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Localization of Differential Operators and of Higher Order de Rham Complexes.

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ABSTRACT - We study localization properties of some algebraic differential complexes associated to an arbitrary commutative algebra which are higher order (in the sense of differential operators) analogues of the ordinary de Rham complex. These results should be considered, in the spirit of [11], as preliminaries to the study of the cohomological invariants provided by these higher de Rham complexes for singular varieties.

Notations and Conventions.

K: a commutative ring with unit;

A: a commutative, associative K-algebra with unit;

A-**Mod**: the category of A-modules;

K-**Mod**: the category of K-modules;

$\mathbf{DIFF}_A$: the category whose objects are A-modules and whose morphisms are differential operators (Section 1), of any (finite) order, between them;

Ens: the category of sets;

$[\boldsymbol{C}, \boldsymbol{C}]$: the category of functors $\boldsymbol{C} \to \boldsymbol{C}$, $\boldsymbol{C}$ being any category;

$\mathrm{Ob}(\boldsymbol{C})$: the objects of $\boldsymbol{C}$, $\boldsymbol{C}$ being any category; we will write $C \in \mathrm{Ob}(\boldsymbol{C})$ to mean that C is an object of $\boldsymbol{C}$;

(A, A)-**BiMod**: the category of (A, A)-bimodules, whose objects are *ordered couples* (P, P^+) of A-modules and whose morphisms are the usual morphisms of bimodules;

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$\boldsymbol{K}(A\text{-}\mathbf{Mod})$ (resp. $\boldsymbol{K}(K\text{-}\mathbf{Mod})$, resp. $\boldsymbol{K}(\mathbf{DIFF}_A)$): the category of complexes in A-**Mod** (resp. K-**Mod**, resp. $\mathbf{DIFF}_A$);

If $\mathfrak{D}$ is a full subcategory of A-**Mod**, a functor $T\colon \mathfrak{D} \to \mathfrak{D}$ will be said *strictly representable* in $\mathfrak{D}$ if it exists $\tau \in \mathrm{Ob}(\mathfrak{D})$ and a functorial isomorphism $T \simeq \mathrm{Hom}_A(\tau, \cdot)$ in $[\mathfrak{D}, \mathfrak{D}]$;

If T_1 and T_2 are strictly representable functors $\mathfrak{D} \to \mathfrak{D}$ with representative objects τ_1 and τ_2, respectively, and $\varphi\colon T_1 \to T_2$ is a morphism in $[\mathfrak{D}, \mathfrak{D}]$ then its *dual* (*representative*) is the morphism $\varphi^{\vee} \doteq \varphi(\tau_1)(\mathrm{id}_{\tau_1}) \in \mathrm{Hom}_{\mathfrak{D}}(\tau_2, \tau_1)$;

(A, A)-$\mathbf{BiMod}_{\mathfrak{D}}$ (resp. $\boldsymbol{K}(\mathbf{DIFF}_{A, \mathfrak{D}})$) will be the subcategory of (A, A)-**BiMod** whose objects are couples of objects in $\mathfrak{D}$ (resp. the subcategory of $\boldsymbol{K}(\mathbf{DIFF}_A)$ whose objects are complex of objects in $\mathfrak{D}$);

A sequence $T_1 \to T_2 \to T_3$ of functors $T_i\colon \mathfrak{D} \to \mathfrak{D}$, $i = 1, 2, 3$, (and functorial morphisms) with $\mathfrak{D}$ an abelian subcategory of A-**Mod**, will be said *exact in* $[\mathfrak{D}, \mathfrak{D}]$ if it is exact in $\mathfrak{D}$ when applied to any object of $\mathfrak{D}$.

1. – Introduction.

In [14] A. M. Vinogradov associated to any commutative algebra A and any differentially closed (see Section 1) subcategory $\mathfrak{D}$ of A-**Mod**, some natural algebraic differential complexes that generalize the well known de Rham and Spencer's ones (see for example [7] or [1]). One of their features is the fact that their differentials may be differential operators of arbitrary, finite, order. Recently, in [11] (see also [10] or the shorter version [12]), it has been proved that under appropriate smoothness assumptions on the ambient subcategory of A-**Mod** (satisfied, with appropriate choices of $\mathfrak{D}$, for example, by smooth real manifolds of finite dimension and by regular affine varieties over algebraically closed fields of zero characteristic), the «higher» de Rham complexes are all quasi-isomorphic to the usual (differential-geometric and algebraic) one. This result suggests to look at the «tower» of all these higher de Rham cohomologies in the singular case, to understand which kind of informations and invariants they yield. This article is intended as a preliminary step in this direction. In fact, we prove a rather elementary and intuitive result: the higher de Rham complexes (of the whole algebra A) «localized» with respect to an arbitrary multiplicative part S are isomorphic to the higher de Rham complexes of the localized algebra A_S. Therefore these cohomologies can be associated uniquely [1] to a given singularity. This result also shows that the constructions of all these higher complexes fit in the framework of sheaf theory.

We address the reader to [2], where similar questions about singularities are considered and localizations' results for ordinary Spencer cohomology are stated.

2. – Definitions.

We briefly recall the relevant definitions from [14] and [11] (see also [10]). Let K be a commutative ring with unit and A a commutative, associative unitary K-algebra.

If P and Q are A-modules and $a \in A$ we define:

$$\delta_a : \mathrm{Hom}_K(P, Q) \to \mathrm{Hom}_K(P, Q),$$

$$\Phi \mapsto \{\delta_a \Phi : p \mapsto \Phi(ap) - a\Phi(p)\}, \qquad p \in P,$$

(where juxtaposition indicates both A-module multiplications in P and Q). For each $a \in A$, δ_a is a morphism of K-modules, and A being commutative, we get:

$$\delta_{a_1} \circ \delta_{a_2} = \delta_{a_2} \circ \delta_{a_1}, \qquad \forall a_1, a_2 \in A.$$

DEFINITION 2.1. *A (K-)differential operator (DO) of order* $\leqslant s$ *from the A-module P to the A-module Q, is an element* $\Delta \in \mathrm{Hom}_K(P, Q)$ *such that:*

$$[\delta_{a_0} \circ \delta_{a_1} \circ \ldots \circ \delta_{a_s}](\Delta) = 0, \qquad \forall \{a_0, a_1, \ldots, a_s\} \subset A.$$

The *set* $\mathit{Diff}_k(P, Q)$ of differential operators of order $\leqslant k$ from P to Q comes equipped with two different A-module structures:

(i) $(\mathit{Diff}_k(P, Q), \tau) \doteq \mathit{Diff}_k(P, Q)$ (left),

$$\tau : A \times \mathit{Diff}_k(P, Q) \to \mathit{Diff}_k(P, Q) : (a, \Delta) \mapsto \tau(a, \Delta) : p \mapsto a\Delta(p);$$

(ii) $(\mathit{Diff}_k(P, Q), \tau^+) \doteq \mathit{Diff}_k^+(P, Q)$ (right),

$$\tau^+ : A \times \mathit{Diff}_k(P, Q) \to \mathit{Diff}_k(P, Q) : (a, \Delta) \mapsto \tau^+(a, \Delta) : p \mapsto \Delta(ap).$$

[1] Remember that (higher) de Rham cohomologies are K-modules and not A-modules, if A is a K-algebra, so we cannot directly localize them with respect to a multiplicative part of A.

