ARTICLE

New linking theorems

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New Linking Theorems.

MARTIN SCHECHTER (*)

SUMMARY - We prove new linking theorems related to those of Schechter-Tintarev which allow us to obtain linking for sets which did not link under the older theories. This allows us to prove new theorems for nonlinear problems.

1. – Introduction.

Let $G$ be a $C^1$ functional on a Banach space $E$, and assume that $E = M \oplus N$, where $M, N$ are closed subspaces, one of which is finite dimensional. Assume that for some $\delta > 0$, where

$$a_0 := \sup_{M \cap \partial B_\delta} G \leq b_0 := \inf_{N} G$$

for some $\delta > 0$, where

$$B_r = \{u \in E: \|u\| < r\}.$$

One of the results of the present paper is

**Theorem 1.1.** Under the above hypotheses, there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad b_0 \leq c < \infty, \quad (1 + \|u_k\|) G'(u_k) \to 0.$$

Interest in such a theorem stems from the fact that for many applica-

(*) Indirizzo dell' A.: University of California, Irvine, CA 92697-3875, USA.
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tions, (1.3) implies the existence of a solution of

\[ G(u) = c, \quad G'(u) = 0. \]

We shall present some of these applications here. When \( \dim M < \infty \), Theorem 1.1 is well known (cf. [Ra, Theorem 4.6]). However, the proof rests completely on the fact that \( M \) is finite dimensional. This is so much so, that no one seems to have suspected that the theorem is true even when \( \dim M = \infty \). We shall show that this indeed is the case. As a result we can solve problems which could not be considered before.

We apply Theorem 1.1 to semilinear boundary value problems. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and let \( A \) be a selfadjoint operator on \( L^2(\Omega) \) with compact resolvent and eigenvalues

\[ 0 < \lambda_0 < \lambda_1 < \ldots < \lambda_j < \ldots. \]

We assume that the eigenfunctions of \( A \) are bounded. Let \( f(x, t) \) be a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying

\[ |f(x, t)| \leq C|t| + V(x), \quad x \in \Omega, \quad t \in \mathbb{R} \]

and

\[ f(x, t)/t \to \alpha_{\pm}(x) \quad \text{a.e. as } t \to \pm \infty \]

where \( V(x) \in L^2(\Omega) \) and the only solution of

\[ Au = \alpha_+ u^+ - \alpha_- u^-, \quad u^\pm = \max\{\pm u, 0\} \]

is \( u \equiv 0 \). We let

\[ F(x, t) = \int_0^t f(x, s) \, ds. \]

We have

**Theorem 1.2.** Assume that for some \( l > 0 \) there are constants \( \nu > \lambda_{l-1} \) and \( \delta > 0 \) such that

\[ \nu t^2 \leq 2F(x, t), \quad x \in \Omega, \quad t \in \mathbb{R}, \]

\[ \lambda_l t^2 \leq 2F(x, t), \quad x \in \Omega, \quad |t| < \delta, \]

\[ \alpha_\pm(x) \leq \lambda_l, \quad x \in \Omega. \]
Then the equation
\begin{equation}
Au = f(x, u)
\end{equation}
has at least one nontrivial solution.

We also have

**Theorem 1.3.** Assume that for some \( l \geq 0 \) there are constants \( \nu < \lambda_{l+1} \) and \( \delta > 0 \) such that
\begin{align}
2F(x, t) &\leq \nu t^2, \quad x \in \Omega, \ t \in \mathbb{R}, \\
2F(x, t) &\leq \lambda_l t^2, \quad x \in \Omega, \ |t| < \delta, \\
\lambda_l &\leq \alpha_+(x), \quad x \in \Omega.
\end{align}

Then (1.11) has at least one nontrivial solution.

The equation (1.6) approximates (1.11) when \( |u(x)| \) is large. Theorem 1.2 cannot be proved by using previous linking theorems. On the other hand, Theorem 1.3 does follow [Si, Theorem 1.15]. It is included here because of its similarity to Theorem 1.2.

Theorem 1.1 is proved in Section 4 along with other theorems on linking stated in Section 2. Theorems 1.2 and 1.3 are proved in Section 3. They are based on a slight variation of Theorem 1.1. Other linking methods can be found in [MW, BN, Ra, Si].

2. – The method.

