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A Note on the Fixed Point for the Polynomials of a Boolean Algebra with an Operator of Endomorphism.

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SUMMARY - We study fixed point problem for Boolean algebras with an endomorphism operator k. In particular we prove that in every finite algebra of this kind there exist polynomials $f(x)$ (in which x only occurs within the scope of k) without fixed point. Some examples of Boolean algebras with endomorphism operator k in which every polynomial $f(x)$ (in which x only occurs within the scope of k) has a fixed point are also given. Also we establish some properties of the algebras of this kind.

0. Introduction.

Let we call k-algebra a Boolean algebra $\mathfrak{A} = \langle A, +, \cdot, \neg, 0, 1, k \rangle$, endowed with an endomorphism k (see [4] for notations and terminology).

In this note we study the problem of the existence of fixed points for the polynomials $f(x)$ of the k-algebras in which x only occurs within the scope of k.

A similar question has been studied for the τ-algebras (or diagonalizable algebras), that is Boolean algebras $\langle A, +, \cdot, \neg, 0, 1, \tau \rangle$ endowed with an operator τ such that $\tau 1 = 1$, $\tau(p \cdot q) = \tau p \cdot \tau q$, $\tau(\tau p \rightarrow p) \leq \tau p$ for every $p, q \in A$.

In [1], [12], [16] the existence and uniqueness of a fixed point for those polynomials has been proved. Moreover the relationship

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between diagonalizable algebras and fixed point algebras (introduced by C. Smorynski in [17]) have been studied in [9].

It is obvious that there are k-algebras that do not have the fixed point property: it suffices to consider a Boolean algebra with the identity as an endomorphism.

Therefore the problem has to be posed in different terms: we study the problem of the existence of k-algebras with the fixed point property.

In this note we prove that every finite k-algebra contains polynomials which do not have any fixed point and we give examples of k-algebras whose polynomials $f(x)$ (in which x only occurs within the scope of k) have fixed point. We study also some properties of those algebras.

1. Boolean algebras with additional operations.

Boolean algebras endowed with additional operations have been widely studied (see [8] and [15]). In this context modal algebras and diagonalizable algebras are of particular interest. We remember that modal algebras are boolean algebras $\langle A, +, \cdot, \neg, 0, 1, \tau \rangle$ with an unary operation τ having the property that $\tau(p \cdot q) = \tau p \cdot \tau q$ for any $p, q \in A$, and $\tau 1 = 1$, and that diagonalizable algebras (introduced by R. Magari in [12]) are modal algebras where τ has the property that $\tau(\tau p \rightarrow p) \leqslant \tau p$ for every $p \in A$.

In some cases, these algebras have been introduced to deal with problems in logic with algebraic tools.

In particular in Peano arithmetic, let $A(x)$ be a PA extensional formula (that is such that if $\vdash_{PA} p \leftrightarrow q$ then $\vdash_{PA} A(\overline{p}) \leftrightarrow A(\overline{q})$): in the Lindenbaum algebra B_{PA} of the sentences we may introduce an unary operation $\widetilde{A}$ defining $\widetilde{A}[p]_{PA} = [A(\overline{p})]_{PA}$ (where $[p]_{PA}$ denote the equivalence class of p with respect to the provability equivalence relation). We obtain a Boolean algebra with an additional unary operation; moreover if $A(x) = \mathscr{P}r_{PA}(x)$ (i.e. the provability predicate for PA) this algebra is diagonalizable.

For a complete study of the problems related to these ideas see [1], [13], [14], [16].

Given a diagonalizable algebra A it is possible to construct a Boolean algebra B in such a way that (A, B) is a fixed point algebra. By this we mean a couple of Boolean algebras (A, B) where each element of B is a function from A to A and

i) B contains every function which is constant on A,

ii) the Boolean operations are defined «pointwise» on A,

iii) B is closed under composition,

iv) each $\alpha \in B$ has a fixed point $a \in A$.

The existence of such an algebra B can be proved using fixed point theorems for particular polynomials of the diagonalizable algebra A (see [17]).