We will often write, to be concise, $\tau(a, \Delta) \equiv a\Delta$ and $\tau^+(a, \Delta) \equiv$ $\equiv a^+\Delta$. As easily seen, $(\mathit{Diff}_k(P, Q), (\tau, \tau^+)) \doteq \mathit{Diff}_k^{(+)}(P, Q)$ turns out to be an (A, A)-bimodule.

REMARK 2.2. *Since*

$$\delta_{a_0}(\Delta) \equiv 0 \Leftrightarrow \Delta(a_0 p) = a_0 \Delta p, \qquad \forall a_0 \in A, \quad \forall p \in P$$

we have $\mathit{Diff}_0(P, Q) \equiv \mathrm{Hom}_A(P, Q)$ *in* **Ens** *and also* $\mathrm{Hom}_A(P, Q) \simeq$ $\simeq \mathit{Diff}_0(P, Q) \simeq \mathit{Diff}_0^+(P, Q)$ *in* A-**Mod**.

The obvious inclusions (of sets):

$$\mathit{Diff}_k(P, Q) \hookrightarrow \mathit{Diff}_l(P, Q), \qquad k \leqslant l$$

induce monomorphisms of (A, A)-bimodules:

$$\mathit{Diff}_k^{(+)}(P, Q) \hookrightarrow \mathit{Diff}_l^{(+)}(P, Q), \qquad k \leqslant l;$$

which form a direct system (over N) in (A, A)-**BiMod** (the category of (A, A)-bimodules):

$$\mathit{Diff}_0^{(+)}(P, Q) \hookrightarrow \mathit{Diff}_1^{(+)}(P, Q) \hookrightarrow \ldots \hookrightarrow \mathit{Diff}_n^{(+)}(P, Q) \hookrightarrow \ldots$$

whose direct limit is the (A, A)-bimodule:

$$\mathrm{dir}\lim_{n \geqslant 0} \mathit{Diff}_n^{(+)}(P, Q) \equiv \bigcup_{n \geqslant 0} \mathit{Diff}_n^{(+)}(P, Q) \doteq \mathit{Diff}^{(+)}(P, Q)$$

filtered by $\{\mathit{Diff}_n^{(+)}(P, Q)\}_{n \geqslant 0}$.

Using the two canonical forgetful functors:

$$\mathbf{Pr}: (A, A) - Bi\mathbf{Mod} \to A\text{-}\mathbf{Mod}: (P, P^+) \mapsto P,$$
$$\mathbf{Pr}^+: (A, A) - Bi\mathbf{Mod} \to A\text{-}\mathbf{Mod}: (P, P^+) \mapsto P^+,$$

we get the two filtered A-modules ($\mathbf{Pr}$ and $\mathbf{Pr}^+$ commutes with direct limits):

$$\mathbf{Pr}(\mathit{Diff}^{(+)}(P, Q)) =$$
$$= \mathit{Diff}(P, Q) \doteq \mathrm{dir}\lim_{n \geqslant 0} \mathit{Diff}_n(P, Q) \equiv \bigcup_{n \geqslant 0} \mathit{Diff}_n(P, Q),$$

$$\mathbf{Pr}^+(\mathit{Diff}^{(+)}(P, Q)) =$$
$$= \mathit{Diff}^+(P, Q) \doteq \mathrm{dir}\lim_{n \geqslant 0} \mathit{Diff}_n^+(P, Q) \equiv \bigcup_{n \geqslant 0} \mathit{Diff}_n^+(P, Q)$$

(where direct limits are to be understood, now, in A-**Mod**).

Putting $Diff_k(A, Q) \doteq Diff_k Q$ and $Diff_k^+(A, Q) \doteq Diff_k^+ Q$, we obtain the functors (2):

$$Diff_k : Q \mapsto Diff_k Q\,,$$

$$Diff_k^+ : Q \mapsto Diff_k^+ Q\,,$$

from A-**Mod** to itself. Remark 2.2 implies $Diff_0^+ = Diff_0 = \mathrm{Id}_{A\text{-}\mathbf{Mod}}$.

Defining $D_{(k)}(Q) \doteq \{\Delta \in Diff_k Q \mid \Delta(1) = 0\}$, which is an A-submodule of $Diff_k Q$ but not of $Diff_k^+ Q$, we get a functor $D_{(k)} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$, together with the short exact sequence:

$$(1) \qquad 0 \longrightarrow D_{(k)} \xrightarrow{i_k} Diff_k \xrightarrow{p_k} \mathrm{Id}_{A\text{-}\mathbf{Mod}} \longrightarrow 0$$

in $[A\text{-}\mathbf{Mod}, A\text{-}\mathbf{Mod}]$, where i_k is the obvious functorial inclusion and p_k is defined by:

$$p_k(Q) : Diff_k Q \to Q : \Delta \mapsto \Delta(1)\,, \qquad \Delta \in Diff_k Q$$

for any A-module Q. The functorial monomorphism $\mathrm{Id}_{A\text{-}\mathbf{Mod}} \equiv \equiv Diff_0 \hookrightarrow Diff_k$ splits (1), so that $Diff_k = D_{(k)} \oplus \mathrm{Id}_{A\text{-}\mathbf{Mod}}$. $D_{(1)}(Q)$ coincides with the A-module $\mathrm{Der}_{A/K}(Q)$ of all Q-valued K-linear derivations on A (see [3], for example).

