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Extensions of Unbounded Topological Spaces.

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Introduction.

A method of compactification of locally compact spaces has been proposed in [1]. This method is based on the concept of essential semilattice homomorphism (ESH for short). More precisely, let $X$ be a locally compact (non-compact) Hausdorff space and $K$ a compact Hausdorff space. Let $\mathcal{B}$ be an (open) basis of $K$ closed with respect to finite unions, and let $\mathcal{N}_X$ be the family consisting of the empty set and the open subsets of $X$ which are not relatively compact. A map $\pi: \mathcal{B} \to \mathcal{N}_X$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is an ESH if the following conditions hold:

ESH1) $X - \pi(K) \notin \mathcal{N}_X - \{\emptyset\}$;

ESH2) if $U, V \in \mathcal{B}$ then the symmetric difference

$$\pi(U \cup V) \Delta (\pi(U) \cup \pi(V)) \notin \mathcal{N}_X - \{\emptyset\};$$

ESH3) if $U, V \in \mathcal{B}$ and $\bar{U} \cap \bar{V} = \emptyset$ then $\pi(U) \cap \pi(V) \notin \mathcal{N}_X - \{\emptyset\}$.

If $T_X$ is the topology of $X$ and $S = \{U \cup (\pi(U) \setminus F): U \in \mathcal{B}, F \subseteq X, F \text{ compact}\}$, then $T_X \cup S$ is a basis for a topology on the disjoint union $X \sqcup K$. This new space is a Hausdorff compactification of $X$ with remainder $K$. It is denoted by $X_{\pi K}$ and is called an ESH-compactification of $X$.

In this paper we present a natural generalization of the construction above. We say that a topological space $X$ is locally bounded with respect to a family (of «bounded» sets) $\mathcal{F}_X \subseteq 2^X$ (which is closed under finite

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unions and subsets) if every point of \( X \) has a bounded neighborhood. We note that, if \( \mathcal{F}_X \) is the family of the relatively compact subsets of \( X \) (resp. the relatively Lindelöf subsets of a \( T_3 \)-space \( X \)), we have that local boundedness with respect to \( \mathcal{F}_X \) is equivalent to local compactness (resp. local Lindelöfness) of \( X \).

We construct dense extensions of unbounded spaces, which we call \( B \)-extensions. By adding some requirements, mainly local boundedness of the space, we obtain Hausdorff \( B \)-extensions. This construction is obtained with a method similar to the one used to obtain \( ESH \)-compactifications. This method can be applied, for instance, to construct Lindelöf extensions of non-Lindelöf locally Lindelöf spaces. As a final remark, we mention that Theorem 2.3 of this paper appears to be a generalization of a Tkachuk’s result (see [8], Proposition 1).

1. – \( B \)-extensions with respect to a boundedness.

An extension of a topological space will mean a dense extension.

We recall that a non-empty family \( \mathcal{F}_X \) of subsets of a space \( X \) is said to be a boundedness in \( X \) if \( \mathcal{F}_X \) is closed with respect to finite unions and subsets (see [5]). Elements of \( \mathcal{F}_X \) are called bounded sets of \( X \). Every subset of \( X \) not in \( \mathcal{F}_X \) is called unbounded.

A space \( X \) with boundedness \( \mathcal{F}_X \) is said to be locally bounded if every point of \( X \) has a bounded neighborhood. If \( X \) is \( T_3 \), this is equivalent to say that the family of the closed bounded neighborhoods of each point of \( X \) is a neighborhood base.

We remark that, for a given space \( X \), the family \( \mathcal{C}_X = \{ A \subset X : \overline{A} \text{ is compact} \} \) is a boundedness in \( X \), as well as \( \mathcal{L}_X = \{ A \subset X : \overline{A} \text{ is Lindelöf} \} \). Clearly, a space \( X \) is locally compact iff \( X \) is locally bounded with respect to \( \mathcal{C}_X \).

A space \( X \) is said to be locally Lindelöf if every point of \( X \) has a Lindelöf neighborhood. If \( X \) is \( T_3 \), it is equivalent to say that every point of \( X \) has a Lindelöf closed neighborhood (or to say that the family of the closed Lindelöf neighborhoods of every point of \( X \) is a neighborhood base). Hence, a \( T_3 \)-space is locally Lindelöf iff \( X \) is locally bounded with respect to \( \mathcal{L}_X \).

In [8] Tkachuk defines a space \( X \) to be locally Lindelöf if every point of \( X \) has an open Lindelöf neighborhood. If \( X \) is \( T_{\beta_{\omega_1}} \), the two definitions are equivalent. In fact, if \( x \in X \) and \( U \) is a Lindelöf neighborhood of \( x \),
then there exists \( f \in C(X, [0, 1]) \) such that \( f(x) = 0 \) and \( f(X \setminus U) = 1 \). Hence

\[
x \in Z = f^{-1}(0) \subset W = f^{-1}([0, 1)) \subset U
\]

and \( f^{-1}([0, 1)) \) is Lindelöf since it is an \( F_\sigma \) contained in a Lindelöf subspace.

