RITA GIULIANO ANTONINI

Subgaussian random variables in Hilbert spaces

Rendiconti del Seminario Matematico della Università di Padova,
tome 98 (1997), p. 89-99

<http://www.numdam.org/item?id=RSMUP_1997__98__89_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1997, tous droits réservés.

L’accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l’accord avec les conditions générales d’utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques

http://www.numdam.org/
Subgaussian Random Variables in Hilbert Spaces (*).

RITA GIULIANO ANTONINI (**) 

0. - Introduction.

In the paper [1] the following definition is given:

(0.1) DEFINITION. A real r.v. $X$ is said to be subgaussian if there exists a number $a \geq 0$ such that

$$E[e^{tx}] \leq \exp \left( \frac{1}{2} a^2 t^2 \right), \quad \forall t \in \mathbb{R},$$

If this is the case, the number

$$\tau_{cl}(X) = \inf \left\{ a \geq 0 : E[e^{tx}] \leq \exp \left( \frac{1}{2} a^2 t^2 \right), \forall t \in \mathbb{R} \right\}$$

is called the gaussian standard of $X$.

Denote by $\mathcal{S}_g(\Omega)$ the set of real subgaussian variables. In [1] it is proved that $\mathcal{S}_g(\Omega)$ is a vector space and $\tau_{cl}$ is a norm in it. Moreover $\mathcal{S}_g(\Omega)$, endowed with the norm $\tau_{cl}$, is a Banach space.

In this paper we consider random variables taking their values in a separable Hilbert space $H$, and we give three different definitions of subgaussianity (the first of them is subgaussianity with respect to a linear trace class operator $R$, symmetric and positive definite; the second is subgaussianity with respect to a complete orthonormal system $E$ in

(*) This paper is partially supported by GNAFA, CNR.
(**) Indirizzo dell'A.: Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, 56100 Pisa (Italy).
\( H; \) the last one is subgaussianity tout-court (i.e., our definition will not depend on any \( R \) or \( E \)). We investigate the relations between these concepts; moreover, we show that the set of \( E \)-subgaussian variables, endowed with a suitable norm, is a Banach space; for subgaussian variables in the other two senses, we prove the same thing when \( H \) is finite dimensional.

1. – Subgaussianity with respect to an operator.

Let \( H \) be a Hilbert space (finite or infinite dimensional), and denote by \( \langle \cdot, \cdot \rangle \) its inner product. Let \( X \) be an \( H \)-valued random variable and \( R \) a linear operator on \( H \). Suppose that \( R \) is a trace class operator, symmetric and positive definite. We shall denote by \( \mathcal{L}_1 \) the set of such operators.

We give the following

\((1.1)\) Definition. We say that \( X \) is subgaussian with respect to \( R \in \mathcal{L}_1 \) (or \( R \)-subgaussian) if there exists \( a \geq 0 \) such that

\[
E[e^{\langle x, X \rangle}] \leq \exp \left( \frac{1}{2} a^2 \langle Rx, x \rangle \right) \quad \text{for every } x \in H.
\]

If this is the case, we put

\[
\sigma_R(X) = \inf \left\{ a \geq 0 : E[e^{\langle x, X \rangle}] \leq \exp \left( \frac{1}{2} a^2 \langle Rx, x \rangle \right) \quad \text{for every } x \in H \right\}.
\]

\((1.3)\) Remark. It is clear that

(i) \[
\sigma_R(X) = \sup_{x \neq 0} \frac{\tau_{\text{cl}}(\langle x, X \rangle)}{\langle Rx, x \rangle^{1/2}},
\]

(ii) \[
E[e^{\langle x, X \rangle}] \leq \exp \left( \frac{1}{2} \sigma_R(X)^2 \langle Rx, x \rangle \right) \quad \text{for every } x \text{ in } H.
\]

\((1.4)\) Remark. In [4] the following definition is given: \( X \) is a subgaussian variable if there exists an \( H \)-valued gaussian vector \( G \) such that, for every \( x \) in \( H \), we have

\[
E[e^{\langle x, X \rangle}] \leq E[e^{\langle x, G \rangle}] ;
\]
now, according to the results of [2], we have

\[ E[e^{\langle x, G \rangle}] = \exp \left( \frac{1}{2} E[\langle x, G \rangle^2] \right) = \exp \left( \frac{1}{2} \langle S_G x, x \rangle \right), \]

where \( S_G \) is the covariance operator of \( G \).

