assume, in all that follows, that $L = [I, R]$ for some non-zero two-sided ideal I of R (see for instance Lemma 2 and Proposition 1 in [6]).

In this case L is invariant under any inner automorphism induced by an invertible (or quasi-invertible) element of R .

Moreover, if $Z(R) \neq 0$, then we can consider

$$\bar{R} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\}$$

$$\bar{L} = \{uz^{-1} : u \in L, z \in Z(R) - \{0\}\}$$

which are the localizations at $Z(R)$ of R and L respectively. Since $L = [I, R]$ is a $Z(R)$ -submodule of R , we have

REMARK 2.

- 1) $\bar{L}$ is a non-central Lie ideal of the prime ring $\bar{R}$
- 2) the derivation d extends uniquely to a derivation on $\bar{R}$ as follows

$$d(rz^{-1}) = (d(r)z - rd(z))z^{-2}$$

- 3) the derivation d , defined on $\bar{R}$, satisfies our assumptions on $\bar{L}$, that is for any $\bar{u} \in \bar{L}$, there exist $n = n(\bar{u}) \geq 1$, $k = k(\bar{u}) \geq 1$ such that $[d(\bar{u}^n), \bar{u}^n]_k = 0$.

LEMMA 3.1. *Let $a \in R$. If a is invertible then $d(a)a^{-1} \in H_R(L)$, if a is quasi-invertible then $d(a)(1+a)^{-1} \in H_R(L)$.*

PROOF. First we assume that a is invertible, then, as we said above, $aLa^{-1} = L$, hence for any $u \in L$ there exist $m, k, n, h \geq 1$ such that

$$0 = [d((a^{-1}ua)^m), (a^{-1}ua)^m]_k = 0$$

and

$$[d(u^n), u^n]_h = 0.$$

Hence for $s = nm$ and $t = \max\{h, k\}$ we also have:

$$[d((a^{-1}ua)^s), (a^{-1}ua)^s]_t = 0 = [d(u^s), u^s]_t.$$

It follows that:

$$\begin{aligned} 0 &= [d(a^{-1}u^s a), a^{-1}u^s a]_t = \\ &= [d(a^{-1})u^s a + a^{-1}d(u^s)a + a^{-1}u^s d(a), a^{-1}u^s a]_t = \\ &= [-a^{-1}d(a)a^{-1}u^s a + a^{-1}u^s d(a), a^{-1}u^s a]_t + [a^{-1}d(u^s)a, a^{-1}u^s a]_t = \\ &= -a^{-1}[d(a)a^{-1}u^s - u^s d(a)a^{-1}, u^s]_t a = -a^{-1}[d(a)a^{-1}, u^s]_{t+1} a. \end{aligned}$$

Hence $[d(a)a^{-1}, u^s]_{t+1} = 0$ that is $d(a)a^{-1} \in H_R(L)$.

A similar proof holds if a is a quasi-invertible element of R . ■

We remark that any square-zero element a of R is quasi-invertible with quasi-inverse $-a$.

Therefore, by [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ and we are done or $d(a)a^{-1} \in Z(R)$, that is the derivation d satisfies the hypothesis of Chuang's result. In this case one of the three conclusions of the Proposition 2.1 must hold.

Now we treat each case separately.

Of course if $U = M_2(C)$, the ring of all 2×2 matrices over C , then it satisfies the standard identity $S_4(x_1, x_2, x_3, x_4)$ and we are done again, since $R \subseteq U$.

In the second case we have:

PROPOSITION 3.1. *If the derivation d is the inner derivation defined by a square-zero element c in U , satisfying $xc = cy = 0$ for any $x, y \in Q$, with $xy = 0$, then d vanishes identically on R .*

PROOF. Since c is an element of the left Utumi quotient ring of R , there exists a left dense ideal M of R such that $Mc \subseteq R$ (see proposition 2.1.7 in [1]).

Moreover, since R is a prime ring, IM is again a left dense ideal of R and, of course, $IMc \subseteq IR \subseteq I$.