Let P and P^+ be the left and right A-modules corresponding to an (A, A)-bimodule $P^{(+)} \equiv (P, P^+)$ (P and P^+ coincide as K-modules, hence as sets). Let's denote by $Diff_k^\bullet(P^+)$ (resp. $D_{(k)}^\bullet(P^+)$) the A-module which coincides with $Diff_k(P^+)$ (resp. $D_{(k)}(P^+)$) as a K-module and whose A-module structure is inherited by that of P (and not of P^+) (3). For an A-submodule $S \subset P$ we define submodules:

$$Diff_k^\bullet(S \subset P^+) \doteq \{\Delta \in Diff_k^\bullet(P^+) \mid \Delta(A) \subset S\} \underset{A\text{-}\mathbf{Mod}}{\subset} Diff_k^\bullet(P^+)\,,$$

$$D_{(k)}^\bullet(S \subset P^+) \doteq \{\Delta \in D_{(k)}^\bullet(P^+) \mid \Delta(A) \subset S\} \underset{A\text{-}\mathbf{Mod}}{\subset} D_{(k)}^\bullet(P^+)\,.$$

DEFINITION 2.3. *Let* $\sigma \equiv (\sigma_1, \sigma_2, \dots \sigma_n, \dots) \in N_+^\infty \doteq \mathrm{inv}\lim_{n>0} N_+^n$. *Writing* $\sigma(n)$ *for* $(\sigma_1, \dots, \sigma_n)$, *we define inductively, the functors* $D_{\sigma(n)} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$, *by:*

$$D_{\sigma(1)} \doteq D_{(\sigma_1)}\,,$$

$$D_{\sigma(n)} : P \mapsto D_{(\sigma_1)}^\bullet(D_{(\sigma_2, \dots, \sigma_n)}(P) \subset Diff_{\sigma_2, \dots, \sigma_n}^+(P))\,,$$

(2) We omit the obvious rules on morphisms.

(3) These A-module structures are well defined due to the fact that $(P, P^+) \equiv P^{(+)}$ is a bimodule.

where, to simplify the notation, we put $Diff^+_{\sigma_2, \dots, \sigma_n}$ *in place of* $Diff^+_{\sigma_2} \circ \dots \circ Diff^+_{\sigma_n}$.

For each $\sigma \in N^\infty_+$ and each $n \in N_+$, we have an exact sequence in [A-**Mod**, A-**Mod**]:

$$(2) \qquad 0 \longrightarrow D_{\sigma(n)} \xrightarrow{I_{\sigma(n)}} D^\bullet_{\sigma(n-1)} \circ Diff^+_{\sigma_n} \xrightarrow{\pi_{\sigma(n)}} D_{(\sigma_1, \dots, \sigma_{n-2}, \sigma_{n-1} + \sigma_n)}$$

where $I_{\sigma(n)}$ is the natural inclusion and $\pi_{\sigma(n)}$ arises from the «glueing» functorial morphism

$$g_{\sigma_{n-1}, \sigma_n} : Diff^+_{\sigma_{n-1}} \circ Diff^+_{\sigma_n} \to Diff^+_{\sigma_{n-1} + \sigma_n},$$

$$[g_{\sigma_{n-1}, \sigma_n}(P)](\Delta)(a) \doteq [\Delta(a)](1),$$

$$\Delta \in Diff^+_{\sigma_{n-1}}(Diff^+_{\sigma_n}(P)), \quad a \in A, \quad P \in \mathrm{Ob}(A\text{-}\mathbf{Mod}).$$

Let $\mathfrak{D}$ be a *differentially closed subcategory* of A-**Mod** (4) ([11]) and $P \in \mathrm{Ob}(\mathfrak{D})$: the functor $\mathfrak{D} \to \mathfrak{D} : Q \mapsto Diff_k(P, Q)$ is then (strictly) represented by $\boldsymbol{J}^k_{\mathfrak{D}}(P) \in \mathrm{Ob}(\mathfrak{D})$ and there is a universal DO $j^{\mathfrak{D}}_k(P) \in Diff_k(P, \boldsymbol{J}^k_{\mathfrak{D}}(P))$ such that the map

$$h \mapsto h \circ j^{\mathfrak{D}}_k(P)$$

establishes an A-module isomorphism between $\mathrm{Hom}_A(\boldsymbol{J}^k_{\mathfrak{D}}(P), Q)$ and $Diff_k(P, Q)$, natural in Q. $\boldsymbol{J}^k_{\mathfrak{D}}(P)$ is called the *k-jet module of P* in $\mathfrak{D}$ (or, in Grothendieck's terminology, [5], the module of *principal parts* of order k of P); note that $\boldsymbol{J}^k_{\mathfrak{D}} \doteq \boldsymbol{J}^k_{\mathfrak{D}}(A)$ has also another A-module structure:

$$A \times \boldsymbol{J}^k_{\mathfrak{D}} \to \boldsymbol{J}^k_{\mathfrak{D}} : (a_0, a(j^{\mathfrak{D}}_k(A))(b)) \mapsto a(j^{\mathfrak{D}}_k(A))(a_0 b), \qquad a_0, a, b \in A,$$

which is denoted by $\boldsymbol{J}^k_{\mathfrak{D}, +}$ and makes $(\boldsymbol{J}^k_{\mathfrak{D}}, \boldsymbol{J}^k_{\mathfrak{D}, +})$ into an object of (A, A)-$\mathbf{BiMod}_{\mathfrak{D}}$.

The (strictly) representative objects $\boldsymbol{A}^{\sigma(n)}_{\mathfrak{D}}$ of the functors $D_{\sigma(n)}$ in $\mathfrak{D}$ are likewise defined ($\mathfrak{D}$ being differentially closed) by

$$D_{\sigma(n)} \simeq \mathrm{Hom}_A(\boldsymbol{A}^{\sigma(n)}_{\mathfrak{D}}, \cdot)$$

in $[\mathfrak{D}, \mathfrak{D}]$ and are higher order analogues of the standard modules of differential forms ([14], [10]). We also put, for the sake of uniformity,

(4) This means that $\mathfrak{D}$ is full, abelian and all the differential functors A-**Mod** $\to$ A-**Mod**, when restricted to $\mathfrak{D}$, have values and are strictly representable in $\mathfrak{D}$. A-**Mod** is itself differentially closed.

$\boldsymbol{\Lambda}_{\mathfrak{D}}^{\emptyset} \doteq A$. $\mathfrak{D}$ is called *smooth* if $\boldsymbol{\Lambda}_{\mathfrak{D}}^{(1)}$ is a projective A-module of finite type.