We remark that \( Z \) can be chosen to be a zero-set neighborhood of the point \( x \). In fact, it is sufficient to consider the map \( g = (2f - 1) \vee 0 \).

Therefore, a locally Lindelöf \( T_{3\frac{1}{2}} \)-space \( X \) is locally bounded with respect to the boundedness

\[
\mathcal{L}_X = \{ A \subset X : A \subset f^{-1}(0), \ f^{-1}([0, 1)) \text{ is Lindelöf} \ & f \in C(X, [0, 1]) \}.
\]

We note that, if \( X \) is locally bounded with respect to a boundedness \( \mathcal{F}_X \), then \( C_X \subset \mathcal{F}_X \). If \( \mathcal{F}_X \) is also closed with respect to countable unions, then \( \mathcal{L}_X \subset \mathcal{F}_X \) too.

If \( aX \) is an extension of \( X \), then there is a natural boundedness in \( X \) associated to \( aX \). In fact, if we define

\[
\mathcal{K}_X(aX) = \{ A \subset X : Cl_X A = Cl_{aX} A \},
\]

then \( \mathcal{K}_X(aX) \) is a boundedness in \( X \). We remark that if \( aX \) is \( T_3 \) and \( aX \setminus X \) is closed, or \( aX \) is \( T_2 \) and \( aX \setminus X \) is compact, then \( X \) is also locally bounded with respect to \( \mathcal{K}_X(aX) \).

Now, let \( X \) be an unbounded space with respect to \( \mathcal{F}_X \) and let \( \mathcal{N}_X \) be the collection consisting of the empty set and the unbounded open subsets of \( X \). Let \( \mathcal{B} \) be a basis for the open subsets of a topological space \( Y \), and assume that \( Y \in \mathcal{B} \) and \( \mathcal{B} \) is closed with respect to finite unions.

We say that \( \pi = \pi_{\mathcal{B}, \mathcal{F}_X} : \mathcal{B} \to \mathcal{N}_X \), with \( \pi(U) \neq \emptyset \) for every \( U \neq \emptyset \), is a \( B \)-map, if it satisfies the following conditions:

B1) if \( \{ U_i \}_{i \in A} \subset \mathcal{B} \) is a cover of \( Y \), then \( X \setminus \bigcup_{i \in A} \pi(U_i) \in \mathcal{F}_X \);

B2) if \( U, V \in \mathcal{B} \) then

\[
\overline{\pi(U \cup V) \setminus \Delta(\pi(U) \cup \pi(V))} \in \mathcal{F}_X;
\]

B3) if \( U, V \in \mathcal{B} \) and \( \overline{U} \cap \overline{V} = \emptyset \) then \( \overline{\pi(U) \cap \pi(V)} \in \mathcal{F}_X \).

In the following, a \( B \)-map \( \pi : \mathcal{B} \to \mathcal{N}_X \), with \( \mathcal{B} \) closed with respect to unions of cardinality \( < \alpha \), will be also called an \( \alpha \)-\( B \) map.
Now, a topological extension of $X$ can be constructed by means of a $B$-map. If $T_X$ is the topology of $X$ and $s = \{U \cup (\pi(U)\setminus F) : U \in \mathcal{B}, F = f^{-1}(U) \in \mathcal{F}_X\}$, then $T_X \cup s$ is a basis for a topology on the disjoint union $X \cup Y$. To prove this, it is sufficient to imitate the proof given in [1] (see p. 852).

The set $X \cup Y$, endowed with the topology generated by $T_X \cup s$, will be denoted by $X \cup Y$ and will be called a $B$-extension of $X$.

We observe that $X$ is open in $X \cup Y$, and the topologies of the subspaces $X, Y$ coincide with the original topologies. If $\emptyset \neq U \in \mathcal{B}$, then $\pi(U) \notin \mathcal{F}_X$. Hence $\pi(U) \setminus F \neq \emptyset$ for every $F \in \mathcal{F}_X$. It follows that $X$ is dense in $X \cup Y$.

If $X$ is a space with boundedness $\mathcal{F}_X$, we say that a continuous map $f : X \to Y$ is $B$-singular (with respect to $\mathcal{F}_X$) if, for every non-empty $U \in T_Y$, $f^{-1}(U)$ is unbounded in $X$. We note that, if $f : X \to Y$ is $B$-singular, then $\pi = f^{-1} : T_Y \to \mathcal{N}_X$ is a $B$-map.