In other words we can assume that there exists M left dense ideal of R such that $Mc \subseteq I$ and so $[Mc, Mc] \subseteq L$. Therefore for any $x, y \in M$ there exist $m = m(c, x, y) \geq 1$, $k = k(c, x, y) \geq 1$ such that

$$[d([xc, yc]^m), [xc, yc]^m]_k = 0.$$

Moreover $d([xc, yc]^m) = [c, [xc, yc]^m] = c[xc, yc]^m$. Therefore

$$\begin{aligned} 0 &= [c[xc, yc]^m, [xc, yc]^m]_k = \\ &= \sum_{h=0}^k \binom{k}{h} (-1)^h [xc, yc]^{mh} (c[xc, yc]^m)[xc, yc]^{m(k-h)} = c[xc, yc]^{m(k+1)}. \end{aligned}$$

Thus $[xc, yc][xc, yc]^{m(k+1)} = 0$, that is $[xc, yc]$ is a nilpotent quasi-invertible element. By lemma 3.1 and the main theorem in [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or $d([xc, yc]) = \alpha(1 + [xc, yc])$, where $\alpha \in Z(R)$.

In the first case R is a prime PI ring and so, by Posner's theorem, $RC = S = Q = U$ is a central simple algebra finite dimensional over its center C .

Since U satisfies $S_4(x_1, x_2, x_3, x_4)$, if c is a non-zero square-zero element, then we have $U = M_2(C)$, the ring of 2×2 matrices over C .

Since $xc = cy = 0$, for any $x, y \in Q = U$ such that $xy = 0$, then $e_{11}c = e_{21}c = e_{22}c = e_{12}c = 0$, that is $M_2(C)c = 0$ and so $c = 0$, a contradiction.

In the second case we know that $d([xc, yc]) = [c, [xc, yc]] = c[xc, yc]$.

$$\text{So } \alpha^2(1 + [xc, yc])^2 = (\alpha(1 + [xc, yc]))^2 = (d([xc, yc]))^2 = (c[xc, yc])^2 = 0.$$

Since $[xc, yc]$ is quasi-invertible, then $\alpha = 0$. Thus $0 = d([xc, yc]) = c[xc, yc]$. Hence, for any x, y in M , $c[xc, yc] = 0$.

Let x, y, z be in R , t in M . Since M is a left dense ideal of R , we have that xt, yt, zt fall in M and so $ztc[xtc, ytc] = 0$, that is R is GPI [2].

In this case, by Martindale's result the central closure $S = RC$ is a primitive ring, containing a minimal right ideal eS , such that eSe is a division algebra finite dimensional over C , for any minimal idempotent e of S [9, theorem 1.3.2].

If $e = 1$ then S is a finite dimensional division algebra over C . Therefore S is PI and so R is PI too. As we said above in this case $S = Q = U$ and so $c \in S$ which is a division ring. Hence $c = 0$ and consequently $d = d_c = 0$.

Now we may suppose $e \neq 1$. We know that $xc = cy = 0$, for any x, y in Q , with $xy = 0$.

Let $x^2 = 0$. Since $xc = cx = 0$ then $d(x) = [c, x] = 0$, that is c commutes with every square-zero element x in Q .

Let A be the subring generated by the elements of square zero. A is invariant under all automorphisms of Q . By our assumption there are non-trivial idempotent in the prime ring Q and so A contains a non-zero ideal J of Q by [11].

Now, since $0 = d(A) \supseteq d(J) \supseteq d(JQ) = Jd(Q)$, by the primeness of Q we obtain $d(Q) = 0$, that is $d = 0$ in Q and so in R too. ■

REMARK. The last case is the one in which Q is a reduced ring. Since Q is also a prime ring then it must be a domain. In fact, let $x, y \in Q$ be such that $xy = 0$ and $y \neq 0$. Then, for any $z \in Q$, we have $(yzx)^2 = yzxyzx = 0$ and so $yzx = 0$, that is $yQx = 0$ and $x = 0$ because Q is prime.

DEFINITION. For $a \in R$ let

$$H(a) = \{r \in R : [r, a]_m = 0 \text{ for some integer } m = m(r) \geq 1\}.$$

Of course $H(a)$ is a subring of R .