EXAMPLE 2.4. (i) *If* $\mathfrak{D} = A$-**Mod** *and* $\sigma(n) = (1, \dots, 1)$ *(n times) then* $\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)}$ *coincides with* $\Omega^n_{A/K}$ ([4] *and* [9]) *and* $\boldsymbol{\Lambda}_{\mathfrak{D}}^{(k)} \simeq I/I^{k+1}$, *I being the kernel of the multiplication* $A \otimes_K A \to A$;

(ii) *If* $K = \boldsymbol{R}$ *(the field of real numbers),* $A = C^\infty(M; \boldsymbol{R})$ *(the algebra of real valued smooth functions on a differentiable* (5) *manifold* M*),* $\mathfrak{D}$ *is the category of geometric* (6) *A-modules and* $\sigma(n) = (1, \dots, 1)$, *then* $\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)}$ *is the* $C^\infty(M; \boldsymbol{R})$*-module of n-th order differential forms on M. Note however that if* σ *is arbitrary, we may have* $\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)} \neq (0)$ *even if* $n > \dim_{\boldsymbol{R}} M$;

(iii) *If A is noetherian (resp. complete local noetherian) then* $\mathfrak{D} = A$-$\mathbf{Mod}_N$, *the subcategory of noetherian A-modules (resp.* $\mathfrak{D} = A$-$\mathbf{Mod}_{c.s.}$, *the subcategory of complete separable A-modules) is differentially closed* ([2]).

REMARK 2.5. *If A is the affine algebra of a regular algebraic variety over a characteristic zero field (resp. the algebra of Example* 2.4 (ii)) *then* $\mathfrak{D} = A$-**Mod** *(resp* $\mathfrak{D}$ *of Example* 2.4 (ii)) *is smooth* (7).

Now we can associate to any $\sigma \in \boldsymbol{N}_+^\infty$ a de Rham-like complex of differential operators $\boldsymbol{dR}_\sigma(\mathfrak{D}) \in \boldsymbol{K}(\mathbf{DIFF}_{A,\mathfrak{D}})$ as follows:

$$(3) \qquad \boldsymbol{dR}_\sigma(\mathfrak{D})\colon 0 \to A \xrightarrow{d^{\mathfrak{D}}_{\sigma(1)}} \boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(1)} \to \dots \to \boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)} \xrightarrow{d^{\mathfrak{D}}_{\sigma(n+1)}} \boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n+1)} \to \dots$$

with $d^{\mathfrak{D}}_{\sigma(n+1)} \doteq I^\vee_{\sigma(n+1)} \circ j^{\mathfrak{D}}_{\sigma(n+1)}(\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)})$, $I^\vee_{\sigma(n+1)}\colon \boldsymbol{J}_{\mathfrak{D}}^{\sigma_{n+1}}(\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)}) \to \boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n+1)}$ being the dual-representative of $I_{\sigma(n+1)}$ in (2). The «higher» differential $d^{\mathfrak{D}}_{\sigma(n+1)}$ is a differential operator of order $\leq \sigma_{n+1}$ and $\boldsymbol{dR}_\sigma(\mathfrak{D})$ is called the *higher de Rham complex of type* σ *in* $\mathfrak{D}$.

In the situations of the above examples, (3) coincides with the canonical «algebraic» ((i)) and «differential geometric» ((ii)) de Rham complex, respectively. We emphasize that the complexes $\boldsymbol{dR}_\sigma(\mathfrak{D})$, $\mathfrak{D}$ being the category of geometric $C^\infty(M; \boldsymbol{R})$-modules, are *natural* in the category of smooth manifolds.

(5) Our differentiable manifolds are Hausdorff and with a countable basis.

(6) A $C^\infty(M; \boldsymbol{R})$-module P is called *geometric* if each of its elements is uniquely defined by its values on the points of $\tilde{M} \doteq \operatorname{Spec}(C^\infty(M; \boldsymbol{R}))$ i.e. if $\bigcap_{\mathfrak{p} \in \tilde{M}} \mathfrak{p}P = (0)$, see Section 5.

(7) For the algebraic part, see [6] or [9].

We also mention that the modules $\boldsymbol{\Lambda}_{\mathfrak{D}}^{(n)}$, $n \geq 1$, may be used to define *n-th order connections* on A-modules, in exactly the same way as it is usual with $n = 1$, both in the algebraic and in the differential geometric context.

In [11] (see also [12] and [10]) it is proved that if $\mathfrak{D}$ is smooth then all the complexes $\boldsymbol{dR}_\sigma(\mathfrak{D})$ are *quasi-isomorphic*: this is the case (see Remark 2.5), for example, of a regular affine algebraic variety over a field of characteristic zero with $\mathfrak{D} = A$-**Mod**, A being the corresponding affine algebra, or of a differentiable manifold M of finite dimension with $\mathfrak{D} = A\text{-}\mathbf{Mod}_{\text{geom}}$, $A = C^\infty(M; \boldsymbol{R})$.

REMARK 2.6. *We give here three equivalent descriptions of differential operators between (strict) representative objects.*

*We work in a fixed differentially closed subcategory $\mathfrak{D}$ of A-**Mod**; all representative objects will be understood in $\mathfrak{D}$. Let F_1 and F_2 be strict representative object of the functors $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Suppose that $\mathcal{F}_1$ has an associated functor* (8) *$\mathcal{F}_1^\bullet$, with domain $(A, A) - \mathbf{BiMod}_{\mathfrak{D}}$, such that $\mathcal{F}_1^\bullet \circ \mathit{Diff}_k^{(+)}$ is strictly representable by $\boldsymbol{J}^k(F_1)$: this is the case, for example, of $\mathcal{F}_1 = D_{\sigma(n)}$ or Diff_n. Let*

$$\Delta : F_1 \to F_2 \tag{4}$$

*be a DO of order $\leq k$. Then, there exists a unique A-**Mod**-morphism*

$$\varphi_\Delta : \boldsymbol{J}^k(F_1) \to F_2 \tag{5}$$

which represents Δ by duality. Since $\boldsymbol{J}^k(F_1)$ is a representative object of $\mathcal{F}_1^\bullet \circ \mathit{Diff}_k^{(+)}$, φ_Δ gives a unique morphism in $[\mathfrak{D}, \mathfrak{D}]$:

$$\varphi^\Delta : \mathcal{F}_2 \to \mathcal{F}_1^\bullet \circ \mathit{Diff}_k^+ \,. \tag{6}$$

Formulas (4), (5) *and* (6) *give us three different descriptions of a DO between (strict) representative objects. Formula* (6) *allows us to identify it with a functorial morphism which, as a rule, may be established in a straightforward way and can, then, be used to get a natural DO using* (4). *The following examples show this procedure at work in two canonical cases; we assume for simplicity* $\mathfrak{D} = A$-**Mod**

(8) For a more rigorous statement, see [11].

(i) Higher de Rham differential $d_{\sigma(n)}$.