Since \( G \) is gaussian, \( S_G \) is in \( \mathcal{L}_1 \) (see [2]); hence \( X \) is \( S_G \)-subgaussian.

Conversely, if \( X \) is \( R \)-subgaussian, the operator \( \sigma^2_R (X) \) is in \( \mathcal{L}_1 \), hence it is the covariance operator of some gaussian vector \( G \), and we have

\[ E[\langle x, G \rangle^2] = \langle S x, x \rangle. \]

Then

\[ E[e^{\langle x, X \rangle}] \leq \exp \left( \frac{1}{2} \langle S x, x \rangle \right) = \exp \left( \frac{1}{2} E[\langle x, G \rangle^2] \right) = E[e^{\langle x, G \rangle}], \]

and \( X \) is subgaussian in the sense of [4].

(1.5) REMARK. Definition (1.1) is a generalization of the one given in [3] for the case \( H = \mathbb{R}^n \).

We shall denote by \( S_{\mathbb{R}} (\Omega) \) the set of \( H \)-valued \( R \)-subgaussian variables. By recalling that \( \tau_{cl} \) is a norm in the space of real subgaussian variables (see [1]), Remark (1.3)(i) yields immediately that \( S_{\mathbb{R}} (\Omega) \) is a vector space and \( \sigma_R \) is a norm in it, i.e. \( (S_{\mathbb{R}} (\Omega), \sigma_R) \) is a metric space. As we shall see in Section 3, it is a Banach space when \( H \) is finite dimensional.

2. - Subgaussianity with respect to a complete orthonormal system.

Let \( E = \{e_n\} \) be a complete orthonormal system (C.O.N.S.) in \( H \).

(2.1) DEFINITION. We say that \( X \) is subgaussian with respect to \( E \) (or \( E \)-subgaussian) if the two following conditions are verified:

(i) For every \( x \in H \), the real random variable \( \langle x, X \rangle \) is subgaussian;

(ii) We have

\[ \tau^2_E (X) \equiv \sum_n \tau^2_{cl} (\langle e_n, X \rangle) < + \infty. \]
We shall denote by $\mathcal{S}_E(\Omega)$ the set of $H$-valued $E$-subgaussian variables. (This notation is quite similar to the one introduced in Section 1 for the set of variables which are subgaussian with respect to an operator, but this should cause no confusion).

We shall prove the following

(2.2) **Theorem.** $\mathcal{S}_E(\Omega)$ is a vector space and $\tau_E$ is a norm in $\mathcal{S}_E(\Omega)$; moreover, $(\mathcal{S}_E(\Omega), \tau_E)$ is a Banach space.

The proof of (2.2) is a straightforward application of the following general result:

(2.3) **Theorem.** Let $(B, \nu)$ be a Banach space, and consider the set

$$B_N^2 = \left\{ x = (x_1, x_2, \ldots) \in B^N \text{ and } \sum_n \nu^2(x_n) < +\infty \right\}.$$

Then $B_N^2$ (with sum and product by a scalar defined in the usual way) is a Banach space with norm

$$\varrho(x) = \left( \sum_n \nu^2(x_n) \right)^{1/2}.$$

Theorem (2.3) is standard. Anyway, by the sake of completeness, we sketch the proof in the appendix.

