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On a Construct of Closure Spaces.

JOSEF ŠLAPAL (*)

ABSTRACT - We introduce and study a special construct of closure spaces. In particular, we show that this construct is an exponential superconstruct of the construct of finitely generated topological spaces.

It is well known that topological constructs having function spaces for all pairs of objects, i.e. being cartesian closed, are useful for applications to many branches of mathematics. But for some applications also such constructs are useful that have function spaces only for some pairs of objects. In particular, it is worthwhile to deal with such constructs $\mathbf{K}$ for which there is a full subconstruct $\mathbf{L}$ of $\mathbf{K}$ such that function spaces G^H in $\mathbf{K}$ exist whenever G is an object of $\mathbf{L}$. One construct having this property is introduced and studied in the presented paper.

It is a usual procedure to replace the construct of topological spaces and continuous maps (which is not cartesian closed) with a superconstruct of its in order to receive a construct with function spaces. This procedure will be used also in this note.

By *closure spaces* we mean the spaces that are studied in [2], i.e. pairs (X, u) where X is a set and $u: \exp X \rightarrow \exp X$ is a map (the so called *closure operation* on X) fulfilling the following three axioms: $u\emptyset = \emptyset$, $A \subseteq X \Rightarrow A \subseteq uA$, and $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$. The construct of closure spaces and continuous maps, i.e. maps $f: (X, u) \rightarrow (Y, v)$ with $f(uA) \subseteq v f(A)$ whenever $A \subseteq X$, will be denoted by *Clo*. Further, we denote by

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Clo_T the full subconstruct of Clo given by the spaces $(X, u) \in Clo$ having the following property:

if $A \subseteq X$ is a subset and $x \in uA$ a point, then there exist an ordinal $i_1 > 0$ and a sequence $(x_i \mid i < i_1) \in A^{i_1}$ such that $x_{i_0} \in u\{x_i : i < i_0\}$ for each ordinal i_0 , $0 < i_0 < i_1$, and $x \in u\{x_i : i < i_1\}$.

Finally, we denote by Clo_S the full subconstruct of Clo given by the objects $(X, u) \in Clo$ that fulfil $uuA = uA = \bigcup_{x \in A} u\{x\}$ whenever $A \subseteq X$. Then, whenever $(X, u) \in Clo_S$, the closure operation u fulfils the Kuratowski closure axioms, i.e. (X, u) is a topological space (in the usual sense of Bourbaki [1]). The objects of Clo_S are precisely the topological spaces that are called finitely generated in [6], quasi-discrete in [3] and S -spaces in [7].

It is evident that Clo_S is a (full) subconstruct of Clo_T .

EXAMPLE. 1) Let $X = \{a, b, c\}$ and define $u: \exp X \rightarrow \exp X$ as follows: $u\emptyset = \emptyset$, $u\{a\} = \{a, b\}$, $u\{b\} = \{b\}$, $u\{c\} = \{c\}$, $u\{a, b\} = u\{a, c\} = uX = X$, $u\{b, c\} = \{b, c\}$. Then $(X, u) \in Clo_T$.

2) Let ω denote the least infinite ordinal and let $(\omega + 1, u)$ be the topological space defined by $u\emptyset = \emptyset$, $uA = \omega$ whenever $A \subseteq \omega$ and $0 < \text{card } A < \omega$, and $uA = \omega + 1$ otherwise. Then $(\omega + 1, u) \in Clo_T$ and it can be easily seen that the closure operation u is additive, i.e. $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq \omega + 1$.

Let us introduce the following denotation:

If $(X, u) \in Clo$, $A \subseteq X$ and $n > 1$ is a cardinal, then we set $\tau(u, A, n) = \{x \in X : \text{there is } (x_i \mid i < n) \in X^n \text{ such that } x_{i_0} \in u\{x_i : i < i_0\} \text{ for each ordinal } i_0, 0 < i_0 < n, \text{ and there is an ordinal } i_1, 0 < i_1 < n, \text{ such that } x = x_{i_1} \text{ and } x_i \in A \text{ for every ordinal } i < i_1\}$. Obviously, the inclusion $\tau(u, A, n) \subseteq uA$ is always satisfied.

The following statement is evident:

LEMMA. Let $(X, u) \in Clo$. Then the identity map $\text{id}_X: (X, u^*) \rightarrow (X, u)$ where u^* is the closure operation on X given by $A \subseteq X \Rightarrow u^*A = \bigcup \{\tau(u, A, n) : n > 1 \text{ a cardinal}\}$ is a coreflection of (X, u) in Clo_T .

