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Rendiconti del Seminario Matematico della Università di Padova, tome 97 (1997), p. 73-79

<http://www.numdam.org/item?id=RSMUP_1997__97__73_0>

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Minimal Non-Totally Minimal Topological Rings.

MICHAEL MEGRELIISHVILI (*)

ABSTRACT - We establish the existence of minimal non-totally minimal topological rings with a unit answering a question of Dikranjan. The Pontryagin duality and a generalization of Ursul's «semidirect product type» construction play major roles in the construction.

Introduction.

A Hausdorff topological ring $R$ is called minimal if its topology is minimal in the sense of Zorn among all Hausdorff ring topologies on $R$. If $R/J$ is minimal for every closed ideal $J$, then $R$ is called totally minimal [2].

The induced topology of a nontrivial valuation on a field is (totally) minimal (see [10, 6]). Some generalizations and related results in the context of fields or divisible rings may be found in [11,13,14]. For more general cases we refer to [1,2,3,9]. Recall [2,3] for instance that the class of all minimal rings with a unit is closed under forming topological products, direct sums and matrix rings. If $P$ is a non-zero prime ideal of finite index in a Dedekind ring, then the $P$-adic topology is minimal.

The question about existence of minimal non-totally minimal rings with a unit is discussed by Dikranjan in [2,3].

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Partially supported by the Israel Ministry of Sciences, Grant No. 3505.
1991 Mathematics Subject Classification: 16W80, 54H13, 13J99, 22B99.
Conventions and preliminaries.

As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of all natural, integer and real numbers, respectively. The unit circle group $\mathbb{R}/\mathbb{Z}$ will be denoted by $T$ and the $n$-element cyclic ring by $\mathbb{Z}_n$.

All rings are assumed to be associative. A ring $R$ is unital if it has a unit. The zero-element will be denoted by $0$. By $\text{char}(R)$ we indicate the minimal natural number (if it exists) $n$ such that $nx = 0$ for every $x \in R$. Otherwise we write $\text{char}(R) = 0$. Clearly, $\text{char}(R) = n > 0$ iff $R$ is a (left) $\mathbb{Z}_n$-algebra in a natural way $(k, x) \mapsto x + x + \ldots + x$ ($k$ terms) for each $(k, x) \in \mathbb{Z}_n \times R$.

For a locally compact Abelian group $G$, denote by $G^*$ the dual group $H(G, \mathbb{T})$ of all continuous characters endowed with the compact open topology. If $R$ is a locally compact ring, then $R^*$ is a topological $(R, R)$-bimodule [12].

If $P$ is a subgroup of a topological group $(G, \tau)$, then $\tau|_P$ will denote the relative topology on $P$, and $\tau/P$ will be the quotient topology on the left coset space $G/P$. The following useful result is well known.

**Merzson's Lemma** [8] (See also [4], Lemma 7.2.3 for a proof). Let $P$ be a subgroup of a group $G$, and let $\tau'$ and $\tau$ be (not necessarily Hausdorff) group topologies on $G$ with the properties: $\tau' \subseteq \tau$, $\tau'|_P = \tau|_P$ and $\tau'/P = \tau/P$. Then $\tau' = \tau$.

Main results.

Recall a construction from [12]. Let $R$ be a topological ring and $X$ a topological $(R, R)$-bimodule. On the product $R \times X$ of topological groups $R$ and $X$, consider the multiplication

$$(r_1, x_1)(r_2, x_2) = (r_1 r_2, r_1 x_2 + x_1 r_2), \quad r_1, r_2 \in R, \quad x_1, x_2 \in X.$$

Then $R \times X$ becomes a topological ring which is denoted by $R \ltimes X$. For details and a particular case of $R \ltimes R^*$ see Ursul [12].