If $\mathcal{F}_2 \doteq D_{\sigma(n)}$, $\mathcal{F}_1 \doteq D_{\sigma(n-1)}$ *and* $k = \sigma_n$, *we can take as* (6) *the natural inclusion*

$$D_{\sigma(n)} \hookrightarrow D^{\bullet}_{\sigma(n-1)} \circ \mathit{Diff}^{+}_{\sigma_n} ;$$

(4) *is then* $d_{\sigma(n)} \colon \boldsymbol{\Lambda}^{\sigma(n-1)} \to \boldsymbol{\Lambda}^{\sigma(n)}$.

(ii) «Absolute» jet-operator j_k.

In this almost tautological case, $\mathcal{F}_1 \doteq \mathrm{Hom}_A(A, \cdot)$ *and* $\mathcal{F}_2 \doteq \mathit{Diff}_k \equiv \equiv \mathrm{Hom}^{\bullet}_A(A, \cdot) \circ \mathit{Diff}^{+}_k$; *if we start from the identity*

$$\mathrm{Id} \colon \mathrm{Hom}^{\bullet}_A(A, \cdot) \circ \mathit{Diff}^{+}_k \equiv \mathit{Diff}_k \to \mathrm{Hom}^{\bullet}_A(A, \cdot) \circ \mathit{Diff}^{+}_k \equiv \mathit{Diff}_k$$

then (4) *becomes* $j_k \colon A \to \boldsymbol{J}^k$.

The A-modules $\boldsymbol{\Lambda}^{\sigma(n)}_{\mathfrak{D}}$ are generated by elements $d_{\sigma(n)}(a_1 d_{\sigma(n-1)} \cdot \cdot (a_2 \ldots d_{\sigma(1)}(a_n)\ldots))$, $a_1, a_2, \ldots, a_n \in A$ (reference to $\mathfrak{D}$ will be omitted, unless it will be necessary).

EXAMPLE 2.7. *If* $A = K[x_1, \ldots, x_n]$, K *being any commutative ring, and* $q > 0$, *then* $\boldsymbol{\Lambda}^{(q)}_{A\text{-Mod}} \simeq I/I^{q+1}$ *is a free A-module on the set of monomials* ($\underline{n} \doteq \{1, \ldots, n\} \subset \boldsymbol{N}$):

$$\{[\hat{d}(x_{i_1})], [\hat{d}(x_{j_1}) \cdot \hat{d}(x_{j_2})], \ldots, [\hat{d}(x_{r_1}) \ldots \hat{d}(x_{r_q})] \mid i_1, \ldots, r_q \in \underline{n},$$
$$j_1 \leqslant j_2, \ldots, r_1 \leqslant r_2 \leqslant \ldots \leqslant r_q\}$$

where $\hat{d} \colon A \to I \colon a \mapsto 1 \otimes a - a \otimes 1$ *and* $[\xi]$ *denotes the class modulo* I^{q+1} *of an element* ξ *of* I. *Moreover, setting*

$$\varepsilon_{i_1} \doteq [\hat{d}(x_{i_1})], \ldots, \varepsilon_{r_1, \ldots, r_q} \doteq [\hat{d}(x_{r_1}) \ldots \hat{d}(x_{r_q})],$$

we have for any $f \in A$:

$$d_{(q)}(f) = \sum \nabla_{i_1}(f)\, \varepsilon_{i_1} + \ldots + \sum \nabla_{r_1, \ldots, r_q}(f)\, \varepsilon_{r_1, \ldots, r_q} \tag{7}$$

where the elements $\nabla_{i_1}(f), \ldots, \nabla_{r_1, \ldots, r_q}(f) \in A$ *are defined by the following identity:*

$$f(x_1 + t_1, \ldots, x_n + t_n) - f(x_1, \ldots. x_n) =$$
$$= \sum \nabla_{i_1}(f)\, t_{i_1} + \ldots + \sum \nabla_{r_1, \ldots, r_q}(f) t_{r_1} \ldots t_{r_q}$$

(for example, if $n = 1$, *we have* $\nabla_{\underbrace{1, \ldots, 1}_{i \text{ times}}}(x^s) = \binom{s}{i} x^{s-i}$, *and* $\nabla_{\underbrace{1, \ldots, 1}_{i \text{ times}}}$ *is a*

derivation of order $\leqslant i$). $\boldsymbol{\Lambda}^{(q)}$ *is also free over the set:*

$$\{d_{(q)}(x_{i_1}),\ d_{(q)}(x_{j_1}x_{j_2}),\ \dots,\ d_{(q)}(x_{r_1}\dots x_{r_q})\,|\,i_1,\dots,r_q\in\underline{n},$$

$$j_1\leqslant j_2,\dots,r_1\leqslant r_2\leqslant\dots\leqslant r_q\}$$

and there is a more complicated formula analogous to (7).

If σ, $\tau\in\boldsymbol{N}_+^\infty$ with $\sigma\geqslant\tau$ (i.e. $\sigma_i\geqslant\tau_i$, $\forall i\geqslant 1$), then we have a sequence of monomorphisms in $[\mathfrak{D},\mathfrak{D}]$, $D_{\tau(n)}\hookrightarrow D_{\sigma(n)}$ (since a DO of order $\leqslant k$ is also a DO of order $\leqslant k'$, $\forall k'\geqslant k$); this induces a sequence of $\mathfrak{D}$-epimorphisms $\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)}\to\boldsymbol{\Lambda}_{\mathfrak{D}}^{\tau(n)}$, on representatives. All these epimorphisms, $\forall n>0$, commutes with higher de Rham differentials and so define a morphism in $\boldsymbol{K}(\mathbf{DIFF}_{A,\mathfrak{D}})$

$$\boldsymbol{dR}_\sigma(\mathfrak{D})\to\boldsymbol{dR}_\tau(\mathfrak{D}) \tag{8}$$

(if $\sigma\geqslant\tau$). We may then consider the ($\mathfrak{D}$-epimorphic) inverse system $\{\boldsymbol{dR}_\sigma(\mathfrak{D})\}_{\sigma\in\boldsymbol{N}_+^\infty}$ and give the following:

DEFINITION 2.8. *The* infinitely prolonged (*or, simply*, infinite) de Rham complex *of the K-algebra A in* $\mathfrak{D}$, *is the complex in* $\boldsymbol{K}(\mathbf{DIFF}_{A,\mathfrak{D}})$

$$\begin{cases}\boldsymbol{dR}_\infty(\mathfrak{D};A)\doteq\operatorname{inv}\lim\limits_{\sigma\in\boldsymbol{N}_+^\infty}\boldsymbol{dR}_\sigma(\mathfrak{D}),\\[2ex] \boldsymbol{dR}_\infty(\mathfrak{D};A):0\to A\xrightarrow{d_{(\infty)}}\boldsymbol{\Lambda}_{\mathfrak{D}}^{(\infty)}\xrightarrow{d_{(\infty,\infty)}}\boldsymbol{\Lambda}_{\mathfrak{D}}^{(\infty,\infty)_2}\to\dots\to\boldsymbol{\Lambda}_{\mathfrak{D}}^{(\infty,\dots,\infty)_n}\to\dots\end{cases} \tag{9}$$

where $\boldsymbol{\Lambda}_{\mathfrak{D}}^{(\infty,\dots,\infty)_n}\doteq\operatorname{inv}\lim\limits_{\sigma(n)\in\boldsymbol{N}_+^n}\boldsymbol{\Lambda}_{\mathfrak{D}}^{\sigma(n)}$, $\forall n>0$.