Theorem (2.2) follows from Theorem (2.3) by identifying $X$ with the vector $((X, e_1), (X, e_2), \ldots)$ and by taking $\nu = \tau_{el}$ (recall that, by the results of [1], the set of real subgaussian variables is a Banach space with norm $\tau_{el}$).

3. - Relation between $R$-subgaussianity and $E$-subgaussianity.

(3.1) **Proposition.** Let $X$ be subgaussian with respect to an operator $R \in \mathcal{L}_1$, and let $E$ be any C.O.N.S. in $H$. Then $X$ is subgaussian with respect to $E$ and

$$\tau_E^2(X) \leq \sigma_R^2(X) \text{tr}(R) < +\infty,$$

where $\text{tr}(R)$ denotes the trace of $R$.

**Proof.** It is clear by (1.2) and (1.3)(ii) that, for every $x$ in $H$, $\langle x, X \rangle$ is subgaussian and

$$\tau_{el}^2(\langle x, X \rangle) \leq \sigma_R^2(X) \langle Rx, x \rangle.$$
Then
\[ \tau_{cl}^2(\langle e_n, X \rangle) \leq \sigma_R^2(X) \langle Re_n, e_n \rangle. \]

We get the conclusion by summing over \( n \) and recalling that
\[ \text{tr}(R) = \sum_n \langle Re_n, e_n \rangle. \]

(3.2) **Remark.** Proposition (3.1) says that \((S_{\mathcal{D}R}(\Omega), \sigma_R)\) can be continuously imbedded in \((S_{\mathcal{D}E}(\Omega), \tau_E)\).

We now drop for a moment the assumption that \(R \in \mathcal{E}_1\), and let \(E_R = \{f_n\} \) be the set of normalized eigenvectors of \(R\). We are going to compare \(\sigma_R\) with \(\tau_E\). To this extent, we need the following

(3.3) **Lemma.** Let \(E = \{e_n\}\) by any C.O.N.S. in \(H\). Then, for every \(x \in H\),
\[ \tau_{cl}(\langle x, X \rangle) \leq \sum_n |\langle x, e_n \rangle| \tau_{cl}(\langle e_n, X \rangle). \]

**Proof.** For every \(\omega \in \Omega\) and every \(n\), put
\[ Y_n(\omega) = \sum_{k=1}^{n} \langle x, e_k \rangle \cdot \langle e_k, X(\omega) \rangle. \]

Then \(Y_n(\omega)\) converges to \(\langle x, X(\omega) \rangle\) for each \(\omega\) in \(\Omega\), as \(n \to \infty\).

Moreover, by the triangular inequality for \(\tau_{cl}\),
\[
\tau_{cl}(Y_n) = \tau_{cl}^2 \left( \sum_{k=1}^{n} \langle x, e_k \rangle \cdot \langle e_k, X \rangle \right) \leq \left[ \sum_{k=1}^{n} |\langle x, e_k \rangle| \tau_{cl}(\langle e_k, X \rangle) \right]^2 \leq \left[ \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \right] \left[ \sum_{k=1}^{n} \tau_{cl}^2(\langle e_k, X \rangle) \right] \leq \left[ \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \right] \left[ \sum_{k=1}^{\infty} \tau_{cl}^2(\langle e_k, X \rangle) \right] = ||x||^2 \tau_E^2(X) < +\infty.
\]