We also have:

LEMMA 3.2. *Let R be a domain of characteristic zero and let d be the derivation satisfying our assumption. If a is an element of I such that*

$$[d(a), a]_l = 0 \text{ for some } l = l(a) \geq 1$$

then $H(a)$ is invariant under the derivation d and moreover $d(a)$ is in the center of $H(a)$.

PROOF. By localizing at non-zero integers we may assume that R is an algebra over the field of the rational numbers.

By [3, assertion 2] it follows that $H(a)$ is invariant under d . Now, we put $\delta = d_a$, the inner derivation induced by a .

Of course the derivation δ restricted to $H(a)$, which we also denote δ ,

is nil and hence for any integer λ , the derivation $\lambda\delta$ is also nil on $H(a)$.

Since $H(a)$ is an algebra over the field of the rational numbers, the map $\exp(\lambda)$ is an automorphism of $H(a)$ (see [3, proposition 2]), hence the map $d_\lambda = \exp(\lambda\delta) d \exp(-\lambda\delta)$ is a derivation of $H(a)$.

Obviously $I \cap H(a)$ is a two-sided ideal of $H(a)$ which is invariant under the action of $\exp(\lambda\delta)$.

Hence $L_1 = [I \cap H(a), H(a)] \subseteq [I, R] = L$ is a Lie ideal of $H(a)$, moreover, for any $u \in L_1$, there exist some integers $n = n(u) \geq 1$, $k = k(u) \geq 1$ such that $[d_\lambda(u^n), u^n]_k = 0$.

Now, given $u \in L_1$, there exist integers $n = n(u) \geq 1$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, $h = h(u) \geq 1$ such that

$$[d(u^m), u^m]_h = 0 = [d_\lambda(u^n), u^n]_k$$

hence, as in the proof of lemma 3.1, for $s = nm$ and $t = \max\{h, k\}$ we also have

$$[d(u^s), u^s]_t = [d_\lambda(u^s), u^s]_t$$

that is $[(d_\lambda - d)(u^s), u^s]_t = 0$.

By [3, proposition 2, (3)] the derivation $d_\lambda - d$ is the inner derivation induced by the element $b_\lambda = \sum_{n \geq 1} ([\delta(\lambda a), \lambda a]_{n-1})/n!$, and so b_λ is in the generalized hypercentralizer of L_1 in $H(a)$.

If $I \cap H(a)$ is the zero ideal of $H(a)$, then $a = 0$ since it is in $I \cap H(a)$ and of course $d(a) = 0 \in Z(H(a))$.

If $I \cap H(a)$ is non-zero then, by [5, proposition 4.1], either $H_{H(a)}(L_1) = Z(H(a))$ or $H(a)$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case we may conclude, by a Vandermonde determinant argument, $d(a) \in Z(H(a))$.

In the other case, by localizing at the center of $H(a)$, we may assume that $H(a)$ is a division algebra of dimension at most 4 over its center $Z(H(a))$. It follows that there exists $m \geq 1$ such that $\delta^m(r) = 0$, for any $r \in H(a)$, that is δ is a nil of bounded index on $H(a)$.

By [9, lemma 1.1.9] there exists $z \in Z(H(a))$ such that $a - z$ is nilpotent and so $a - z = 0$, because $H(a)$ is a division ring. Hence $a \in Z(H(a))$. Therefore, for any $r \in H(a)$, $0 = d([r, a]) = d(ra - ar) = [r, d(a)]$, that is $d(a) \in Z(H(a))$. ■

LEMMA 3.3. Let R be a domain. For any $x, y \in I$ there exists $m = m(x, y) \geq 1$ such that $C_R([x, y]^m) = \{r \in R : [[x, y]^m, r] = 0\}$ is invariant under derivation d , that is $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$.

PROOF. If $\text{char. } R > 0$ then, as we said in Remark 1, our assumption about the derivation d implies that for any $x, y \in R$ there exists $m = m(x, y) \geq 1$ such that $d([x, y]^m) = 0$. For any $r \in C_R([x, y]^m)$ we have

$$0 = d([[x, y]^m, r]) = [[x, y]^m, d(r)]$$

that is $d(r) \in C_R([x, y]^m)$.