Obviously, Clo has products: if $(X, u), (Y, v) \in Clo$ are objects, then their product in Clo is the object $(X \times Y, u \times v)$ where the closure operation $u \times v$ on $X \times Y$ is given by $(u \times v)D = u \text{pr}_X(D) \times v \text{pr}_Y(D)$ for each subset $D \subseteq X \times Y$. For any pair of objects $(X, u), (Y, v) \in Clo_T$ let

us define a map $t_{u,v} : \exp(X \times Y) \rightarrow \exp(X \times Y)$ by $D \subseteq X \times Y \Rightarrow t_{u,v} D = \bigcup \{ \tau(u \times v, D, n) : n > 1 \text{ a cardinal} \}$ where $u \times v$ is the product closure operation in Clo . Then the Lemma immediately results in:

PROPOSITION. *Given objects $(X, u), (Y, v) \in Clo_T$, $(X \times Y, t_{u,v})$ is their product in Clo_T .*

If K is a construct and $G, H \in K$ a pair of its objects, then by a *power* G^H in K we understand any object of K whose underlying set is $\text{Mor}(H, G)$ (i.e. the set of all morphisms from H into G in K). If G^H is a power in a construct with finite products, then the map $e : H \times G^H \rightarrow G$ given by $e(y, f) = f(y)$ is called the *evaluation map* for G^H .

Regarding the well-known notion of an exponential subconstruct of a construct (see e.g. [8]), we introduce a notion of an exponential superconstruct of a construct:

DEFINITION (cf. [9]). If K is a construct with finite products and L is a full isomorphism closed subconstruct of K , then K is called an *exponential superconstruct* of L provided that for any two objects $G \in L, H \in K$ there exists a power G^H in K with the property that $G^H \in L$ and the pair (G^H, e) , where e is the evaluation map for G^H , is a co-universal map for G with respect to the functor $H \times - : K \rightarrow K$.

The substantial property of the exponentiability defined is the validity of the so-called first exponential law $(G^H)^K \simeq G^{H \times K}$ for the powers. Obviously, if a construct K is exponential for itself, then it is cartesian closed, i.e. the functor $H \times - : K \rightarrow K$ has a right adjoint for each object $H \in K$ (see [4],[5]).

Of course, the powers from the definition are unique up to the isomorphisms that are identity maps and thus they are unique whenever K is transportable.

In the proof of the next statement we will use the evident fact that the following two conditions are equivalent whenever $(X, u) \in Top_S$, $n > 1$ is a cardinal and $(x_i \mid i < n) \in X^n$:

- (i) $x_{i_0} \in u\{x_i : i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$,
- (ii) $x_i \in u\{x_0\}$ for each ordinal $i < n$.

THEOREM. *Clo_T is an exponential superconstruct of Clo_S .*

PROOF. By the Proposition, Clo_T has finite products. For any pair of objects $G = (X, u) \in Clo_S$ and $H = (Y, v) \in Clo_T$ put $G^H = (Z, w)$ where

$Z = \text{Mor}(H, G)$ and $w: \exp Z \rightarrow \exp Z$ is given by $C \subseteq Z \Rightarrow wC = \{g \in Z: \text{there exists } f \in C \text{ such that } g(y) \in u\{f(y)\} \text{ for each } y \in Y\}$. It is evident that $G^H \in \text{Top}_S$.

Let $e: H \times G^H \rightarrow G$ be the evaluation map for G^H , let $D \subseteq Y \times Z$ and $(y, g) \in t_{v,w} D$. Then there is a cardinal $n > 1$ such that $(y, g) \in \tau(v \times \times w, D, n)$. Consequently, there is $((y_i, g_i) | i < n) \in (Y \times Z)^n$ such that $y_{i_0} \in v\{y_i: i < i_0\}$ and $g_{i_0} \in w\{g_i: i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$, and there is an ordinal $i_1, 0 < i_1 < n$, such that $(y, g) = (y_{i_1}, g_{i_1})$ and $(y_i, g_i) \in D$ for every ordinal $i < i_1$. As $(Z, w) \in \text{Top}_S$, we have $g_i \in w\{g_0\}$ for each ordinal $i < n$. Since $g_0 \in \text{Mor}(H, G)$, it holds $g_0(y_{i_0}) \in u\{g_0(y_i): i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$. Hence $g_0(y_i) \in u\{g_0(y_0)\}$ whenever $i < n$. Now, for any ordinal $i < n$ we have $e(y_i, g_i) = g_i(y_i) \in u\{g_0(y_i)\} \subseteq u\{g_0(y_0)\} = u\{e(y_0, g_0)\}$. Especially, $e(y, g) = e(y_{i_1}, g_{i_1}) \in u\{e(y_0, g_0)\} \subseteq ue(D)$. We have shown that $e: H \times G^H \rightarrow G$ is a continuous map.