The cohomology of this complex (or of some «finite» [9] version of this) should contain interesting differential invariants of the singularity (see the Introduction), when A is taken to be the corresponding local ring: we plan to return on this question in a subsequent article.

3. – Change of rings and localization of differential operators.

From now on, every representative object will be tacitly referred to the *whole* category of A-(B-)modules, i.e. $\mathcal{D}=A$-**Mod** (B-**Mod**), $\boldsymbol{\Lambda}_{A/K}^{\sigma(n)}\doteq\boldsymbol{\Lambda}_{A\text{-}\mathbf{Mod}}^{\sigma(n)}$ and $\boldsymbol{\Lambda}_{B/K}^{\sigma(n)}\doteq\boldsymbol{\Lambda}_{B\text{-}\mathbf{Mod}}^{\sigma(n)}$.

[9] «Finite» in the sense that we may take the inverse limit only over bounded (resp. bounded in the derived category) higher $\boldsymbol{dR}$-complexes.

Let $K \to A \xrightarrow{h} B$ be unitary commutative ring morphisms, $d_{(q),A/K} : A \to \boldsymbol{\Lambda}^{(q)}_{A/K}$ and $d_{(q),B/K} : B \to \boldsymbol{\Lambda}^{(q)}_{B/K}$ be the canonical derivations and $\mathcal{C}_{A\backslash B} : B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ be the «change of ring»-functor. The functor $D_{(q),A/K} \circ \mathcal{C}_{A\backslash B} : B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ can be viewed as a functor $B\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ [10] and, in this form, it is strictly representable (see Proposition 6 of the Appendix) by $B \otimes_A \boldsymbol{\Lambda}^{(q)}_{A/K}$:

$$D_{(q),A/K} \circ \mathcal{C}_{A\backslash B} \simeq \mathrm{Hom}_B(B \otimes_A \boldsymbol{\Lambda}^{(q)}_{A/K}, \cdot).$$

The B-**Mod**-morphism dual to:

$$D_{(q),B/K} \xrightarrow{\varphi_{q,A/B}} D_{(q),A/K} \circ \mathcal{C}_{A\backslash B},$$

$$\nabla \mapsto \nabla \circ h,$$

is just:

$$\varphi^{\vee} \equiv \varphi^{\vee}_{q,A/B} : B \otimes_A \boldsymbol{\Lambda}^{(q)}_{A/K} \to \boldsymbol{\Lambda}^{(q)}_{B/K},$$

$$b \otimes d_{(q)A/K}(a) \mapsto b \cdot d_{(q),B/K}(h(a)).$$

Proposition 3.1. *$\varphi^{\vee}_{q,A/B}$ has a left inverse (hence is monic) iff $\forall P \in \underline{\mathrm{Ob}}(B\text{-}\mathbf{Mod})$ and $\forall \nabla \in D_{(q),A/K}(\mathcal{C}_{A\backslash B}(P))$, there exists an «extension» $\overline{\nabla}$ of ∇ to B, i.e. a $\overline{\nabla} \in D_{(q),B/K}(P)$ such that*

$$\begin{tikzcd} A \arrow[r, "\nabla"] \arrow[d, "h"'] & P \\ B \arrow[ur, "\overline{\nabla}"'] & \end{tikzcd}$$

is commutative.

Proof. It is a corollary of Proposition 6 of the Appendix. ∎

Remark 3.2. *If $\lambda_q : \boldsymbol{\Lambda}^{(q)}_{B/K} \to \mathrm{coker}(\varphi^{\vee}_{q,A/B})$ is the natural epimorphism, it is easy to verify that the couple $(B, \lambda_q \circ d_{(q),B/K})$ is universal with respect to B/K-derivations ∇ of order $\leq q$ from B to a B-module P such that $\nabla \circ h = 0$: each of these derivations factorizes uniquely through a B-homomorphism f_∇ as $\nabla = f_\nabla \circ (\lambda_q \circ d_{(q),B/K})$. Since the*

[10] Using the B-module structure of the argument.

canonical B/A-differential of order $\leq q$

$$d_{(q),\,B/A} : B \to \Lambda^{(q)}_{B/A}$$

is also a B/K-derivation of order $\leq q$*, by the universality of* $(B, \lambda_q \circ d_{(q),\,B/K})$*, we get a canonical B-homomorphism*

$$\psi_q : \operatorname{coker}(\varphi^{\vee}_{q,\,A/B}) \to \Lambda^{(q)}_{B/A}$$

which is epic since $\Lambda^{(q)}_{B/A}$ *is generated over B by* $\{d_{(q),\,B/A}(b) \mid b \in B\}$*. Note, however, that while for* $q = 1$ *(ordinary derivations)* ψ_q *is also monic* ([9]) *and hence an isomorphism of B-modules, this is no longer true for* $q > 1$.

We prove now an elementary (and intuitively obvious) result on localization of differential operators:

PROPOSITION 3.3. *Let P and Q be A-modules and S a multiplicative part of A. For each* $\Delta \in \mathit{Diff}_{k,\,A/K}(P, Q)$ *there exists a unique* $\Delta_S \in \mathit{Diff}_{k,\,A_S/K}(P_S, Q_S)$ *such that the following diagram is commutative:*

$$\begin{tikzcd} P \arrow[r, "\Delta"] \arrow[d] & Q \arrow[d] \\ P_S \arrow[r, "\Delta_S"'] & Q_S \end{tikzcd} \tag{10}$$

(*where the vertical arrows denote localization morphisms*).

Moreover, if $\Delta \in D_{(k),\,A/K}(Q)$ *then* $\Delta_S \in D_{(k),\,A_S/K}(Q_S)$.