We present a refined version of the new linking concept introduced in [ST]. Let \( E \) be a Banach space and let \( \Phi \) be the set of all continuous maps \( \Gamma = \Gamma(t) \) from \( E \times [0, 1] \) to \( E \) such that

1) \( \Gamma(0) = I \), the identity map.

2) For each \( t \in [0, 1] \), \( \Gamma(t) \) is a homeomorphism of \( E \) onto \( E \) and \( \Gamma^{-1}(t) \in C(E \times [0, 1], E) \).

3) \( \Gamma(1)E \) is a single point in \( E \) and \( \Gamma(t)A \) converges uniformly to \( \Gamma(1)E \) as \( t \to 1 \) for each bounded set \( A \subset E \).
4) For each \( t_0 \in [0, 1) \) and each bounded set \( A \subset E \)

\[
\sup_{0 \leq t \leq t_0} \{ \| \Gamma(t) u \| + \| \Gamma^{-1}(t) u \| \} < \infty .
\]

**DEFINITION.** For \( A, B \subset E \) we say that \( A \) links \( B \) if

a) \( A \cap B = \emptyset \),

b) for each \( \Gamma \in \Phi \) there is a \( t \in (0, 1] \) such that

\[
\Gamma(t) A \cap B \neq \emptyset .
\]

We have

**PROPOSITION 2.1.** If \( A, B \subset E \) are closed and bounded, \( E \setminus A \) is path-wise connected and \( A \) links \( B \), then \( B \) links \( A \).

**PROPOSITION 2.2.** If \( F \in C(E, \mathbb{R}^n) \) and \( Q \subset E \) is such that \( F_0 = F \mid Q \) is a homeomorphism of \( Q \) onto the closure of a bounded open subset \( \Omega \) of \( \mathbb{R}^n \), then \( \partial Q \equiv F^{-1}_0(\partial \Omega) \) links \( F^{-1}(p) \) for each \( p \in \Omega \).

**THEOREM 2.3.** Let \( G \) be a \( C^1 \) functional on \( E \), and let \( A, B \) be subsets of \( E \) such that \( A \) is bounded and links \( B \). Assume

\[
a_0 := \sup_A G \leq b_0 := \inf_B G ,
\]

\[
a := \inf_{\Gamma \in \Phi} \sup_{0 \leq s \leq 1} G(\Gamma(s) u) < \infty .
\]

Let \( \psi(t) \) be a positive nonincreasing function on \( (0, \infty) \) such that

\[
\int_0^1 \psi(r) \, dr = \infty .
\]

Then there is a sequence \( \{ u_k \} \subset E \) such that

\[
G(u_k) \to a , \quad G'(u_k) = o(\psi(\|u_k\|)) .
\]

**COROLLARY 2.4.** Under the hypotheses of Theorem 2.3, there is a sequence \( \{ u_k \} \subset E \) such that

\[
G(u_k) \to a , \quad (1 + \|u_k\|) G'(u_k) \to 0 .
\]
PROPOSITION 2.5. Let $H$ be a homeomorphism of $E$ onto itself such that $H$ and $H^{-1}$ map bounded sets into bounded sets. If $A, B \subset E$ and $A$ links $B$, then $HA$ links $HB$.

PROPOSITION 2.6. Let $A, B_n, n = 1, 2, \ldots$, be subsets of $E$ such that $A$ is bounded and links $B_n$ for each $n$. Suppose

$B_n = B_n' \cup B_n''$

where

$d(B_n'', 0) \to \infty$ as $n \to \infty$

and there is a set $B \subset E$ such that

$A \cap B = \emptyset, \quad B_n' \subset B, \quad n = 1, 2, \ldots$.

Then $A$ links $B$.

COROLLARY 2.7. Let $M, N$ be closed subspaces of $E$, one of which is finite dimensional and such that

$E = M \oplus N$.

If

$B_R := \{ u \in E: \|u\| < R \}$

then $M \cap \partial B_R$ links $N$ for each $R > 0$.

COROLLARY 2.8. Let $M, N$ be closed subspaces of $E$ such that (2.11) holds with one of them being finite dimensional. Let $w_0$ be an element of $M \setminus \{0\}$, and let $0 < r < R$,

$A = \{ w \in M: \|w\| = R \},$

$B = \{ v \in N: \|v\| \geq r \} \cup \{ u = v + sw_0: v \in N, s \geq 0, \|u\| = r \}$.

Then $A$ links $B$.

3. – The application.

We now give the proof of Theorems 1.2, 1.3.