Now we generalize this construction in two directions. The first change is minor. Let $K$ be a commutative unital Hausdorff topological ring, $(R, \tau)$ a topological $K$-algebra, and $(S, \nu)$ be a topological $K$-module. Instead of $R^* = H_\mathbb{Z}(R, \mathbb{T})$, consider the $K$-module $H_K(R, S)$ of all continuous $K$-homomorphisms $R \rightarrow S$. As in the case of $R^*$, the left and right multiplications in $R$ induce the $(R, R)$-bimodule structure in $H_K(R, S)$. The second modification is more essential. We add to $R \ltimes H_K(R, S)$ a supplementary coordinate. Denote by $M_K(R, S)$ the product $R \times H_K(R, S) \times S$ of $K$-modules. The multiplication we define
by the rule:

\[(r_1, f_1, s_1)(r_2, f_2, s_2) = (r_1, f_2, r_1f_2 + f_1r_2, f_2(r_1) + f_1(r_2))\]

where \(r_1, r_2 \in R, f_1, f_2 \in H_K(R, S)\) and \(s_1, s_2 \in S\). Simple computations show that \(M_K(R, S)\) becomes a \(K\)-algebra. Let \(H_K(R, S)\) carry a \(K\)-module topology \(\sigma\) such that its \((R, R)\)-bimodule structure is topological too. Moreover, suppose that the evaluation mapping

\[\omega: H_K(R, S) \times R \to S, \quad \omega(f, r) = f(r)\]

is continuous with respect to the triple \((\sigma, r, v)\) of Hausdorff topologies. Then \((M_K(R, S), \gamma)\) is a Hausdorff topological \(K\)-algebra with respect to the product topology \(\gamma\). In particular, if \(R\) is a locally compact ring, \(S = T\) and \(K = \mathbb{Z}\), then one gets a locally compact topological ring \(M_Z(R, T) = R \times R^* \times T\) which will be denoted by \(M(R)\).

Furthermore, we identify \(R, H_K(R, S)\) and \(H_K(R, S) \times S\) with the corresponding subsets of \(M_K(R, S)\). We will keep below our assumptions about \((M_K(R, S), \gamma)\).

**Proposition 1.** Let \(\gamma'\) be a new ring topology on \(M_K(R, S)\) such that the canonical group retraction \(q: H_K(R, S) \times S \to S\) is continuous for the topologies \(\gamma'\big|_{H_K(R, S) \times S}\) and \(v\). Then the evaluation mapping \(\omega\) is continuous with respect to the triple of topologies \(\gamma'\big|_{H_K(R, S) \times S}\), \(\gamma'/H_K(R, S) \times S\) and \(v\).

**Proof.** Fix \(\varphi_0 \in H_K(R, S), r_0 \in R\) and a \(v\)-neighborhood \(O\) at \(\varphi_0(r_0)\) in \(S\). By the continuity of \(q\), we may choose a \(\gamma'\)-neighborhood \(U\) of the element \(z_0 = (0, \varphi_0 r_0, \varphi_0(r_0)) \in M_K(R, S)\), such that \(q(U \cap (H_K(R, S) \times S)) \subset O\).

By our assumption, the ring multiplication is \(\gamma'\)-continuous. Therefore, there exist \(\gamma'\)-neighborhoods \(V, W\) of the elements \((0, \varphi_0, 0)\) and \((r_0, 0, 0)\) respectively, such that \(V \cdot W\) is contained in the chosen \(\gamma'\)-neighborhood \(U\) of \(z_0 = (0, \varphi_0, 0)(r_0, 0, 0)\).

For every \(\varphi \in V \cap H_K(R, S)\) and every \((r, f, s) \in W\), we have

\[(0, \varphi, 0)(r, f, s) = (0, \varphi r, \varphi(r)) \in U \cap (H_K(R, S) \times S).\]

Clearly, \(\varphi(r) = \omega(\varphi, r) = \omega(V \cap H_K(R, S), pr(W))\), where \(pr\) denotes the projection \(M_K(R, S) \to R\) on the first coordinate.