The algebraic approach to the study of some problems in logic has, in some cases, the advantage of translating some «metamathematical» theorems in simple form, and to allow applying the methods of universal algebra. On the other hand this approach has the disadvantage that $A(x)$, in order to be applicable, has to be an extensional formula.

As already noted, the operator τ in a diagonalizable algebra translates in algebraic terms the provability predicate.

On the other hand it is well known that it is not possible to introduce a truth predicate in arithmetics.

Some authors have proposed some formal theories in which they add to PA a predicate T which embodies some aspect of the truth predicate (see [6] and [7]).

In our opinion k-algebras may constitute a possible answer to this problem, since the properties of the unary operator k may simulate in algebraic terms the main properties of the intuitive meaning of «truth».

It is in this regard that study the fixed point problem becomes relevant. Moreover k-algebras are in the «intersection» of modal algebras and of Boolean algebras with additive operators (see [10] and [11]) and this is another reasons for their interest.

We mention that the results regarding provability in PA which have been obtained with an algebraic approach, may be proved also through the study of some suitable modal logics (see [3], [18], [19]).

2. Finite k-algebras.

In this paragraph we will be dealing with the problem of the fixed point for the polynomials of finite k-algebras.

Let's begin with the following simple

LEMMA 2.1. *Let $\mathfrak{A}$ be a k-algebra. Let's suppose that there exists an element $a \in A$ such that $\neg ka = a$. Then for each natural number n different from 0 you have:*

$$\neg k^n a = \begin{cases} a & \text{for } n \text{ odd}, \\ \neg a & \text{for } n \text{ even}, \end{cases}$$

PROOF. From the hypothesis you have: $k^2 a = k(\neg a) = a$.
Now, if n is even $k^n a = (k^2 \circ \ldots \circ k^2) a = a$. So $\neg k^n a = \neg a$.
If n is odd, by exploiting the previous part, we get

$$k^n a = k(k^{n-1} a) = ka = \neg a$$

that is to say $\neg k^n a = a$.

LEMMA 2.2. *Let $\mathfrak{A}$ be a k-algebra. The polynomials $\neg k^n x$ have a fixed point if and only if the same is true for the polynomials $\neg k^{(2^m)} x$ (n, m natural numbers and n different from 0).*

PROOF. Let us consider $\neg k^n x$. If n is odd and a is a fixed point for $\neg k^{(2^0)} x$, then, by the previous lemma, it is also a fixed point for $\neg k^n x$.

If n is even, then there exist two numbers p, q with q odd, so that $n = 2^p q$. By hypothesis, the polynomial $\neg k^{(2^p)} x$ has a fixed point. Let's put $k^{(2^p)} = h$. Obviously h is an endomorphism and $k^n x = h^q x$. If a is a fixed point of $\neg k^{(2^p)} x$, then (as in the Lemma 2.1), being $\neg ha = a$, we have $\neg h^q a = a$. That is to say a is a fixed point of $\neg k^n x$.

The opposite is obvious.

THEOREM 2.1. *Let $\mathfrak{A}$ be a k-algebra. If $n \neq m$ (n, m natural numbers different from 0), each fixed point of $\neg k^{(2^n)} x$ is different from each fixed point of $\neg k^{(2^m)} x$ (if the fixed points exist).*

PROOF. Let's assume $n > m$ that is $n = m + c$ for a suitable c. Posing $k^{(2^m)} = h$ we have $\neg k^{(2^n)} x = \neg h^{(2^c)} x$. Let a be a fixed point of $\neg k^{(2^m)} x = \neg hx$. Being $\neg ha = a$, we have from Lemma 2.1 $\neg h^{(2^c)} a = \neg a$ that is $\neg k^{(2^n)} a = \neg a$. Therefore a is not a fixed point for $\neg k^{(2^n)} x$. (Let's assume that the Boolean algebra $\mathfrak{A}$ is not degenerate, that is $a \neq \neg a$).