PROOF OF THEOREM 1.2. Let

$G(u) = \|u\|_B^p - 2 \int_\Omega F(x, u) \, dx, \quad u \in D$
where $D = D(A^{1/2})$ and

$$
(3.2) \quad \|u\|_D = \|A^{1/2} u\|.
$$

With this norm $D$ becomes a Hilbert space. Under hypothesis (1.4) it is easily shown that $G$ is a $C^1$ functional on $D$ and

$$
(3.3) \quad (G'(u), v)/2 = (u, v)_D - (f(u), v).
$$

From this it follows that $u$ is a solution of (1.10) if

$$
(3.4) \quad G'(u) = 0.
$$

Let $N$ be the subspace spanned by the eigenfunctions of $A$ corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{L}$, and let $E(\lambda_i)$ be the eigenspace of $\lambda_i$. Let $M = N^\perp \cap D$. By (1.8) we have

$$
(3.5) \quad G(v) \leq \|v\|^2_D - v\|v\|^2 \leq \left(1 - \frac{v}{\lambda_{l-1}}\right)\|v\|^2_D, \quad v \in N_{l-1}.
$$

Moreover, I claim that for each $\epsilon > 0$ sufficiently small either (a) there is a $y \in E(\lambda_i) \setminus \{0\}$ satisfying

$$
(3.6) \quad Ay = f(x, y) = \lambda_i y, \quad \|y\|_D = \epsilon
$$

or (b) there is an $\epsilon > 0$ such that

$$
(3.7) \quad G(v + y) \leq -\epsilon, \quad v \in N_{l-1}, \quad y \in E(\lambda_i), \quad \|v + y\|_D = \epsilon.
$$

Assume this for the moment. Since (3.6) exhibits a nontrivial solution of (1.11), we need only address option (b). Let $\epsilon, \epsilon$ be such that (3.7) holds. Let $y_0 \in E(\lambda_i) \setminus \{0\}$ and take

$$
(3.8) \quad B = \{v \in N_{l-1}: \|v\|_D \geq \epsilon\} \cup \{u = v + sy_0: v \in N_{l-1}, s \geq 0, \|u\|_D = \epsilon\}.
$$

Then (3.5) and (3.7) imply

$$
(3.9) \quad G(v) \leq -\epsilon_0, \quad v \in B
$$

for some $\epsilon_0 > 0$. On the other hand, (1.10) implies

$$
(3.10) \quad G(w) \to \infty \quad \text{as} \quad \|w\|_D \to \infty, \quad w \in M_{l-1}.
$$

To see this, let $\{w_k\}$ be any sequence in $M = M_{l-1}$ such that $q_k =
Let \( \tilde{w}_k = w_k / q_k \). Then

\begin{equation}
G(w_k) / q_k^2 = 1 - 2 \int_{\Omega} F(x, w_k) / q_k^2 \, dx .
\end{equation}

(3.11)

Now \( \| \tilde{w}_k \|_D = 1 \). Hence there is a renamed subsequence such that \( \tilde{w}_k \to \tilde{w} \) weakly in \( D \), strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \). Moreover, (1.4) implies

\begin{equation}
| F(x, t) | \leq Ct^2 + V(x) | t |
\end{equation}

(3.12) and (1.5) implies

\begin{equation}
2F(x, t) / t^2 \to \alpha_\pm(x) \text{ a.e. as } t \to \pm \infty .
\end{equation}

Thus

\begin{equation}
2 \int_{\Omega} F(x, w_k) \, dx / q_k^2 \to \int_{\Omega} \{ \alpha_+ (\tilde{w}^+)^2 + \alpha_- (\tilde{w}^-)^2 \} \, dx , \quad k \to \infty
\end{equation}

(3.15) and

\begin{equation}
G(w_k) / q_k^2 \to 1 - \int_{\Omega} \{ \alpha_+ (\tilde{w}^+)^2 + \alpha_- (\tilde{w}^-)^2 \} \, dx , \quad k \to \infty.
\end{equation}

(3.16)

Since \( \| \tilde{w} \|_D \leq 1 \), (1.10) implies that the right hand side of (3.16) is \( \geq 0 \). The only way it could vanish is if \( \tilde{w} \in E(\lambda_i) \) and

\begin{equation}
\int_{\Omega} \{ (\lambda_i - \alpha_+) (\tilde{w}^+)^2 + (\lambda_i - \alpha_-) (\tilde{w}^-)^2 \} \, dx = 0 .
\end{equation}