Then,

\[\varphi(r) \in \omega(V \cap H_K(R, S), pr(W)) \subset q(U \cap (H_K(R, S) \times S)) \subset O.\]

Since \(pr(W)\) is a \(\gamma'/H_K(R, S) \times S\)-neighborhood of the point \(r_0\) and
Let \((F, \sigma), (S, \nu)\) be Abelian Hausdorff groups. A continuous mapping \(\omega: F \times E \to S\) is called biadditive if the induced mappings \(\omega_x: F \to S, \omega_f: E \to S\) are homomorphisms for every \(x \in E\) and every \(f \in F\). We say that a coarser pair \((\sigma', \tau') \leq (\sigma, \tau)\) of group topologies is \(\omega\)-compatible if \(\omega\) remains continuous with respect to the triple \((\sigma', \tau', \nu)\). If \(\omega\) is separated (i.e., if the annihilators of \(E\) and \(F\) are both zero), then the Hausdorff property of \(\nu\) implies that every \(\omega\)-compatible pair \((\sigma', \tau')\) is necessarily Hausdorff. Following [7], we say that \(\omega\) is minimal if for every \(\omega\)-compatible pair \((\sigma', \tau') \leq (\sigma, \tau)\), we have necessarily \(\sigma' = \sigma, \tau' = \tau\).

**Lemma 2** [7, Proposition 1.10]. For every Hausdorff locally compact Abelian group \(G\), the evaluation mapping \(G^* \times G \to T\) is minimal.

Another example of a minimal biadditive mapping is the canonical duality \(E^* \times E \to R\) for a normed space \(E\).

**Proposition 3.** Let the evaluation mapping

\[\omega: (H_K(R, S), \sigma) \times (R, \tau) \to (S, \nu)\]

be minimal, and let \(\gamma' \subset \gamma\), be a coarser Hausdorff ring topology on \(M_K(R, S)\) such that \(\gamma'\) and \(\gamma\) coincide on \(H_K(R, S) \times S\). Then \(\gamma' = \gamma\).

**Proof.** Because \(\gamma'\) and \(\gamma\) agree on \(H_K(R, S) \times S\), then, in particular, the mapping

\[q: H_K(R, S) \times S \to S\]

is continuous with respect to the pair \((\gamma' |_{H_K(R, S) \times S}, \nu)\). So, we can apply Proposition 1. Then \(\gamma' |_{H_K(R, S)}, \gamma'/H_K(R, S) \times S\) is a \(\omega\)-compatible pair of group topologies. The minimality of \(\omega\) implies \(\gamma'/H_K(R, S) \times S = \tau = \gamma/H_K(R, S) \times S\). Now Merzon’s Lemma finishes the proof.

As a corollary we get

**Proposition 4.** Let the evaluation mapping \(\omega\) be minimal and let \(S\) and \(H_K(R, S)\) be compact. Then \(M_K(R, S)\) is a minimal ring.
THEOREM 5. Let $R$ be a discrete ring. Then the topological ring $M(R) = R \times R^* \times T$ is minimal. Hence, every (commutative) discrete ring is a continuous ring retract of a minimal (commutative) locally compact ring.

PROOF. By Pontryagin's Theorem, $R^*$ is compact iff $R$ is discrete. Now the minimality of $M(R)$ follows from Lemma 2 and Proposition 4. The canonical retraction $pr: M(R) \to R$ is the desired one. ■

The ring $M(R)$ from Theorem 5 is not unital. In order to «improve» this, we use a well known unitalization procedure. Let $R$ be a topological $K$-algebra. Consider a new $K$-algebra

$$R_+ = \{ r + a1_+ | r \in R, a \in K \}$$

adjoining a unit $1_+$. More precisely, $R_+$ is a topological $K$-module sum $R \oplus K$, and we identify $(r, a) = r + a1_+$. A multiplication on $R_+$ is defined in the following manner:

$$(a + a1_+)(b + b1_+) = ab + ab + \beta a + \alpha b1_+$$

where $\alpha, \beta \in K$ and $a, b \in R$. The following lemma is trivial.

**Lemma 6.** If $J$ is a (closed) ideal in $R$, then $J$ is a (closed) ideal in $R_+$ and $R_+/J = (R/J)_+$.

In the following result we use a method familiar from the theory of minimal topological groups (see, for example, [5]).