Hence, for every \(t \in \mathbb{R}\) and every \(\varepsilon > 0\),
\[
\sup_n E[(e^{iy_n})^{1+\varepsilon}] = \sup_n E[e^{it(1+\varepsilon)y_n}] \leq \sup_n \exp \left( \frac{1}{2} t^2 (1+\varepsilon)^2 \tau_{cl}^2(Y_n) \right) < +\infty,
\]
so that the r.v. \(e^{iy_n}\) are uniformly integrable and, by Lebesgue theorem,
we have
\[ E[e^{t x_n X}] = E[e^{t \lim n Y_n}] = E[\lim n e^{t Y_n}] = \lim n E[e^{t Y_n}] \leq \exp \left( \frac{1}{2} t^2 \sup \tau_{cl}^2 (Y_n) \right). \]
It follows
\[ \tau_{cl} (\langle x, X \rangle) \leq \sup \tau_{cl} (Y_n) = \sup \tau_{cl} \left( \sum_{k=1}^n \langle x, e_k \rangle \cdot \langle e_k, X \rangle \right) \leq \sup \sum_{k=1}^n |\langle x, e_k \rangle| \tau_{cl} (\langle e_k, X \rangle) = \sum_{k=1}^\infty |\langle x, e_k \rangle| \tau_{cl} (\langle e_k, X \rangle). \]

(3.4) **Proposition.** Let \( 0 < \alpha_1 \leq \alpha_2 \leq \ldots \) be the eigenvalues of \( R \), and assume that \( X \) is \( E_R \)-subgaussian. Then
\[ \tau_{cl}^2 (\langle x, X \rangle) \leq \frac{1}{\alpha_1} \tau_{cl}^2 (X) \langle Rx, x \rangle. \]

**Proof.** By Lemma (3.3), we have
\[ \tau_{cl}^2 (\langle x, X \rangle) \leq \left[ \sum_{k=1}^\infty |\langle x, f_k \rangle| \tau_{cl} (\langle f_k, X \rangle) \right]^2 = \left[ \sum_{k=1}^\infty \sqrt{\alpha_k} |\langle x, f_k \rangle| \frac{1}{\sqrt{\alpha_k}} \tau_{cl} (\langle f_k, X \rangle) \right]^2 \leq \left[ \sum_{k=1}^\infty \alpha_k |\langle x, f_k \rangle|^2 \right] \left[ \sum_{k=1}^\infty \frac{1}{\alpha_k} \tau_{cl}^2 (\langle f_k, X \rangle) \right] = \langle Rx, x \rangle \left[ \sum_{k=1}^\infty \frac{1}{\alpha_k} \tau_{cl}^2 (\langle f_k, X \rangle) \right] \leq \langle Rx, x \rangle \frac{1}{\alpha_1} \tau_{cl}^2 (X). \]

If \( H \) is finite dimensional, Proposition (3.4) yields the following upper bound for \( \sigma_R (X) \):

(3.5) **Proposition.** We have
\[ \sigma_R^2 (X) \leq \frac{1}{\alpha_1} \tau_{cl}^2 (X). \]
PROOF. For every \( x \in H \) we have, by (3.4),
\[
E[e^{(x, X)}] \leq \exp \left( \frac{1}{2} \tau^2_{\Omega} ((x, X)) \right) \leq \exp \left( \frac{1}{2} \frac{1}{\alpha_1} \tau^2_{E_R} (X) \langle Rx, x \rangle \right),
\]
so that
\[
\sigma^2_R (X) = \inf \left\{ b \geq 0 : E[e^{(x, X)}] \leq \exp \left( \frac{1}{2} b \langle Rx, x \rangle \right) \text{ for every } x \in H \right\} \leq \frac{1}{\alpha_1} \tau^2_{E_R} (X).
\]

Propositions (3.1) and (3.4), together with Theorem (2.2), allows us to state the following result

(3.6) THEOREM. If \( H \) is finite dimensional and \( R \) is injective, then \( S_{\Omega} (\Omega) = S_{E_R} (\Omega) \) (this is a set-theoretical inclusion). Moreover the two norms \( \tau_{E_R} \) and \( \sigma_R \) are equivalent; hence \( (S_{\Omega} (\Omega), \sigma_R) \) is a Banach space.

4. – The space of subgaussian variables.

The two definitions of subgaussianity we have given in Section 1 and 2 depend strongly on the operator \( R \) in \( \mathcal{L}_1 \) and the C.O.N.S. \( E \) respectively. Here we give a definition which will not depend on such objects.