(3.17)

Since the integrand in (3.17) is nonnegative, we must have

\[ \alpha_+(x) \equiv \lambda_i \quad \text{when } \tilde{w}(x) > 0 , \]

\[ \alpha_-(x) \equiv \lambda_i \quad \text{when } \tilde{w}(x) < 0 . \]

From this it follows that \( \tilde{w} \) is a solution of (1.6). By hypothesis, this implies that \( \tilde{w} \equiv 0 \), showing that the right hand side of (3.16) does not vanish. Hence the left hand side of (3.16) converges to a positive limit for every such sequence, showing that (3.10) holds. Once we know this, we take \( R \) such that

\begin{equation}
G(w) \geq 0 , \quad w \in M_{i-1} \cap \partial B_R \equiv A .
\end{equation}

(3.18)
Let $G_1(u) = -G(u)$. Then

$$\sup_A G_1 \leq 0 < \epsilon_0 \leq \inf_B G_1.$$  

By Corollary 2.8, $A$ links $B$. Hence there is a sequence $\{u_k\} \subset D$ such that

$$G_1(u_k) \to c_1, \quad \epsilon_0 \leq c_1 < \infty, \quad G_1'(u_k) \to 0.$$  

Thus

$$\|u_k\|_D^2 - 2 \int_\Omega F(x, u_k) \, dx \to -c_1$$  

and

$$\langle u_k, v \rangle_D - \langle f(u_k), v \rangle \to 0, \quad v \in D$$  

where we write $f(u)$ in place of $f(x, u)$. If $Q_k = \|u_k\|_D \to \infty$, we let $\tilde{u}_k = u_k/Q_k$. Then $\|\tilde{u}_k\|_D = 1$ and there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ weakly in $D$, strongly in $L^2(\Omega)$ and a.e. in $\Omega$. We then obtain

$$\int_\Omega \{\alpha_+ (\tilde{u}^+)^2 + \alpha_- (\tilde{u}^-)^2\} \, dx = 1$$  

from (3.21) and

$$\langle \tilde{u}, v \rangle_D = \int_\Omega (\alpha_+ \tilde{u}^+ - \alpha_- \tilde{u}^-) \, v \, dx, \quad v \in D$$  

from (3.22). This implies that $\|\tilde{u}\|_D^2$ equals the left hand side of (3.23) and that $\tilde{u}$ is a solution of (1.6). Thus by hypothesis we must have $\tilde{u} \equiv 0$. But this contradicts (3.23). Hence the $Q_k$ are bounded. We can now apply well known techniques to show that there is a solution of (1.11) satisfying $G_1(u) = c_1 \geq \epsilon_0 > 0$. Since $G_1(0) = 0$, we see that $u$ is a nontrivial solution of (1.11).

It remains to prove (3.7). First we have

**Lemma 3.1.** If (1.9) holds, then for each $\varrho > 0$ sufficiently small there is a positive $\epsilon$ such that

$$G(v + y) \leq -\epsilon \|y\|_D^2, \quad v \in N_{l-1}, \quad y \in E(\lambda_i), \quad \|v + y\|_D \leq \varrho.$$  

**Proof.** Since $E(\lambda_i) \subset L^\infty(\Omega)$, there is a $\varrho > 0$ such that

$$\|y\|_D \leq \varrho \text{ implies } \|y\|_\infty \leq \delta/2, \quad y \in E(\lambda_i).$$
where \( \delta \) is the constant in (1.9). Let \( w = v + y \), where \( v \in N_{i-1}, y \in E(\lambda_i) \).

If

\[
(3.27) \quad \|w\|_D \leq \varrho \quad \text{and} \quad |w(x)| \geq \delta
\]

then

\[
(3.28) \quad \delta \leq |w(x)| \leq |v(x)| + |y(x)| \leq |v(x)| + \delta/2 .
\]

Consequently, (3.27) implies

\[
(3.30) \quad |w(x)| \leq 2|v(x)| .
\]