Let $K = (P, s) \in \text{Clo}_T$ be an object and let $f: H \times K \rightarrow G$ be a continuous map. Let $f^*: P \rightarrow X^Y$ be the map given by $f^*(p)(y) = f(y, p)$. Let $p \in P$, $B \subseteq Y$ and $y \in vB$. Then there is a cardinal $n > 1$ and $(y_i | i < n) \in Y^n$ such that $y_{i_0} \in v\{y_i: i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$, and there is an ordinal $i_1, 0 < i_1 < n$, such that $y = y_{i_1}$ and $y_i \in B$ for every ordinal $i < i_1$. Consequently, $((y_i, p) | i < n) \in (Y \times P)^n$, $(y_{i_0}, p) \in (v \times s)\{(y_i, p): i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$, $(y, p) = (y_{i_1}, p)$ and $(y_i, p) \in B \times \{p\}$ for every ordinal $i < i_1$. Therefore $(y, p) \in t_{v,s}(B \times \{p\})$. As $f: H \times K \rightarrow G$ is continuous, we get $f(y_{i_0}, p) = f^*(p)(y_{i_0}) \in u\{f(y_i, p): i < i_0\} = u\{f^*(p)(y_i): i < i_0\}$ whenever $0 < i_0 < n$. Since $f^*(p)(y) = f^*(p)(y_{i_1})$ and $f^*(p)(y_i) \in f^*(p)(B)$ for every ordinal $i < i_1$, it follows $f^*(p)(y) \in uf^*(p)(B)$. Therefore $f^*(p): H \rightarrow G$ is continuous.

Let $D \subseteq P$ and $p \in sD$. Then there is a cardinal $n > 1$ and $(p_i | i < n) \in P^n$ such that $p_{i_0} \in s\{p_i: i < i_0\}$ for each ordinal $i_0, 0 < i_0 < n$, and there is an ordinal $i_1, 0 < i_1 < n$, such that $p = p_{i_1}$ and $p_i \in D$ for every ordinal $i < i_1$. Consequently, for any $y \in Y$ we have $((y, p_i) | i < n) \in (Y \times P)^n$, $(y, p_{i_0}) \in (v \times s)\{(y, p_i): i < i_0\}$ whenever $0 < i_0 < n$, $(y, p) = (y, p_{i_1})$ and $(y, p_i) \in \{y\} \times D$ for every ordinal $i < i_1$. Thus, for any $y \in Y$ we get $(f(y, p_i) | i < n) \in X^n$ and $f(y, p_{i_0}) \in u\{f(y, p_i): i < i_0\}$ whenever $0 < i_0 < n$. As $(X, u) \in \text{Clo}_S$, there holds $f(y, p_i) = f^*(p_i)(y) \in u\{f(y, p_0)\} = u\{f^*(p_0)(y)\}$ for every $y \in Y$ and every ordinal $i < n$. Especially, we have $f^*(p)(y) = f^*(p_{i_1})(y) \in u\{f^*(p_0)(y)\}$ for each $y \in Y$. Since $f^*(p_0) \in f^*(D)$, it follows $f^*(p) \in uf^*(D)$. We have shown that $f^*: K \rightarrow G^H$ is a continuous map, and it is obvious that f^* is the only continuous map fulfilling $e \circ (\text{id}_Y \times f^*) = f$. In other words, we have proved that the pair (G^H, e) is a

co-universal map for G with respect to the functor $H \times - : Clo_T \rightarrow Clo_T$. This results in the statement. ■

REMARK. *a)* It is well known that Clo_S is an exponential superconstruct of itself, i.e. a cartesian closed construct (as it is isomorphic with the cartesian closed construct of preordered sets — see [3]). The Theorem 2 can be considered as a generalization of this fact.

b) If a space $(X, u) \in Clo_T$ fulfils $A \subseteq X \Rightarrow uuA = uA$, then $(X, u) \in Clo_S$. On the other hand, the full subcategory of Clo_T given by those objects $(X, u) \in Clo_T$ that satisfy $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ is not finitely productive in Clo_T . Each of these two facts means no new results could be attained by restricting our considerations to topological spaces.

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