COROLLARY 2.1. *Let $\mathcal{A}$ be a finite k-algebra. Then there exists a natural number n, different from zero, such that the polynomial $\neg k^n x$ does not have a fixed point (in $\mathcal{A}$).*

PROOF. Obvious.

REMARK. In order to obtain the previous results it is sufficient to consider a set A with an unary operation $\neg$ so that for each element a of A we have $\neg(\neg a) = a$, $\neg a \neq a$ and an endomorphism (with respect to $\neg$).

3. – Infinite k-algebras.

We have seen, in the previous paragraph, that in each finite k-algebra there exists at least one polynomial of the kind $\neg k^n x$ which does not have a fixed point. In this paragraph we will give examples of k-algebras where every polynomial of the type $\neg k^n x$ has a fixed point and examples where every polynomial has a fixed point.

EXAMPLE 1. Let's indicate with β' the free Boolean algebra generated by a numerable set. Let $X = (x_i \mid i \in N - \{0\})$ be a family of free generators for β' (N is the set of natural numbers).

Let's indicate with k the endomorphism of β' which extends the following function f from X to β':

$$fx_i = \begin{cases} \neg x_{2^{n-1}}, & \text{if } i = 2^n - 1 \text{ (for a suitable } n \in N - \{0\} \text{ and } i \in N - \{0\}), \\ x_{i+1}, & \text{otherwise}. \end{cases}$$

Let β the k-algebra formed by β' and k.

We claim that, relatively to the k-algebra β, each polynomial of the kind $\neg k^n x$ has a fixed point. In fact, due to Lemma 2.2, it is sufficient to prove that each polynomial of the type $\neg k^{(2^m)} x$ has a fixed point and one can easily prove that x_{2^m} is a fixed point for $\neg k^{(2^m)} x$.

EXAMPLE 2. Let's indicate now with Z the set of integers numbers and let's consider the Boolean algebra $\mathcal{C}' = \langle \mathcal{P}(Z), \cup, \cap, \neg, Z, \emptyset \rangle$ (where by $\neg$ we indicate the operation of complementation).

Let $k\colon \mathcal{P}(Z) \to \mathcal{P}(Z)$ be the function defined by

$$kY = \{z \in Z\colon \exists y \in Y \text{and} z = y + 1\}, \qquad (Y \subset Z).$$

It can be easily proved that k is an endomorphism of $\mathcal{C}'$.

We observe that in the corresponding k-algebras every polynomial of the type $\neg k^n x$ has a set A^n of fixed points, where:

$$A^n = \{x \in Z: x \equiv y (\mathrm{mod}\ 2n),\ y \in \{0, 1, \dots, n-1\}\}.$$

EXAMPLE 3. We will demonstrate now that the Lindenbaum's algebra of the Peano's arithmetics can be endowed with an endomorphism k in such a way that, relatively to the k-algebra thus obtained, each polynomial $f(x)$ (in which x only occurs within the scope of k) has a fixed point.

Referring to the Theorem 3.17 of [5] we can obtain the following results (we express it in the notations of [5]):

1) If $\vdash_{PA} p$ then $\vdash_{PA} \mathcal{P}_{r_\beta}(\overline{p})$;
2) $\vdash_{PA} \mathcal{P}_{r_\beta}(\overline{p}) \wedge \mathcal{P}_{r_\beta}(\overline{p \rightarrow q}) \rightarrow \mathcal{P}_{r_\beta}(\overline{q})$;
3) $\vdash_{PA} \mathcal{P}_{r_\beta}(\overline{p}) \vee \mathcal{P}_{r_\beta}(\overline{\neg p})$;
4) $\vdash_{PA} \neg \mathcal{P}_{r_\beta}(\overline{p \wedge \neg p})$.