By (3.12)

\[
G(w) \leq \|w\|_D^2 - \lambda_i \int_{|w| < \delta} w^2 \, dx + C \int_{|w| > \delta} (w^2 + |w|) \, dx \leq \\
\leq \|w\|_D^2 - \lambda_i \|w\|^2 + C' \int_{|w| > \delta} (w^2 + |w|) \, dx \leq \\
\leq \|v\|_D^2 - \lambda_i \|v\|^2 + C'' \int_{2|v| > \delta} |v|^\sigma \, dx \leq \left(1 - \frac{\lambda_i}{\lambda_{i-1}} - C''\|\varrho\|_D^{-2}\right)\|\varrho\|_D^2
\]

where \( \sigma > 2 \). If we take \( \varrho \) sufficiently small, this implies (3.25). \( \blacksquare \)

Once we have inequality (3.25), we prove (3.7) by assuming, on the contrary, that there is a sequence \( \omega_k = v_k + y_k, v_k \in N_{i-1}, y_k \in E(\lambda_i) \) such that \( \|w_k\|_D = \varrho \) and

\[
(3.32) \quad G(w_k) \to 0 .
\]

By (3.25) we see that \( v_k \to 0 \). Thus \( \|y_k\|_D \to \varrho \). Since \( E(\lambda_i) \) is finite dimensional, there is a renamed subsequence such that \( y_k \to y \) in \( E(\lambda_i) \) and \( \|y\|_D = \varrho \). By (3.32)

\[
(3.33) \quad G(y) = 0 .
\]

If \( \varrho \) is such that (3.27) holds, then (1.9) implies

\[
(3.34) \quad \lambda_i y(x)^2 \leq 2F(x, y(x)), \quad x \in \Omega .
\]

But (3.33) says

\[
(3.35) \quad \int_{\Omega} \{\lambda_i y(x)^2 - 2F(x, y(x))\} \, dx = 0 .
\]
From (3.34) and (3.35) we see that
\[ \lambda_i y(x)^2 \equiv F(x, y(x)), \quad x \in \Omega. \]
Let \( \zeta(x) \) be any function in \( C_0^\infty(\Omega) \). Then for \( t > 0 \) sufficiently small
\[ \lambda_i \left[ (y + t\zeta)^2 - y^2 \right]/t \leq 2[F(x, y + t\zeta) - F(x, y)]/t. \]
If we take the limit as \( t \to 0 \), we have
\[ \lambda_i y(x)^2 \zeta(x) \leq f(x, y(x))\zeta(x), \quad x \in \Omega. \]
From this we conclude that
\[ \lambda_i y(x)^2 \equiv f(x, y(x)), \quad x \in \Omega. \]
Since \( y \in E(\lambda_i) \), this implies that \( y \) is a solution of (3.6), the option which we discarded. This completes the proof of the theorem. 

**Proof of Theorem 1.3.** We only sketch the proof because of its similarity to that of Theorem 1.2. We use the notation of the proof of Theorem 1.2. By (1.12)

\[
(3.36) \quad G(w) \geq ||w||_D - v||w||^2 \geq \left(1 - \frac{v}{\lambda_{i+1}}\right)||w||^2 D, \quad w \in M.
\]

Moreover, (1.13) implies that for each \( Q > 0 \) sufficiently small, either (a) there is a solution \( y \in E(\lambda_i) \setminus \{0\} \) of (3.6) or (b) there is such that

\[
(3.37) \quad G(w + y) \geq \varepsilon, \quad w \in M, y \in E(\lambda_i), \quad ||w + y|| = Q.
\]

This is proved by the same method used in the proof of (3.7). Since the existence of a solution of (3.6) implies the conclusion of the theorem, we may assume that (3.37) holds. Let \( Q, \varepsilon \) be such that (3.37) holds and let \( y_0 \in E(\lambda_i) \setminus \{0\} \) be fixed. Take

\[
(3.38) \quad B = \{w \in M: ||w||_D \geq \varepsilon\} \cup \{u = w + sy_0: w \in M, \ s \geq 0, \ ||u||_D = Q\}.
\]

Then (3.36) and (3.37) imply that

\[
(3.39) \quad G(w) \geq \varepsilon_0, \quad w \in B
\]

for some \( \varepsilon_0 > 0 \). Moreover, (1.14) implies

\[
(3.40) \quad G(v) \to -\infty \quad \text{as} \quad ||v||_D \to \infty, \ v \in N
\]
Again, this is proved in the same way that we proved (3.10). Next we take $R$ so large that

\begin{equation}
G(v) \leq 0, \quad v \in N \cap \partial B_R \equiv A.
\end{equation}

Then

\begin{equation}
\sup_A G \leq 0 < \epsilon_0 \leq \inf_B G.
\end{equation}

We know that $A$ links $B$ (this follows from Corollary 2.8, but it was known previously [Si, Lemma 1.14]). Thus by Theorem 2.3 there is a sequence $\{u_k\} \subset E$ such that (3.20) holds with $G_1$ replaced by $G$. The rest of the proof proceeds as before.