If $\mathcal{F}_X$ is a boundedness in $X$, then $\tilde{\mathcal{F}}_X = \{F \in \tilde{\mathcal{F}}_X : F \in \mathcal{F}_X\}$ is also a boundedness and one has that $F \in \tilde{\mathcal{F}}_X$ iff $\tilde{\pi}(F) \in \mathcal{F}_X$.

A boundedness $\mathcal{G}_X$ with the property that $F \in \mathcal{G}_X$ iff $\tilde{F} \in \mathcal{G}_X$ (that is $\mathcal{G}_X = \tilde{\mathcal{G}}_X$) will be called a closed boundedness. Clearly, $\mathcal{C}_X, \mathcal{L}_X, \mathcal{Z}_X$ and $\mathcal{K}_X(aX)$ are closed boundednesses.

Now, if $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a $B$-map, then $\tilde{\pi} = \pi_{\mathcal{F}_X, \tilde{\mathcal{F}}_X}$, defined by $\tilde{\pi}(U) = \pi(U)$ for every $U \in \mathcal{B}$, is also a $B$-map and we have that $X \cup Y = X \cup Y$. In fact, $\pi(U) \notin \mathcal{F}_X$ implies $\tilde{\pi}(U) \notin \tilde{\mathcal{F}}_X$ and the closed unbounded subsets with respect to $\tilde{\mathcal{F}}_X$ are the closed unbounded subsets with respect to $\mathcal{F}_X$. Hence, every $B$-extension of $X$ can be considered as a $B$-extension with respect to a closed boundedness. Therefore, we can assume, without restriction from the standpoint of $B$-extension, that every boundedness we consider is a closed boundedness.

Now, let $aX$ be a $B$-extension of $X$. We show there is a maximal boundedness $\mathcal{M}_X$ and a $B$-map $\pi' = \pi_{\mathcal{B}, \mathcal{M}_X}$ such that $aX = X \cup Y$.

**Proposition 1.1.** Let $aX = X \cup Y$ be a $B$-extension of $X$, with $B$-map $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$. Then $\mathcal{F}_X \subset \mathcal{K}_X(aX)$ and there is a $B$-map $\pi' = \pi_{\mathcal{B}, \mathcal{M}(aX)}$ such that $aX = X \cup Y$. If $Y$ is compact, then $\mathcal{F}_X = \mathcal{K}_X(aX)$.

**Proof.** If $A \in \mathcal{F}_X$, then $\pi(Y) \setminus \overline{A}$ is an open neighborhood of $Y$ that does not meet $A$. Hence $Cl_XA = Cl_{aX}A$ and so $A \in \mathcal{K}_X(aX)$. 

Now, we observe that if $\emptyset \neq U \in \mathcal{B}$ then $\pi(U) \not\in \mathcal{C}_X(aX)$. Otherwise $U \cup \pi(U) \setminus \text{Cl}_X \pi(U) = U \cup \pi(U) \setminus \text{Cl}_X \pi(U) = U$ would be a non-empty open subset of $aX$ contained in $aX \setminus X$. Therefore, we can consider the $B$-map $\pi' = \pi \in \mathcal{C}_X(aX)$ defined by $\pi'(U) = \pi(U)$ for every $U \in \mathcal{B}$. Since $\mathcal{F}_X \subseteq \mathcal{C}_X(aX)$, then we have $T_{X \cup Y} \subseteq T_{X \cup Y}$. On the other hand, every basic open of $X \cup Y$ of the form $U \cup \pi'(U) \setminus F = U \cup \pi(U) \setminus F$ is also open in $aX = X \cup Y$, because $F = \text{Cl}_X F = \text{Cl}_{aX} F$. Hence $T_{X \cup Y} = T_{X \cup Y}$.

Now, suppose that $Y$ is compact and let $A \in \mathcal{C}_X(aX)$. For every $y \in Y$, there is a basic open $U_y \cup \pi(U_y) \setminus F_y$ containing $y$ such that $(U_y \cup \pi(U_y) \setminus F_y) \cap A = \emptyset$. If $\{U_{y_1}, \ldots, U_{y_n}\}$ is a finite subfamily of $\{U_y\}_{y \in Y}$ that covers $Y$, then

$$A \subseteq \left( \bigcup_{i=1}^{n} \pi(U_{y_i}) \right) \cup F_{y_1} \cup \ldots \cup F_{y_n} \in \mathcal{F}_X. \quad \blacksquare$$