**Theorem 7.** Let $R$ be a complete $K$-algebra such that $(R, \tau)$, $(K, \sigma)$ are minimal topological rings. Then the K-unitalization $R_+$ is a minimal topological ring.

**Proof.** Denote by $\gamma$ the given product topology on $R_+$ and suppose that $\gamma' \subseteq \gamma$ is a new Hausdorff ring topology. Since $(R, \tau)$ is a minimal ring, $\gamma'|_R = \gamma|_R = \tau$. By our assumption, $(R, \tau)$ is complete. Therefore, $R$ is a closed ideal in $(R_+, \gamma')$. Consider the Hausdorff ring topology $\gamma'/R$ on $K$. Since $\gamma'/R \subseteq \gamma/R = \sigma$ and $(K, \sigma)$ is a minimal ring, then $\gamma'/R = \gamma/R$. By Merzon's Lemma we get $\gamma' = \gamma$. ■

**Corollary 8.** Let $R$ be a minimal complete ring with char $(R) = n > 0$. Then the $\mathbb{Z}_n$-unitalization $R_+$ of $R$ is a minimal ring.

**Theorem 9.** Let $R$ be a discrete ring with char $(R) = n > 0$. Then the $\mathbb{Z}_n$-unitalization $R_+$ of $R$ is a continuous ring retract of a minimal locally compact unital ring $M_+$. 
PROOF. Apply our construction for the situation $S = K = Z_n$ and consider the $Z_n$-algebra $M := M_{Z_n}(R, Z_n) = R \times H_{Z_n}(R, Z_n) \times Z_n$. Denote by $M_+$ the $Z_n$-unitalization of $M$. Since $\text{char}(R) = n > 0$, then every character $\xi : R \to T$ can actually be considered as a restricted homomorphism $R \to Z_n \subset T$ identifying $Z_n$ with the $n$-element cyclic subgroup of $T$. It is also clear that every homomorphism $R \to Z_n$ is even a morphism of $Z_n$-algebras. Therefore, $H_{Z_n}(R, Z_n)$ and $R^* = H(R, T)$ coincide algebraically. Endow $H_{Z_n}(R, Z_n)$ with the compact topology $\sigma$ of $R^*$. Eventually, the mapping $\omega : (R, \tau) \times (H_{Z_n}(R, Z_n), \sigma) \to Z_n \subset T$ is minimal, because of Lemma 2. By Proposition 4, the ring $M$ is minimal. Since $M$ is a $Z_n$-algebra, then Corollary 8 and Lemma 6 complete the proof.

COROLLARY 10. For every nonnegative integer $n$ which is not equal to 1 there exists a minimal non-totally minimal separable metrizable locally compact unital ring with $\text{char}(R) = n$.

PROOF. Fix a natural number $n \geq 2$. Let $F_i = Z_n$ for every $i \in \mathbb{N}$. Consider the topological ring product $\left( \prod_{i \in \mathbb{N}} F_i, \sigma \right)$ and the dense countable topological subring $\left( \sum_{i \in \mathbb{N}} F_i, \tau \right)$. Denote by $\tau_d$ the discrete topology on $R := \sum_{i \in \mathbb{N}} F_i$. Clearly, the $Z_n$-unitalization $R_+$ of $(R, \tau_d)$ is not a minimal ring because we can take on $R_+$ the (strictly coarser) ring topology of the $Z_n$-unitalization for $(R, \tau)$. On the other hand, by Theorem 9, the discrete non-minimal ring $R_+$ is a continuous ring retract of a minimal ring $M_+$. Eventually, $M_+$ is the desired ring.

For the case $n = 0$, consider the ring product $R \times M_+$, where $M_+$ is a minimal ring constructed for the case $n \geq 2$, and use the productivity of the class of minimal unital rings [3].

Acknowledgement. I would like to express my gratitude to D. Dikranjan for helpful comments and suggestions.

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Manoscritto pervenuto in redazione l’1 luglio 1995.