\section*{4. -- The linking theorems.}

In this section we give the proof of the theorems of Section 2. Proofs of Proposition 2.1 and 2.2 were given in [ST] (the definition of the set $\Phi$ given there was slightly different from that given in Section 2, but the proofs are not affected.)

\textbf{PROOF OF THEOREM 2.3.} If the theorem were false, there would be a $\delta > 0$ and a $\psi$ satisfying (2.5) such that

\begin{equation}
\psi(\|u\|) \leq \|G'(u)\|
\end{equation}

when

\begin{equation}
 u \in Q := \{u \in E: |G(u) - a| \leq 3\delta\}.
\end{equation}

Assume first that $b_0 < a$, and reduce $\delta$ so that $3\delta < a - b_0$. Since $G \in C^1(E, R)$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\overline{E} = \{u \in E: G'(u) = 0\}$ into $E$ such that

\begin{equation}
\|Y(u)\| \leq 1, \quad \theta \|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \overline{E}
\end{equation}

holds for some $\theta > 0$. Let

\begin{align*}
Q_0 &= \{u \in E: |G(u) - a| < 2\delta\}, \\
Q_1 &= \{u \in E: |G(u) - a| < \delta\}, \\
Q_2 &= E \setminus Q_0, \quad \eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)],
\end{align*}

and let $\sigma(t)$ be the flow generated by

\begin{equation}
W(u) = -\eta(u) Y(u).
\end{equation}
The mapping $W(u)$ is locally Lipschitz continuous on the whole of $E$ and is bounded in norm by 1. We have

$$dG(\sigma(t))/dt = \eta(\sigma(t) u)(G'(\sigma(t) u), Y(\sigma(t) u)) \leq$$

$$\leq -\theta\eta(\sigma)\|G'(\sigma)\| \leq -\theta\eta(\sigma)\psi(\|\sigma\|) \leq -\theta\eta(\sigma)\psi(\|u\| + t)$$

since

$$\|\sigma(t) u - u\| \leq t, \quad t > 0$$

and $\psi$ is nonincreasing. By the definition (2.4) of $a$, there a $T \in \Phi$ such that

$$G(\Gamma(s)u) < a + \delta, \quad s \in [0, 1], \quad u \in A.$$ 

Let

$$M = \sup \{\|\Gamma(s)u\| : s \in [0, 1], u \in A\}.$$ 

Since $A$ is bounded, $M$ is finite by the definition of $\Phi$. Let $T$ be such that

$$2\delta < \theta \int_0^T \psi(t) dt.$$ 

This can be accomplished because $\psi$ satisfies (2.5). Let $v = \Gamma(s)u$, where $s \in [0, 1]$ and $u \in A$. If there is a $t_1 < T$ such that $\sigma(t_1)v \not\in Q_1$, then

$$G(\sigma(T)v) < a - \delta$$

by (4.5) and (4.7). Otherwise, $\sigma(t)v \in Q_1$ for all $t \in [0, T]$ and

$$G(\sigma(T)v) \leq a + \delta - \theta \int_0^T \psi(M + t) dt < a - \delta.$$ 

Hence

$$G(\sigma(T)\Gamma(s)u) < a - \delta, \quad s \in [0, 1], \quad u \in A.$$ 

Let

$$\Gamma_1(s) = \begin{cases} 
\sigma(2sT), & 0 \leq s \leq \frac{1}{2}, \\
\sigma(T)\Gamma(2s - 1), & \frac{1}{2} < s \leq 1.
\end{cases}$$