We note that, if $\pi = \pi \in \mathcal{F}_X$ is a $B$-map and $X \cup Y$ is $T_2$-compact, then $\mathcal{F}_X = \mathcal{C}_X(aX) = \mathcal{C}_X$. Moreover, if $X$ is $T_2$-locally compact and $Y$ is $T_2$-compact, then the $B$-maps (with respect to $\mathcal{C}_X$) are exactly the ESH’s as defined in [1]. In fact, let $\pi: \mathcal{B} \rightarrow \mathcal{N}_X$ be such that $X \setminus \pi(Y) \in \mathcal{C}_X$ and $\pi(U \cup V) \Delta(\pi(U) \cup \pi(V)) \in \mathcal{C}_X$ for every $U, V \in \mathcal{B}$. If $U = \{U_i\}_{i \in A} \subseteq \mathcal{B}$ is a cover of $Y$ and $\{U_{i_1}, \ldots, U_{i_r}\}$ is a finite subcover of $Y$, then $X \setminus \pi(Y) = X \setminus (\bigcup_{k=1}^{r} \pi(U_{i_k})) \subseteq \mathcal{C}_X$ and $\pi(Y) \setminus (\bigcup_{k=1}^{r} \pi(U_{i_k})) \subseteq \mathcal{C}_X$ imply that $X \setminus (\bigcup_{k=1}^{r} \pi(U_{i_k})) = X \setminus \left( \bigcup_{k=1}^{r} \pi(U_{i_k}) \right) \subseteq \mathcal{C}_X$. Hence $X \setminus \bigcup_{i \in A} \pi(U_i) \subseteq \mathcal{C}_X$ too.

If $X$ is a space with boundedness $\mathcal{F}_X$, then the relation defined by $A \sim B$ iff $A \Delta B \subseteq \mathcal{F}_X$ is an equivalence relation in $2^X$. Finite unions and intersections are compatible with it. Moreover, if $\mathcal{F}_X$ is closed under unions of cardinality $\gamma$, then one has that also unions of cardinality $\gamma$ are compatible with $\sim$.

**Proposition 1.2** (see Prop. 1.1 in [2]). Let $X$ be a locally bounded space with respect to $\mathcal{F}_X$. If $\pi = \pi \in \mathcal{F}_X$ is a $B$-map, then every map $\pi': \mathcal{B} \rightarrow \mathcal{N}_X$ such that $\pi(U) \Delta \pi'(U) \subseteq \mathcal{F}_X$, for every $U \in \mathcal{B}$, is also a $B$-map such that $X \cup Y \equiv X \cup Y$.

**Proof.** It is easily seen that $\pi'$ is a $B$-map such that $T_{X \cup Y} = T_{X \cup Y}$. \quad \blacksquare
Now, we will see that the Hausdorff property of $X \cup \pi Y$ is ensured under the assumption that $X$ is locally bounded, namely that $X$ is locally bounded with respect to a closed boundedness.

The proof of the following statement is straightforward (see the proof of Prop. 1 in [1]).

**Proposition 1.3.** Let $X$ and $Y$ be Hausdorff spaces. If $X$ is locally bounded with respect to a (closed) boundedness $\mathcal{F}_X$ and $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a $B$-map, then $X \cup \pi Y$ is a Hausdorff space containing $X$ as a dense subspace.

With an argument suggested by the last part of the proof of Prop. 1.1, one can also prove the following proposition.

**Proposition 1.4.** If $aX = X \cup \pi Y$ is Hausdorff, with $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$, and $Y$ is compact, then $X$ is locally bounded with respect to $\mathcal{F}_X$.

**Theorem 1.5.** Let $aX$ be a $T_4$-extension of $X$ such that $aX \setminus X$ is closed and 0-dimensional. Then $aX$ is a $B$-extension of $X$.

**Proof.** We denote by $\mathcal{B}$ the basis of $Y = aX \setminus X$ consisting of the clopen subsets of $Y$. If $U \in \mathcal{B}$, by the normality of $aX$, we have that there is an open subset $A$ of $aX$ such that $A \cap (aX \setminus X) = U$ and $aX \setminus A$ is a neighborhood of $Y \setminus A$. In fact, $U$ and $Y \setminus U$ are closed (and disjoint) in $aX$. For every $U \in \mathcal{B}$, choose such a set $A$. Observe that, if $B$ is another subset of $aX$ that satisfies the same conditions, then $(A \cap X) \Delta (B \cap X) \in \mathcal{X}(aX)$. Moreover, $(\text{Cl}_{aX}(A \cap X)) \setminus X = U$ implies that if $U \neq \emptyset$ then $A \cap X \notin \mathcal{X}(aX)$. Hence, we can define $\pi = \pi_{\mathcal{B}, \mathcal{X}(aX)}$ by setting $\pi(U) = U \cap A \cap X$. We note that $X$ is locally bounded with respect to $\mathcal{X}(aX)$.

Now, we check the $B$-properties of $\pi$. First, suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of $Y$ of members of $\mathcal{B}$. Since every $U_i \cup \pi(U_i)$ is open in $aX$, then $X \setminus \bigcup_{i \in I} \pi(U_i) = aX \setminus \left( \bigcup_{i \in I} U_i \cup \pi(U_i) \right)$ is closed in $aX$ and so it belongs to $\mathcal{X}(aX)$.