(4.1) DEFINITION. We say that \( X \) is subgaussian if there exists \( R \in \mathcal{L}_1 \) such that \( X \) is \( R \)-subgaussian. We shall denote by \( S_{\Omega} (\Omega) \) the set of such variables, and define the quantity
\[
\sigma (X) = \sup \{(\text{tr} R)^{1/2} \sigma_R (X) ; \ R \in \mathcal{L}_1 \}.
\]

By virtue of the results of Section 2, \( \sigma \) is obviously a norm in \( S_{\Omega} (\Omega) \). Our aim is now to prove the following

(4.2) THEOREM. If \( H \) is finite dimensional, \( (S_{\Omega} (\Omega); \sigma) \), is a Banach space.

PROOF. Let \( \{X_n\} \) be a Cauchy sequence in \( (S_{\Omega} (\Omega); \sigma) \). The inequality
\[
(\text{tr} R)^{1/2} \sigma_R (X_n - X_m) \leq \sigma (X_n - X_m)
\]
yields that, for every \( R \) in \( \mathcal{L}_1 \), \( \{X_n\} \) is Cauchy in \( (S_{E_R} (\Omega), \sigma_R) \). Since the
last space is Banach, for each \( R \) there exists \( Y^{(R)} \) is such that

\[(\text{tr} R)^{1/2} \sigma_R(X_n - Y^{(R)}) \to 0.\]

Let now \( E = \{e_n\} \) be any C.O.N.S. in \( H \). From Proposition (3.1) it follows that \( \{X_n\} \) \( \tau_E \)-converges to \( Y^{(R)} \), so that \( Y^{(R)} \) cannot depend on \( R \); let's call it \( Y \) from now on.

From the triangular inequality we now deduce that

\[
(\text{tr} R)^{1/2} \sigma_R(X_n - Y) \leq (\text{tr} R)^{1/2} \sigma_R(X_n - X_m) + (\text{tr} R)^{1/2} \sigma_R(X_m - Y) \leq \\
\leq \sigma(X_n - X_m) + (\text{tr} R)^{1/2} \sigma_R(X_m - Y).
\]

By interchanging the roles of \( n \) and \( m \) we get

\[
|\text{tr} R|^{1/2} \sigma_R(X_n - Y) - (\text{tr} R)^{1/2} \sigma_R(X_m - Y) | \leq \sigma(X_n - X_m).
\]

The above inequality yields that \( (\text{tr} R)^{1/2} \sigma_R(X_n - Y) \to 0 \) uniformly in \( R \), and this in turn implies that \( \sigma(X_n - Y) \to 0 \).

(4.3) REMARK. It is easy to see that \( X \) is subgaussian in the sense of Definition (4.1) if and only if there exists a C.O.N.S. \( E \) such that \( X \) is \( E \)-subgaussian (in the sense of (2.1)). In \( S_{\sigma}(\Omega) \) one can then consider the quantity

\[
\tau(X) = \sup \{\tau_E(X); \ E \text{ C.O.N.S. in } H\}.
\]

It is immediate to see that \( \tau \) is a norm in \( S_{\sigma}(\Omega) \) and, by arguments similar to the previous ones (using (3.4) instead of (3.1)), one can easily show that \( (S_{\sigma}(\Omega); \tau) \) is a Banach space. By Proposition (3.1), \( (S_{\sigma}(\Omega); \sigma) \) can be imbedded continuously in \( (S_{\sigma}(\Omega); \tau) \).

5. – A condition for subgaussianity with respect to an operator.