Now let $\text{char. } R = 0$. For any $r \in C_R([x, y]^m)$ one has

$$0 = d([[x, y]^m, r]) = [d([x, y]^m), r] + [[x, y]^m, d(r)].$$

Since by previous lemma $d([x, y]^m) \in Z(H([x, y]^m))$ then $[[x, y]^m, d(r)] = 0$, that is $d(r) \in C_R([x, y]^m)$. ■

The last step in our proof is the following:

PROPOSITION 3.2. Let Q be a domain, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.

PROOF. First we show that for all $x, y \in I$ one has:

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In fact given $x, y \in I$, by previous lemma there exists an integer $m = m(x, y) \geq 1$ such that $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$, and of course we can assume $[x, y] \neq 0$. We denote $A = C_R([x, y]^m)$, therefore $[x, y]^m$ is a non-zero element of $Z(A)$ and $I \cap A$ is a non-zero two-sided ideal of A . By localizing A at $Z(A)$ we obtain a domain D whose center is a field containing $[x, y]^m$, moreover $D = \{rz^{-1} : r \in A, z \in Z(A) - \{0\}\}$. As we said in Remark 2 d extends uniquely to a derivation on D , which we will also denote d and moreover d satisfies our assumption on D with respect to the Lie ideal $\bar{L}$ which is the localization of $[I \cap A, A] \subseteq [I, R] = L$.

Of course $[x, y]$ is invertible in D , therefore by lemma 3.1 and main result in [5], either $d([x, y]) = \alpha[x, y]$, for some $\alpha \in Z(A)$ or D satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case $[[x, y], d([x, y])] = 0$ and a fortiori

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In the second case D is a division algebra of dimension at most 4 over its center. Moreover we know that in this case, for any $a, b \in D$, $[a, b]^2 \in Z(D)$.

This implies $[[x, y], d([x, y])]^2 \in Z(A)$, because $[x, y] \in A \subseteq D$ and $d([x, y]) \in A \subseteq D$.

In particular the following holds

$$[[x, y], d([x, y])]^2, [x, y] = 0.$$

Therefore, in any case, we have

$$[[x, y], [d(x), y] + [x, d(y)]]^2, [x, y] = 0$$

for all $x, y \in I$.

In other words

$$\phi(x_1, x_2, d(x_1), d(x_2)) = [[x_1, x_2], [d(x_1), x_2] + [x_1, d(x_2)]]^2, [x_1, x_2]$$

is a differential identity for I .

Because any non-zero two-sided ideal of a prime ring R is also a dense (or rational, see [8] page 50) R -submodule of U , then, by [16, theorem 2], $\phi(x_1, x_2, d(x_1), d(x_2))$ is a differential identity for U .

By theorem 1 of [16] (or theorem 2 in [14]) it follows that either d is an inner derivation of U or U satisfies the polynomial identity

$$\phi(z_1, z_2, z_3, z_4) = [[z_1, z_2], [z_3, z_2] + [z_1, z_4]]^2, [z_1, z_2].$$

If d is an inner derivation induced by some $q \in U$ then

$$[[x, y], [q, [x, y]]]^2, [x, y] = 0$$

for all $x, y \in U$.

In particular this one holds in R and so R is a GPI-ring [2], its central closure $S = RC$ is a primitive ring having minimal right ideal, moreover, for any minimal idempotent $e = e^2 \neq 0$, eSe is a division algebra finite dimensional over its center $eCe \cong C$ [9, theorem 1.3.2].

Because $S = RC \subseteq Q$ and Q is a domain then S is a domain and so any idempotent element e of S is trivial.

This implies that S is a division algebra finite dimensional over C , that is R is a PI-ring and C is the quotient field of $Z(R)$.

It follows that $RC = S = Q = U$.

Moreover $RC = S = \overline{R} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\}$ by Posner's theorem and so, for any $u \in \overline{L} = \{uz^{-1} : u \in L, z \in Z(R) - \{0\}\}$, there exist integers m, k such that $[d(u^m), u^m]_k = 0$ (see Remark 2).