Let $U, V \in \mathcal{B}$. If $O = (U \cup \pi(U)) \cup (V \cup \pi(V))$, then $O \cap (aX \setminus X) = U \cup V$ and $aX \setminus O$ is a neighborhood of $Y \setminus (U \cup V)$. Therefore,

$$\left( O \cap X \right) \Delta \pi(U \cup V) = (\pi(U) \cup \pi(V)) \Delta \pi(U \cup V) \in \mathcal{X}(aX) \cdot$$
Now, suppose $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$, that is $U \cap V = \emptyset$. Then
\[
Cl_{aX}(\pi(U) \cap \pi(V)) \times \mathcal{X} \supset (Cl_{aX} \pi(U)) \cap (Cl_{aX} \pi(V)) \times \mathcal{X} = \\
= (Cl_{aX} \pi(U)) \times \mathcal{X} \cap (Cl_{aX} \pi(V)) \times \mathcal{X} = U \cap V = \emptyset
\]
implies $Cl_{aX}(\pi(U) \cap \pi(V)) \subset X$ and so $\pi(U) \cap \pi(V) \in \mathcal{C}_{X}(aX)$.

Finally, we show that $aX = X \cup Y$. Obviously, $T_{X\cup Y} \leq T_{aX}$. Now, let $T$ be an open subset of $aX$ and suppose $y \in T \cap Y$. Choose $U \in \mathcal{B}$ such that $y \in U$ and $U \subset T$. Clearly, $\pi(U) \setminus T$ has no adherence points in $T$ (that is open in $aX$). Also, $\pi(U) \setminus T$ has no adherence points in $Y \setminus T$. In fact,
\[
(Cl_{aX}(\pi(U)) \setminus T)) \times \mathcal{X} \subset (Cl_{aX} \pi(U)) \times \mathcal{X} = U \subset T.
\]
Therefore, $F = \pi(U) \setminus T \in \mathcal{K}_{X}(aX)$ and we have $y \in U \cup \pi(U) \setminus F \subset T$. Hence,
\[
T = \bigcup_{y \in T \cap Y} (U \cup \pi(U) \setminus F) \cup (T \cap X),
\]
and so $T \in T_{X\cup Y}$.

2. – Lindelöf and other special extensions.

The proof of the next theorem is routine (see Proposition 1 in [1]).

**Theorem 2.1.** Let $X$ be a locally bounded $T_{2}$-space with respect to a (closed) boundedness $\mathcal{G}_{X} \subset \mathcal{L}_{X}$ (hence $X$ is locally Lindelöf) and let $Y$ be a Lindelöf $T_{2}$-space. If $\pi = \pi_{\mathcal{B}, \mathcal{G}_{X}}$ is a $B$-map, then $X \cup Y$ is a Lindelöf $T_{2}$-space.

If $X$ is a Hausdorff space, locally bounded with respect to $\mathcal{C}_{X}$, and $Y$ is compact and $T_{2}$, then $X \cup Y$ is a $T_{4}$-space.

Now, we investigate regularity and normality of $X \cup Y$ when $X$ is locally Lindelöf and $Y$ is Lindelöf. Of course, we suppose that $X$ and $Y$ are $T_{3}$-spaces.

**Theorem 2.2.** Let $X$ be a locally bounded $T_{3}$-space with respect to a (closed) boundedness $\mathcal{G}_{X} \subset \mathcal{L}_{X}$. Assume that every $F \in \mathcal{G}_{X}$ is contained in an open subset $A \in \mathcal{G}_{X}$. If $Y$ is a Lindelöf $T_{3}$-space and $\pi = \pi_{\mathcal{B}, \mathcal{G}_{X}}$ is an $\omega_{1}$-$B$ map, then $X \cup Y$ is a $T_{4}$-space.
PROOF. In view of Proposition 1.3 and Theorem 2.1 we have only to show that \( X \cup Y \) is regular.

Let \( I \) be a neighborhood of \( x \in X \) in \( X \cup Y \). If \( U_1 = \text{Cl}_X U_1 \subset I \cap X \) and \( U_2 = \text{Cl}_X U_2 \in \mathcal{S}_X \) are neighborhoods of \( x \) in \( X \), then \( V = U_1 \cap U_2 \subset I \) is a closed bounded neighborhood of \( x \) in \( X \). Hence \( V \) is also a closed neighborhood of \( x \) in \( X \cup Y \).