In this section we are looking for a condition which assures the existence of an operator \( R \) such that \( X \) is \( R \)-subgaussian. We need the following

(5.1) LEMMA. Let \( E = \{e_n\} \) be a C.O.N.S. in \( H \); suppose that \( X \) is subgaussian with respect to \( E \) and that the following assumption holds:

(5.2) For every $n \in \mathbb{N}$ and every $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, we have
\[
\tau^2_{\text{cl}} \left( \sum_{k=1}^{n} \lambda_k \langle e_k, X \rangle \right) = \sum_{k=1}^{n} \lambda_k^2 \tau^2_{\text{cl}} (\langle e_k, X \rangle).
\]

Then
\[
\tau^2_{\text{cl}} (\langle x, X \rangle) = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2 \tau^2_{\text{cl}} (\langle e_k, X \rangle)
\]
(the last series converges since it is not greater than $\|x\|^2 \tau^2_{E} (X)$).

Proof. For every $\omega \in \Omega$ and every $n$, put, as in (3.3),
\[
Y_n (\omega) = \sum_{k=1}^{n} \langle x, e_k \rangle \cdot \langle e_k, X(\omega) \rangle.
\]
We have
\[
\tau^2_{\text{cl}} (Y_n - \langle x, X \rangle) = \\
= \tau^2_{\text{cl}} \left( \sum_{k=n+1}^{\infty} \langle x, e_k \rangle \cdot \langle e_k, X \rangle \right) \leq \left[ \sum_{k=n+1}^{\infty} |\langle x, e_k \rangle|^2 \right] \left[ \sum_{k=n+1}^{\infty} \tau^2_{\text{cl}} (\langle e_k, X \rangle) \right] \to 0,
\]
as $n \to \infty$.

By the continuity of the norm $\tau_{\text{cl}}$, it follows that
\[
\tau_{\text{cl}} (Y_n) \to \tau_{\text{cl}} (\langle x, X \rangle).
\]

On the other hand, by assumption (5.2),
\[
\tau^2_{\text{cl}} (Y_n) = \tau^2_{\text{cl}} \left( \sum_{k=1}^{n} \langle x, e_k \rangle \cdot \langle e_k, X \rangle \right) = \\
= \sum_{k=1}^{n} \langle x, e_k \rangle^2 \tau^2_{\text{cl}} (\langle e_k, X \rangle) \to \sum_{k=1}^{\infty} \langle x, e_k \rangle^2 \tau^2_{\text{cl}} (\langle e_k, X \rangle).
\]

(5.3) Remark. Recall that the variance of the sum of two independent random variables is the sum of their variance. From this point of view, condition (5.2) may be regarded as a sort of independence among the variables $\langle e_n, X \rangle$, $n \in \mathbb{N}$.

(5.4) Proposition. Suppose that $\text{span} (\text{Im} X) = H$. Suppose moreover that there exists a C.O.N.S. $E = \{e_n\}$ such that $X$ is subgaussian with respect to $E$ and (5.2) holds. Then $X$ is subgaussian with respect to

...
the operator \( R \) defined by
\[
R e_n = \tau_{\mathcal{D}}^2 (\langle e_n, X \rangle) e_n.
\]

Moreover \( \sigma_R(X) = 1 \).

**Proof.** It is easily seen that, by Lemma (5.1),
\[
\langle Rx, x \rangle = \tau_{\mathcal{D}}^2 (\langle x, X \rangle).
\]

Since \( \text{span}(\text{Im} X) = H \), the operator \( R \) is definite positive. It is also a trace class operator since
\[
\text{tr}(R) = \sum_n \langle Re_n, e_n \rangle = \tau_{\mathcal{D}}^2 (X) < + \infty.
\]

Then
\[
E[\langle x, X \rangle] \leq \exp \left( \frac{1}{2} \tau_{\mathcal{D}}^2 (\langle x, X \rangle) \right) = \exp \left( \frac{1}{2} \langle Rx, x \rangle \right).
\]

The infimum property of \( \tau_{\mathcal{D}} (\langle x, X \rangle) \) gives the last statement of the proposition.