It makes sense then to define a function k on the Lindenbaum's algebra of Peano's arithmetics posing:

$$k[p] = [\mathcal{P}_{r_\beta}(\overline{p})].$$

The mentioned conditions can then be translated in the following $(0, 1, \rightarrow, \leqslant, \dots$ have the usual meaning):

1′) $k1 = 1$;
2′) $ka \cdot k(a \rightarrow b) \leqslant kb$;
3′) $ka + k(\neg a) = 1$;
4′) $k0 = 0$.

These imply:

5′) $ka \cdot kb = k(a \cdot b)$;
6′) $ka \cdot k(\neg a) = 0$;
7′) $k(\neg a) = \neg ka$.

In fact from 1′) and 2′) it follows the monotony of k.

Let $a \leqslant b$ then $a \rightarrow b = 1$ therefore $ka \cdot k(a \rightarrow b) = ka \leqslant kb$. From the monotony of k we have easily $ka \cdot kb \geqslant k(a \cdot b)$. On the other hand $ka \cdot kb \leqslant$ $\leqslant ka \cdot k(a \rightarrow b) = ka \cdot k(a \rightarrow a \cdot b) \leqslant k(a \cdot b)$. Therefore 5′) is true.

From 5′) and 4′) we have $ka \cdot k(\neg a) = k0 = 0$ that is 6′).

From 3′) and 6′) it obviously follows 7′).

k is therefore an endomorphism.

Let's now consider the k-algebra constitued by the Lindenbaum's algebra of Peano's arithmetics and by the endomorphism k which we have just defined. From the theorem of the fixed point relative to Peano's arithmetics it follows that each polynomial $f(x)$ (in which x only occurs within the scope of k) has a fixed point.

EXAMPLE 4. Let $Z^- = \{a \in Z : a \leqslant 0\}$. Let's consider the algebra

$$\beta(Z^-) = \langle P(Z^-), \cup, \cap, \neg \rangle$$

and let's define $K^*X = \{y \in Z^- : y+1 \in X\}(X \subseteq Z^-)$. We notice that

$$K^*(X \cup Y) = K^*X \cup K^*Y,$$

$$K^*(\neg X) = \neg K^*X,$$

$$K^*(Z^-) = Z^- - \{0\}.$$

Therefore K^* is not an homomorphism in $\beta(Z^-)$.

Let's consider in $P(Z^-)$ the relation $\simeq$ thus defined: $X \simeq Y$ if and only if $X \div Y$ is finite.

It is easy to see that $\simeq$ is a congruence relation in $\langle P(Z^-), \cup, \cap, \neg, K^* \rangle$ and that in the Boolean algebra quotient the operator K defined by $K[X]_\simeq = [K^*X]_\simeq$ is an homomorphism. So we obtain a k-algebra.

Let us prove that every polynomial $f(x)$ (in which x only occurs within the scope of k) has a fixed point. It is sufficient to prove that every polynomial of $\langle P(Z^-), \cup, \cap, \neg, K^* \rangle$ has a fixed point.

First of all let's consider the following lemmas.

I) Given a polynomial $f(x)$ and $X \subset Z^-$, in order to see if x belongs to $f(X)$ or not it is sufficient to decide if $y \in X$ for every $y \geqslant x$. In particular if in $f(x)$ x only occurs within the scope of k, one gets the property:

II) In order to establish if $x \in f(X)$ it is sufficient to know if $y \in X$ for every $y > x$.

From this results it follows that, reasoning on the structure of $f(x)$, we can build a fixed point X for it as follows: from the composition of the polynomial we can decide whether 0 belongs to $f(X)$ or not, whatever X may be; so, according to the answer, we can pose 0 in X or not.

Let's suppose we have established for every $i \leqslant n$ if $(-i) \in X$ or not, at this point we can decide that $(-i) \in X$ if and only if $(-i) \in f(X)$. If we set $i = n+1$, we can decide if $-n-1$ belongs to $f(X)$ or not according to Lemma II.

4. - Filters and congruences.

Let V denote the variety of the k-algebras. Let us give two definitions.