Now, let \( x \in Y \) with \( x \in U \cup \pi(U) \setminus F_1 \). By hypotheses, there is an open subset \( A_1 \) of \( X \) such that \( F_1 \subset A_1 \subset \overline{A}_1 \) and \( G_1 = \overline{A}_1 \in \mathcal{S}_X \). By the regularity of \( Y \), there exist \( W, W_1 \in \mathcal{B} \) such that \( x \in W \subset \overline{W} \subset W_1 \subset \overline{W}_1 \). If \( y \in Y \setminus U \), then \( V_y \in \mathcal{B} \) be such that \( x \in V_y \subset Y \setminus \overline{W}_1 \). Since \( Y \setminus U \) is closed and \( Y \) is Lindelöf, then the open cover \( \{V_y\}_{y \in Y} \) of \( Y \setminus U \) has a countable subcover \( \{V_n\}_{n=1}^{\infty} \), and one has

\[
Y \setminus U \subset V = \bigcup_{n=1}^{\infty} V_n \subset Y \setminus \overline{W}_1.
\]

Now \( U, V \in \mathcal{B} \) and \( U \cup V = Y \) imply \( F_2 = X \setminus (\pi(U) \cup \pi(V)) \in \mathcal{S}_X \). Denote by \( A_2 \) an open subset of \( X \) such that \( F_2 \subset A_2 \subset \overline{A}_2 \) and \( G_2 = \overline{A}_2 \in \mathcal{S}_X \).

Since \( V \cap \overline{W} = \emptyset \) we have that \( \pi(V) \cap \pi(W) \in \mathcal{S}_X \). If we set \( G_3 = \overline{W} \cap \pi(W) \) then one has \( \pi(V) \cap (\overline{W} \cap G_3) = \emptyset \). Now, we claim that

\[
\overline{W} \cap \pi(W) \setminus (G_1 \cup G_2 \cup G_3) \subset U \cup \pi(U) \setminus F_1.
\]

Suppose \( z \notin U \cup \pi(U) \setminus F_1 \) and \( z \in X \). Then

\[
z \in \pi(V) \cup F_1 \cup F_2 \subset \pi(V) \cup A_1 \cup A_2 \subset \pi(V) \cup G_1 \cup G_2,
\]

and \( \pi(V) \cup A_1 \cup A_2 \) is an open neighborhood of \( z \) that does not meet \( W \cup \pi(W) \setminus (G_1 \cup G_2 \cup G_3) \).

Finally, let \( z \notin U \cup \pi(U) \setminus F_1 \) and \( z \in Y \). Then \( z \in V \) and \( V \cup \pi(V) \) is an open neighborhood of \( z \) that does not meet \( W \cup \pi(W) \setminus G_3 \).

By the way, we note that, in the previous theorem, if \( \mathcal{S}_X \) is closed under countable unions, it is sufficient to assume that \( \pi \) is a \( B \)-map (in place of \( \omega_1 \)-map).

Let us see the theory at work in the following example. It also shows that Lindelöf extensions of locally Lindelöf spaces are not always compactifications.

**Example 2.3.** Let \( E \) be an uncountable set, viewed as a discrete space, and consider \( X = S \times E, f : S \times E \to S \) the canonical projection,
with $S$ the Sorgenfrey line. Then $X$ is a non-Lindelöf locally Lindelöf space, which has a Lindelöf non-compact $B$-extension $aX = X \cup S$, with respect to the $B$-map

$$ \pi = f^{-1} : T_S \to \mathcal{N}_X = \{ \emptyset \} \cup \{ M \subset X : M \text{ is not relatively Lindelöf} \}. $$

In fact, for every $U \in T_S \setminus \{ \emptyset \}$, $f^{-1}(U) = U \times E$ has a closure $\overline{U \times E} = \bigcup \{ U \times q : q \in E \}$, namely $U \times E$ has a partition in an uncountable family of non-empty open sets. Then $f^{-1}(U)$ is not relatively Lindelöf and $f$ is $B$-singular. By definition $S$ is closed as a subset of $aX$, therefore $aX$ is not compact. On the other hand, $X$ is obviously $T_{3\frac{1}{2}}$, therefore has $T_\sigma$-compactifications, and in each of them the remainder is not closed, since $X$ is not locally compact. From Theorem 2.2, we know that $X \cup Y$ is a $T_4$-space.

If a space $X$ is $T_{3\frac{1}{2}}$, we have already observed that $X$ is locally bounded with respect to $\mathcal{E}_X$. Since for every $F \in \mathcal{E}_X$ there is an open $A \in \mathcal{E}_X$ containing $F$, we have the following result.

**Theorem 2.3.** Let $X$ be a locally Lindelöf $T_{3\frac{1}{2}}$-space and let $Y$ be a Lindelöf $T_3$-space. If $\pi = \pi_{\infty, \mathcal{E}_X}$ is an $\omega_1$-$B$ map then $X \cup Y$ is a Lindelöf $T_4$-space.

In the previous theorem, if $X$ is a locally Lindelöf $T_4$-space, $\mathcal{E}_X$ can be replaced by $\mathcal{L}_X$. In fact, we have the following proposition.