Because d is the inner derivation in U induced by $q \in U$, we obtain that $q \in H_U(\overline{L})$, that is either $q \in Z(U)$ or U satisfies $S_4(x_1, x_2, x_3, x_4)$. In this last case we are done because $R \subseteq U$. If $q \in Z(U)$ then $d = 0$ in U , and this is a contradiction. Now we have to analyze the only case in which $\phi(z_1, z_2, z_3, z_4)$ is a polynomial identity of U . In this case $R \subseteq U$ satisfies the blended component $[[z_1, z_2], [z_3, z_2]]^2, [z_1, z_2]$ of the polynomial identity $\phi(z_1, z_2, z_3, z_4)$.

Since R is prime there exists a field F such that R and $M_k(F)$, the ring of all $k \times k$ matrices over F , satisfy the same polynomial identities (see [12]).

Suppose $k \geq 3$. Let e_{ij} the matrix unit with 1 in (i, j) entry and 0 elsewhere.

Let $z_1 = e_{13} + e_{22}$, $z_2 = e_{21} + e_{33}$, $z_3 = e_{32} + e_{31}$. By calculation we obtain

$$[z_3, z_2] = -e_{32}$$

$$[z_1, z_2] = e_{21} - e_{23} + e_{13}$$

$$[[z_1, z_2], [z_3, z_2]] = e_{22} - e_{12} + e_{31} - e_{33}$$

$$([z_1, z_2], [z_3, z_2])^2 = e_{22} - e_{12} - e_{32} - e_{31} + e_{33}$$

$$([([z_1, z_2], [z_3, z_2])^2, [z_1, z_2]) = e_{12} - e_{31} \neq 0$$

and this is a contradiction. So $k \leq 2$ and R satisfies $S_4(x_1, x_2, x_3, x_4)$. ■

At this point the proof of our theorem is complete and we state it here again for sake of clearness:

THEOREM 3.1. *Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R , L a non-central Lie ideal of R . If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.*

REFERENCES

- [1] K. I. BEIDAR - W. S. MARTINDALE III - V. MIKHALEV, *Rings with generalized identities*, Pure and Applied Math., Dekker, New York (1996).
- [2] C. L. CHUANG, *GPIs having coefficients in Utumi quotient ring*, Proc. Amer. Math. Soc. 103, no. 3 (1988).
- [3] C. L. CHUANG, *Hypercentral derivations*, J. Algebra, 166 (1994), pp. 34-71.
- [4] C. L. CHUANG - J. S. LIN, *On a conjecture by Herstein*, J. Algebra, 126 (1989), pp. 119-138.
- [5] V. DE FILIPPIS - O. M. DI VINCENZO, *On the generalized hypercentralizer of a Lie ideal in a prime ring*, Rend. Sem. Mat. Univ. Padova, 100 (1998), pp. 283-295.
- [6] O. M. DI VINCENZO, *On the n -th centralizer of a Lie ideal*, Boll. UMI (7), 3-A (1989), pp. 77-85.
- [7] O. M. DI VINCENZO, *Derivations and multilinear polynomials*, Rend. Sem. Mat. Univ. Padova, 81 (1989), pp. 209-219.
- [8] C. FAITH, *Lectures on Injective Modules and Quotient Rings*, Lecture Notes in Mathematics, 49, Springer-Verlag, New York (1967).
- [9] I. N. HERSTEIN, *Rings with involution*, Univ. of Chicago Press, Chicago (1976).
- [10] I. N. HERSTEIN, *On the hypercenter of a ring*, J. Algebra, 36 (1975), pp. 151-157.
- [11] I. N. HERSTEIN, *A theorem on invariant subrings*, J. Algebra, 83 (1983), pp. 26-32.
- [12] N. JACOBSON - P. I. ALGEBRAS, *An Introduction*, Lecture Notes in Mathematics, no. 44, Springer-Verlag, Berlin/New York (1975).
- [13] V. K. KHARCHENKO, *Differential identities of prime rings*, Algebra and Logic, 17 (1978), pp. 155-168.
- [14] V. K. KHARCHENKO, *Differential identities of semiprime rings*, Algebra and Logic, 18 (1979), pp. 86-119.
- [15] J. LAMBEK, *Lectures on Rings and Modules*, Blaisdell Waltham, MA (1966).
- [16] T. K. LEE, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica, 20, no. 1 (1992), pp. 27-38.

Manoscritto pervenuto in redazione il 4 febbraio 1998
 e in forma revisionata il 31 marzo 1998.