DEFINITION 4.1. A k-filter of $\mathcal{A} \in V$ is a Boolean filter F such that

$$\text{if } p \in F \text{ then } kp \in F \quad (p \in A).$$

DEFINITION 4.2. A k-ideal of $\mathcal{A} \in V$ is a Boolean ideal J such that

$$\text{if } p \in J \text{ then } kp \in J \quad (p \in A)$$

The link between congruences of $\mathcal{A}$ and k-filters (or k-ideals) is completely determined by the following

THEOREM 4.1. *The variety V of the k-algebras admits a good theory of the ideals.*

PROOF. The result is easily obtainable with elementary procedings (like in [4]) or by applying the process of A. Ursini [22] to V. We notice that in this case the 1-ideals and the 0-ideals are respectively the k-filters and the k-ideals.

Let's now examine some properties of the k-filters of an algebra $\mathcal{A} \in V$ with the property of the fixed point for the polynomials.

THEOREM 4.2. *a) Each proper k-filter of $\mathcal{A}$ does not contain any fixed point for the polynomials of the kind $\neg k^n x (n \neq 0)$;*

b) Each proper k-filter of $\mathcal{A}$ is extendable to a maximal k-filter;

c) The Boolean ultrafilters of $\mathcal{A}$ are not k-filters;

d) The maximal k-filters are not (Boolean) ultrafilters.

PROOF. *a*) Let F be a k-filter on $\mathcal{A}$ and let p be an element of F that is a fixed point for the polynomial $\neg k^n x$ for a suitable $n \neq 0$. Then $\neg k^n p = p$ so $\neg p$ belongs to F (because F is closed with regard to k), F is therefore the total filter.

b) it follows from Zorn's lemma.

c) Let p be a fixed point for $\neg kx$ and let U be a Boolean ultrafilter of $\mathcal{A}$. If p belonged to U, and if U were closed with regard to k, then $kp = \neg p$ would belong to U, and this is absurd. If p did not belong to U, then $\neg p$ would belong to U, so $k(\neg p) = p$ would belong to U, and this is also absurd.

d) it follows from *c*).

Referring to Example 2 of Paragraph 3 we can make the following remarks.

REMARK 1. The filter of the cofinites is a k-filter.

REMARK 2. Let's indicate with $F = \overline{\{X\}}^k$ the k-filter generated by $X \subseteq Z$. It can be seen that if X is finite then F is the total filter and if X is cofinite then F is a proper k-filter. In this case moreover F does not coincide with the Boolean filter generated by $X \cdot kX$ (differently from what happens for the τ-algebras, see [12]).

5. – The class of the k-algebras with properties of fixed point.

Let's indicate with $\mathcal{K}$ the subclass of V constituted by the k-algebras whose polynomials have a fixed point. The principal properties of $\mathcal{K}$ are summarised in the following:

THEOREM 5.1. *a*) *$\mathcal{K}$ is not empty.*

b) *Each $\mathcal{A} \in \mathcal{K}$ is infinite.*

c) *$\mathcal{K}$ is closed with regard to the operator P.*

d) *$\mathcal{K}$ is closed with regard to the operator H.*

e) *$\mathcal{K}$ is not closed with regard to the operator S.*

f) *$\mathcal{K}$ is not equationally definible.*

PROOF. *a*) follows from example 3.

b) follows from the Corollary 2.1.

c) Let $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ and let $\mathcal{A}_i$ be an algebra belonging to $\mathcal{K}$. Let $p(x)$ be a polynomial. Let a_i denote the fixed points of the realizations of $p(x)$ in $\mathcal{A}_i$, it is easy to see that the element a of $\mathcal{A}$ that has as i^{th}-component a_i is a fixed point of the realization of $p(x)$ in $\mathcal{A}$.

d) It can be proved as *c*).

e) It is sufficient to observe that each endomorphism k in a Boolean algebra is the identity in $\{0, 1\}$, therefore $\langle \{0, 1\}, +, \cdot, \neg, 0, 1, k \rangle$ is a subalgebra of each algebra of $\mathcal{K}$ and obviously does not belongs to $\mathcal{K}$.

f) It is obvious.

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