**Proposition 2.4.** If $X$ is a locally Lindelöf $T_4$-space then $\mathcal{E}_X = \mathcal{L}_X$.

**Proof.** Let $F \in \mathcal{L}_X$. For every $y \in F$, let $U_y$ be an open neighborhood of $y$ with Lindelöf closure. The open cover $\{ U_y \}_{y \in F}$ of $F$ has a countable subcover $\{ U_n \}_{n=1}^\infty$. Then the open subset $U = \bigcup_{n=1}^\infty U_n$ of $X$ contains $F$ and is contained in $U' = \bigcup_{n=1}^\infty U_n$, that is Lindelöf. Now, by the normality of $X$, there is $f \in C(X, [0, 1])$ such that

$$ F \subset f^{-1}(0) \subset f^{-1}([0, 1]) \subset U \subset U'. $$

Then $f^{-1}([0, 1])$ is Lindelöf because it is a cozero-set contained in the Lindelöf set $U'$. 

Now, let $X$ be non-Lindelöf locally Lindelöf, and $Y = \{ \infty \}$. If $X$ is $T_{3\frac{1}{2}}$, let $\pi = \pi_{T_Y, \mathcal{E}_X}$ be the map defined by $\pi(\infty) = X$. Then $X \cup \{ \infty \}$, that
is $T_4$, is just the «single-point Lindelöfication» considered by Tkachuk in [8].

If $X$ is $T_3$, but not $T_{3\frac{1}{2}}$, and $\pi = \pi_{T_4, x}$ is defined by $\pi(\infty) = X$ then $X \cup \{\infty\}$ is a Lindelöf $T_2$-extension of $X$, that is not $T_4$.

We observe that, by Theorem 2.2, it follows that a $T_3$-space $X$ is locally Lindelöf and $T_{3\frac{1}{2}}$ if and only if $X$ is locally bounded with respect to $\mathcal{L}_X$.

If $aX$ is a $T_2$-extension of a space $X$ with $|aX\setminus X| = n$, then $X$ is locally bounded with respect to $\mathcal{L}_X(aX)$. If $aX$ is also Lindelöf, then $\mathcal{L}_X(aX) \subseteq \mathcal{L}_X$ and so $X$ is locally Lindelöf. Now, we show that $aX$ is a $B$-extension of $X$. This result could be proved in a way similar to that of Theorem 1.5. Here, a slightly different proof is presented, which will be useful to prove the next theorem.

**Proposition 2.5.** Let $aX$ be a $T_2$-extension of a space $X$ with $|aX\setminus X| = n$. Then $X$ is locally bounded with respect to $\mathcal{L}_X(aX)$ and $aX$ is a $B$-extension of $X$.

**Proof.** Let $Y = aX\setminus X = \{y_1, \ldots, y_n\}$ and let $U_1, \ldots, U_n$ be mutually disjoint open neighborhoods of $\{y_1, \ldots, y_n\}$ in $aX$. Since $U_i \cap X \notin \mathcal{L}_X(aX)$ for every $i$, we can define $\pi = \pi_{T_4, \mathcal{L}_X(aX)}$ by setting

$$\pi(\{y_{i_1}, \ldots, y_{i_s}\}) = (U_{i_1} \cap X) \cup \ldots \cup (U_{i_s} \cap X).$$

It is easily seen that $\pi$ is a $B$-map. Now, we show that $X \cup Y = aX$. Clearly $T_{X\cup Y} \subseteq T_{aX}$. Conversely, let $A$ be an open subset of $aX$. If $A \subseteq X$, then $A$ is obviously an open subset of $X \cup Y$. Suppose $y_k \in A$ for some $k$. Since $y_i \notin Cl_{ax}(U_i \setminus A)$ for every $i = 1, \ldots, n$, we get $F_k = Cl_X(U_k \setminus A) = Cl_{ax}(U_k \setminus A) \in \mathcal{L}_X(aX)$. Hence, if $A \cap (aX\setminus X) = \{y_{k_1}, \ldots, y_{k_s}\}$, then we have

$$A = \left(\bigcup_{j=1}^s \{y_{k_j}\} \cup \pi(\{y_{k_j}\}) \setminus F_{k_j}\right) \cup (X \cap A),$$

and $A$ is open in $X \cup Y$. □

Now, we characterize the spaces that have a $T_2$-extension with finite remainder (compare [6]).
THEOREM 2.6. A Hausdorff space X has a $T_2$-extension $aX$ with $|aX \setminus X| = n$ iff $X$ is locally bounded with respect to a boundedness $\mathcal{F}_X$ and there exist $n$ mutually disjoint unbounded open sets $A_1, \ldots, A_n \subset X$ and a bounded set $F \subset X$ such that

$$X = A_1 \cup \ldots \cup A_n \cup F.$$ 

PROOF. The proof of the above proposition shows that the condition is necessary. Conversely, suppose that the condition holds. If $y_1, \ldots, y_n$ are $n$ points not in $X$ and $Y = \{y_1, \ldots, y_n\}$, we define $\pi = \pi_{Y, \mathcal{F}_X}$ by $\pi(\{y_1, \ldots, y_n\}) = A_{i_1} \cup \ldots \cup A_{i_n}$. Then $\pi$ is a $B$-map and $aX = X \cup \pi Y$ is a $T_2$-extension with $|aX \setminus X| = n$. 