PROOF. *Uniqueness.* We use induction on the order k of Δ. For $k = 0$ the statement is standard; let us suppose the uniqueness proved for any DO of order $\leq k$. If Δ_S and Δ'_S are two elements in $\mathit{Diff}_{k+1,\,A_S/K}(P_S, Q_S)$ satisfying (10) for a given $\Delta \in \mathit{Diff}_{k+1,\,A/K}(P, Q)$, we have

$$(\Delta_S - \Delta'_S)\left(\frac{p}{1}\right) = 0, \qquad \forall p \in P.$$

Let $\xi = p/s \in P_S$, $p \in P$, $s \in S$; then $(\Delta_S - \Delta'_S)(sp/s) = 0$ so that:

$$(\Delta_S - \Delta'_S)(sp/s) = \left[\delta_{s/1}(\Delta_S - \Delta'_S) + \frac{s}{1}(\Delta_S - \Delta'_S)\right]\left(\frac{p}{s}\right) = 0\,.$$

The commutativity of (10) implies that both $\delta_{s/1}\Delta_S$ and $\delta_{s/1}\Delta'_S$ satisfy the commutativity of (10) when Δ is replaced by $\delta_s \Delta$ (which is of order $\leqslant k$): the induction hypothesis then gives $\delta_{s/1}(\Delta_S - \Delta'_S)(p/s) = 0$. So we get $(s/1)(\Delta_S - \Delta'_S)(p/s) = 0$ and therefore $(\Delta_S - \Delta'_S)(p/s) = 0$: $\Delta_S = = \Delta'_S$.

Existence. We again use induction on $k \in \boldsymbol{N}$.

For $k = 0$, Δ is an A-homomorphism and Δ_S is the usual localization A_S-homomorphism.

Suppose we have defined Δ_S for each $\Delta \in Diff_{k, A/K}(P, Q)$. Let $\Delta \in Diff_{k+1, A/K}(P, Q)$ and define for $p \in P$, $s \in S$

$$\Delta_S\left(\frac{p}{s}\right) \doteq \frac{1}{s}\left[\Delta(p) - (\delta_s \Delta)_S\left(\frac{p}{s}\right)\right] \tag{11}$$

which makes sense by induction hypothesis. We first prove that (11) is well defined: if $\xi = p/s = q/r$, $p, q \in P$, $r, s \in S$ then there exists $t \in S$ such that $trp = tsq$; consider

$$\frac{\Delta(trp)}{1} - (\delta_{tsr}\Delta)_S(\xi) = \frac{(\delta_{tr}\Delta)(p)}{1} + \frac{tr\Delta(p)}{1} - (\delta_{tsr}\Delta)_S(\xi). \tag{12}$$

Localizing the identity $a\delta_b + b^+\delta_a = \delta_{ab}$, $a, b \in A$, we get:

$$(a\delta_b\Delta + b^+\delta_a\Delta)_S \equiv \frac{a}{1}(\delta_b\Delta)_S + \left(\frac{b}{1}\right)^+(\delta_a\Delta)_S = (\delta_{ab}\Delta)_S$$

which can be used in (12), with $a = tr$, $b = s$, to get

$$\frac{(\delta_{tr}\Delta)(p)}{1} + \frac{tr\Delta(p)}{1} - (\delta_{tsr}\Delta)_S(\xi) =$$

$$= \frac{(\delta_{tr}\Delta)(p)}{1} + \frac{tr\Delta(p)}{1} - \frac{tr}{1}(\delta_s\Delta)_S(\xi) - (\delta_{tr}\Delta)_S(s\xi)$$

but $\dfrac{(\delta_{tr}\Delta)(p)}{1} = (\delta_{tr}\Delta)_S\left(\dfrac{p}{1}\right) = (\delta_{tr}\Delta)_S\left(\dfrac{sp}{s}\right)$, by induction, and then

$$\frac{\Delta(trp)}{1} - (\delta_{tsr}\Delta)_S(\xi) = \frac{tr\Delta(p)}{1} - \frac{tr}{1}(\delta_s\Delta)_S\left(\frac{p}{s}\right) = trs\Delta_S\left(\frac{p}{s}\right).$$

But $\dfrac{\Delta(trp)}{1} - (\delta_{tsr}\Delta)_S(\xi) = \dfrac{\Delta(tsq)}{1} - (\delta_{tsr}\Delta)_S(\xi)$ and using the same method we get:

$$\frac{\Delta(tsq)}{1} - (\delta_{tsr}\Delta)_S(\xi) = \frac{ts\Delta(p)}{1} - \frac{ts}{1}(\delta_r\Delta)_S\left(\frac{q}{r}\right) = trs\Delta_S\left(\frac{q}{r}\right)$$

so that $\Delta_S(p/s) = \Delta_S(q/r)$. This prove that (3.11) is well defined.

It is straightforward to verify that (3.11) is then a DO of order $\leq k+1$ from the A_S-module P_S to the A_S-module Q_S.

The last assertion of the Proposition follows from $\Delta_S(1/1) = \Delta(1)/1 = = 0$. ■

COROLLARY 3.4. *If $\Delta \in Diff_{k, A/K}(P, Q)$ and $\nabla \in Diff_{l, A/K}(Q, T)$ then $(\nabla \circ \Delta)_S \equiv \nabla_S \circ \Delta_S$;*

PROOF. It follows immediately from the uniqueness part of the previous proposition. ■

Therefore the usual localization functor $\mathrm{Loc}_S : A\text{-}\mathbf{Mod} \to A_S\text{-}\mathbf{Mod}$ extends to a functor $\mathbf{DIFF}_A \to \mathbf{DIFF}_{A_S}$.

COROLLARY 3.5. *Let S be a multiplicative part of A and P be an A_S-module. Then every $\nabla \in D_{(k), A/K}(P)$ (P viewed as an A-module via the localization morphism $\mathrm{loc}_S : A \to A_S$) admits a unique «extension» $\nabla_S \in D_{(k), A_S/K}(P)$ such that $\nabla = \nabla_S \circ \mathrm{loc}_S$.*

PROOF. Apply Proposition 3.3, observing that the localization with respect to S of P, viewed as an A-module via loc_S, coincides with P (as an A_S-module). ■

The previous results enable us, $\Delta \mapsto \Delta_S$ being additive, to build, for any multiplicative part S of A and $\sigma \in N_+^\infty$, the *S-localized de Rham complex of type σ* of A/K:

$$(13) \qquad (\boldsymbol{dR}_{\sigma, A/K})_S : 0 \to A_S \xrightarrow{(d_{\sigma(1)})_S} (\boldsymbol{\Lambda}^{\sigma(1)}_{A/K})_S \to$$

$$\to \ldots \to (\boldsymbol{\Lambda}^{\sigma(n)}_{A/K})_S \xrightarrow{(d_{\sigma(n+1)})_S} (\boldsymbol{\Lambda}^{\sigma(n+1)}_{A/K})_S \to \ldots .$$

4. - Localization of Higher de Rham Complexes.