(4.12)
Then $\Gamma_1 \in \Phi$. Since
\begin{equation}
G(\sigma(t) u) \leq a_0, \quad t \geq 0, \quad u \in A
\end{equation}
we see by (4.11) that
\begin{equation}
G(\Gamma_1(s) u) < a - \delta, \quad s \in [0, 1], \quad u \in A.
\end{equation}
But this contradicts the definition (2.4) of $a$. Hence (4.1) cannot hold for $u$ satisfying (4.2). If $b_0 = a$, we proceed as before, but we cannot use (4.13) to imply (4.14). However, we note that (4.5) implies
\begin{equation}
G(\sigma(t) u) \leq b_0 - \theta \int_0^t \eta(\sigma(\tau) u) \psi(\|\sigma(\tau) u\|) d\tau
\end{equation}
for $u \in A$. This shows that
\begin{equation}
\sigma(t) A \cap B = \emptyset, \quad t \geq 0.
\end{equation}
For the only way we can have $\sigma(t) u \in B$ is if
\begin{equation}
\eta(\sigma(\tau) u) \equiv 0, \quad 0 \leq \tau \leq t.
\end{equation}
But this implies $\sigma(\tau) u \in \overline{Q}_2$. Consequently,
\begin{equation}
G(\sigma(\tau) u) < a - \delta, \quad 0 \leq \tau \leq t
\end{equation}
which cannot happen if $\sigma(t) u \in B$. Thus (4.16) holds. Similarly, (4.11) shows that
\begin{equation}
\sigma(T) \Gamma(t) A \cap B = \emptyset, \quad t \in [0, 1].
\end{equation}
Combing (4.16) and (4.17), we see that
\begin{equation}
\Gamma_1(s) A \cap B = \emptyset, \quad 0 \leq s \leq 1
\end{equation}
contradicting the fact that $A$ links $B$. This completes the proof of the theorem.

We prove Corollary 2.4 be taking $\psi(r) = 1/(1 + r)$.

**Proof of Proposition 2.5.** Let $\Gamma$ be an arbitrary map in $\Phi$. Then $H^{-1} \Gamma(s) H$ is in $\Phi$. If $A$ links $B$, then there is an $s_1 \in [0, 1]$ such that
\begin{equation}
H^{-1} \Gamma(s_1) HA \cap B \neq \emptyset.
\end{equation}
Thus

$$\Gamma(s_1) \cap HB \neq \emptyset.$$ 

Since $\Gamma$ was arbitrary, $HA$ links $HB$.  ■

**PROOF OF PROPOSITION 2.6.** Let $\Gamma$ be any map in $\Phi$. Then

\begin{equation}
K = \sup \{ \|\Gamma(t) u\| : t \in [0, 1], u \in A \} < \infty.
\end{equation}

For $n$ sufficiently large

\begin{equation}
d(B_n^{-}, 0) > K.
\end{equation}

Now

\begin{equation}
\Gamma(t_1) \cap B_n \neq \emptyset
\end{equation}

for some $t_1 \in [0, 1]$. But

$$\Gamma(t_1) \cap B_n^{-} = \emptyset$$

by (4.18) and (4.19). Hence

$$\Gamma(t_1) \cap B_n^{-} \neq \emptyset.$$ 

Consequently

$$\Gamma(t_1) \cap B \neq \emptyset.$$ 

Thus $A$ links $B$.  ■

**PROOF OF COROLLARY 2.7.** If $\dim M < \infty$, the result follows from Proposition 2.2 if we take $Q = M \cap B_R$ and let $F$ be the projection of $E$ onto $Q$. If $\dim N < \infty$, let $A = M \cap \partial B_R$ and $B_n = \{v \in N : \|v\| \leq n\} \cup \{v + sw_0 : v \in N, s \geq 0, \|v + sw_0\| = n\}, Q = \{v + sw_0 : v \in N, s \geq 0, \|v + sw_0\| \leq n\}$, and

\begin{equation}
F(v + w) = v + \|w\|w_0, \quad v \in N, \quad w \in M
\end{equation}

where $w_0 \in M$ and $\|w_0\| = 1$. It follows from Proposition 2.2 that $B_n$ links $A$ when $R < n$. By Proposition 2.1 we also have that $A$ links $B_n$. If we now take $B = N$, we can apply Propositions 2.6 to conclude that $A$ links $B$.  ■
PROOF OF COROLLARY 2.8. Let

\[ B_n = \{ v \in N : r \leq \|v\| \leq n \} \cup \{ u = v + sw_0 : v \in N, s \geq 0, \|u\| = r \} \cup \]

\[ \cup \{ u = v + sw_0 : v \in N, s \geq 0, \|u\| = n \} . \]

If \( r < R < n \), it follows from Propositions 2.1 and 2.2 that \( A \) and \( B_n \) link each other (no matter which subspace is finite dimensional). We now apply Proposition 2.6 to obtain the desired conclusion. ■

REFERENCES


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