From Prop. 1.1, it follows that in a Hausdorff space $X$, the existence of a boundedness $\mathcal{F}_X$ with the properties as in in Theorem 2.6, implies that $\mathcal{F}_X = \mathcal{K}_X(aX)$ for a suitable $T_2$-extension $aX$ of $X$.

A consequence of Theorem 2.6 and Theorem 2.1 is the following.

COROLLARY 2.7. A Hausdorff space $X$ has a Lindelöf $T_2$-extension $aX$ with $|aX \setminus X| = n$ iff $X$ is locally bounded with respect to a boundedness $\mathcal{G}_X \subset \mathcal{L}_X$ and there exist $n$ mutually disjoint unbounded open sets $A_1, \ldots, A_n \subset X$ and a bounded set $F \subset X$ such that

$$X = A_1 \cup \ldots \cup A_n \cup F.$$ 

Now, we give an example of a Lindelöf $T_2$-extension of a locally Lindelöf space $X$ that cannot be obtained as a $B$-extension with respect to $\mathcal{L}_X$.

EXAMPLE 2.8. Let $E$ be a discrete non countable space, and let $E = S \cup T$ be a partition with $S$ countably infinite. Consider further a two points set $Y = \{a, b\}$ disjoint from $E$, $\bar{S} = S \cup \{a\}$ the one-point compactification of $S$, $\bar{T} = T \cup \{b\}$ the Lindelöf one-point $B$-extension of $T$, with respect to $\pi = \pi_{(b), \mathcal{G}_T}$ defined by $\pi(\{b\}) = T$. Then, the topological sum $L = \bar{S} + \bar{T}$ is $T_2$ Lindelöf and $E$ is dense in $L$. But $L$, that is a $B$-extension of $E$ with respect to the boundedness $\mathcal{K}_E(L)$, cannot be obtained as a $B$-extension of $E$ with respect to the «Lindelöf» boundedness $\mathcal{L}_E$. In fact, $\mathcal{K}_E(L) \neq \mathcal{L}_E$, and, from Prop. 1.1, we have that if $\pi' = \pi_{B, \mathcal{G}_E}$ is a $B$-map such that $L = E \cup Y$ then $\mathcal{F}_E = \mathcal{K}_E(L)$. 

We recall that a space $X$ is said to be $[\theta, \kappa]$-compact if every open cover of $X$ of cardinality $\leq \kappa$ has a subcover of cardinality $< \theta$. If $\theta = \omega$, then $X$ is said to be initially $\kappa$-compact, and if $|X| \leq \kappa$, then $X$ is said to be finally $\theta$-compact. Lindelöf spaces are exactly the finally $\omega_1$-compact spaces.

For a space $X$, $c_X(\theta, \kappa) = \{A \subset X: \overline{A}$ is $[\theta, \kappa]$-compact$\}$ is a boundedness in $X$. We say that a space $X$ is locally $[\theta, \kappa]$-compact if every point of $X$ has a $[\theta, \kappa]$-compact neighborhood. As in the Lindelöf case, if $X$ is $T_3$, then $X$ is locally $[\theta, \kappa]$-compact if and only if $X$ is locally bounded with respect to $c_X(\theta, \kappa)$.

Proposition 2.1 can be easily generalized to $[\theta, \kappa]$-compact case.

**Proposition 2.9.** Let $X$ be a locally bounded $T_2$-space with respect to a (closed) boundedness $S_X \subset c_X(\theta, \kappa)$ (hence $X$ is locally $[\theta, \kappa]$-compact) and let $Y$ be a $[\theta, \kappa]$-compact $T_2$-space. If $\pi = \pi_{S_X}$, $S_X$ is a $B$-map, then $X \cup Y$ is a $[\theta, \kappa]$-compact $T_2$-space.

Theorem 2.2 can be generalized to finally $\theta$-compact case in the following way.

**Theorem 2.10.** Let $X$ be a locally bounded $T_3$-space with respect to a (closed) boundedness $S_X \subset c_X(\theta, \kappa)$, with $|X| \leq \kappa$. Assume that every $F \in S_X$ is contained in an open subset $A \in S_X$. If $Y$ is $T_3$ and $\pi = \pi_{S_X}$, $S_X$ is a $\theta$-$B$ map, then $X \cup Y$ is a $T_3$-space.

Finally, Theorem 2.3 and Corollary 2.7 have similar generalizations to the finally $\theta$-compact case.

**REFERENCES**


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