**Appendix. Proof of Theorem (2.3).**

If \( x \) and \( y \) are two elements of \( B_2^N \), then
\[
\sum_n \nu^2 (x_n + y_n) \leq \sum_n (\nu(x_n) + \nu(y_n))^2 \leq 2 \left[ \sum_n \nu^2 (x_n) + \sum_n \nu^2 (y_n) \right],
\]
so that \( x + y \in B_2^N \).

It is immediate to see that, for every \( \lambda \in \mathbb{R} \), \( \lambda x \in B_2^N \) if \( x \in B_2^N \).

Let's now see that \( \varrho \) is a norm in \( B_2^N \). The only non trivial thing to check is the triangular inequality. We have
\[
\varrho^2 (x + y) = \sum_n \varrho^2 (x_n + y_n) \leq \sum_n (\nu(x_n) + \nu(y_n)) \nu(x_n + y_n) =
\]
\[
= \sum_n \nu(x_n) \nu(x_n + y_n) + \sum_n \nu(y_n) \nu(x_n + y_n) \leq
\]
\[
\leq \left( \sum_n \nu^2 (x_n) \right)^{1/2} \left( \sum_n \nu^2 (x_n + y_n) \right)^{1/2} +
\]
\[
+ \left( \sum_n \nu^2 (y_n) \right)^{1/2} \left( \sum_n \nu^2 (x_n + y_n) \right)^{1/2} = \varrho(x) \varrho(x + y) + \varrho(y) \varrho(x + y),
\]
(where the first \( \leq \) is due to the triangular inequality for \( \nu \) and the second \( \leq \) to the Schwartz inequality).

We now prove that \( \varrho \) is a Banach norm.
Let \((x^{(p)})_p\) be a Cauchy sequence in \(B^N_2\). This means that, for every \(\varepsilon > 0\), there exists \(p_0\) such that, for every \(p, q > p_0\), we have
\[
Q^2(x^{(p)} - x^{(q)}) = \sum_n \nu(x^{(p)}_n - x^{(q)}_n)^2 < \varepsilon,
\]
and it is easy to see that the series \(\sum_n \nu(x^{(p)}_n - x^{(q)}_n)^2\) converges uniformly in \(p, q\).

The inequality
\[
\nu(x^{(p)}_n - x^{(q)}_n)^2 \leq Q^2(x^{(p)} - x^{(q)}) < \varepsilon,
\]
valid for \(p, q > p_0\), implies that, for each \(n\), \((x^{(p)}_n)\) is a Cauchy sequence in \(B\), hence converges in \(B\) (since \(B\) is Banach). Let
\[
y_n = \lim_p x^{(p)}_n, \quad y = (y_1, y_2, \ldots).
\]
Passing to the limit in (A.1) with respect to \(q\), we get, for \(p > p_0\),
\[
\lim_q \sum_n \nu(x^{(p)}_n - x^{(q)}_n)^2 = \sum_n \lim_q \nu(x^{(p)}_n - x^{(q)}_n)^2 = \sum_n \nu(x^{(p)}_n - y_n)^2 \leq \varepsilon
\]
(where the first equality is due to the uniform convergence of the series with respect to \(q\), and the second to the continuity of the norm \(\nu\)).

Hence, for \(p > p_0\),
\[
\sum_n \nu^2(y_n) = \sum_n \nu^2(y_n - x^{(p)}_n + x^{(p)}_n) \leq 2\left[ \sum_n \nu(x^{(p)}_n - y_n)^2 + \sum_n \nu(x^{(p)}_n)^2 \right] \leq 2\varepsilon + 2\sum_n \nu(x^{(p)}_n)^2 < +\infty,
\]
that is, \(y \in B^N_2\). Finally, relation (A.2) may be rephrased as \(\langle x^{(p)} \rangle_p\) converges to \(y\) in norm as \(p \to \infty\).

REFERENCES


Manoscritto pervenuto in redazione il 12 ottobre 1995 e, in forma revisionata, il 18 gennaio 1996.