In this Section we prove the main result of this paper (Proposition 4.3).

PROPOSITION 4.1. *Let S be a multiplicative part of A. Then:*

(i) *for any* $\sigma \in \boldsymbol{N}_+^\infty$ *and* $n \geqslant 1$ *we have a canonical* A_S-**Mod**-*isomorphism* $\varphi_S^{\sigma(n)} : (\boldsymbol{\Lambda}_{A/K}^{\sigma(n)})_S \simeq \boldsymbol{\Lambda}_{A_S/K}^{\sigma(n)}$;

(ii) $\forall k \geqslant 1$ *we have a natural* A_S-**Mod**-*isomorphism* $(\boldsymbol{J}_{A/K}^k)_S \simeq$ $\simeq \boldsymbol{J}_{A_S/K}^k$.

PROOF. We merely sketch the argument. (ii) is a consequence of (i) for $n = 1$, since $\boldsymbol{J}_{A/K}^k \simeq \boldsymbol{\Lambda}_{A/K}^{(k)} \oplus A$. To prove (i) we proceed by induction on n: let us show that (i) holds for $n = 1$. Consider the morphism in $[A_S\text{-}\mathbf{Mod}, A_S\text{-}\mathbf{Mod}]$:

$$\begin{aligned} \varphi &\equiv \varphi_{\sigma_1, A_S/A} : D_{(\sigma_1), A_S/K} \to D_{(\sigma_1), A/K} \circ \mathcal{C}_{A \backslash A_S}, \\ \varphi(P)(\nabla) &\doteq \nabla \circ \mathrm{loc}_S, \end{aligned} \tag{14}$$

($\mathrm{loc}_S : A \to A_S$ being the localization morphism), and its dual A_S-**Mod**-morphism (recall Proposition 6.3):

$$\begin{aligned} \varphi_S^{\sigma(1)} &\doteq \varphi^\vee : (\boldsymbol{\Lambda}_{A/K}^{(\sigma_1)})_S \simeq A_S \otimes_A \boldsymbol{\Lambda}_{A/K}^{(\sigma_1)} \to \boldsymbol{\Lambda}_{A_S/K}^{(\sigma_1)}, \\ \left(\frac{a_0}{s}\right) \otimes d_{(\sigma_1), A/K}(a) &\mapsto \left(\frac{a_0}{s}\right) d_{(\sigma_1), A/K}\left(\frac{a}{1}\right). \end{aligned} \tag{15}$$

Now, Corollary 3.5 tells us that (14) is an isomorphism (the existence part gives surjectivity while the uniqueness gives injectivity) and so, by Proposition 6.1, (15) is an isomorphism, too.

Suppose now that (i) holds for each σ and each $n < k$. We use the functorial isomorphism:

$$\boldsymbol{J}_+^k \overset{\bullet}{\otimes}_A (\cdot) \simeq \boldsymbol{J}^k(\cdot),$$

(where the upper bold dot over $\otimes$ indicate that the A-module structure on the tensor product is inherited by $\boldsymbol{J}^k$ [11]), the short exact sequences dual to (2):

$$\boldsymbol{\Lambda}_{A/K}^{(\sigma(n-2), \sigma_{n-1} + \sigma_n)} \xrightarrow{\pi_{\sigma(n)}^\vee} \boldsymbol{J}_{A/K}^{\sigma_n}(\boldsymbol{\Lambda}_{A/K}^{\sigma(n-1)}) \longrightarrow \boldsymbol{\Lambda}_{A/K}^{\sigma(n)} \to 0$$

and standard properties of localization, to get

$$(\boldsymbol{\Lambda}_{A/K}^{\sigma(k)})_S \simeq \frac{(\boldsymbol{J}_{A/K,+}^{\sigma_k} \overset{\bullet}{\otimes}_A \boldsymbol{\Lambda}_{A/K}^{\sigma(k-1)})_S}{(\mathrm{im}\,(\pi_{\sigma(n), A/K}^\vee))_S} \simeq \frac{\boldsymbol{J}_{A_S/K,+}^{\sigma_k} \overset{\bullet}{\otimes}_{A_S} \boldsymbol{\Lambda}_{A_S/K}^{\sigma(k-1)}}{(\mathrm{im}\,(\pi_{\sigma(n), A/K}^\vee))_S}$$

.

([11]) Remember that $(\boldsymbol{J}^k, \boldsymbol{J}_+^k)$ is an (A, A)-bimodule.

(in the last passage, we used induction hypothesis and (ii)). By localizing

$$\boldsymbol{\Lambda}_{A/K}^{(\sigma(n-2),\,\sigma_{n-1}+\sigma_n)} \xrightarrow{\pi^{\vee}_{\sigma(n)}} \boldsymbol{J}^{\sigma_k}_{A/K}(\boldsymbol{\Lambda}^{\sigma(k-1)}_{A_S/K})$$

with respect to S, using the induction hypothesis and (ii) we finally get

$$(\operatorname{im}(\pi^{\vee}_{\sigma(n),\,A/K}))_S \simeq \operatorname{im}(S^{-1}(\pi^{\vee}_{\sigma(n),\,A/K})) \simeq \operatorname{im}(\pi^{\vee}_{\sigma(n),\,A_S/K}).$$

The explicit expression of $\varphi^{\sigma(n)}_S$ is, then

$$\varphi^{\sigma(n)}_S\left(\frac{a_0\, d_{\sigma(n),\,A/K}(a_1\, d_{\sigma(n-1),\,A/K}(a_2 \ldots (a_{n-1}\, d_{\sigma(1),\,A/K}(a_n))\ldots))}{s}\right) = \tag{16}$$

$$= \frac{a_0}{s}\, d_{\sigma(n),\,A_S/K}\left(\frac{a_1}{1}\, d_{\sigma(n-1),\,A_S/K}\left(\frac{a_2}{1}\ldots\left(\frac{a_{n-1}}{1}\, d_{\sigma(1),\,A_S/K}\left(\frac{a_n}{1}\right)\right)\ldots\right)\right). \quad \blacksquare$$

REMARK 4.2. *In* [16] *Proposition* 4.1 (i) *is stated in the particular case* $\sigma = (1, \ldots, 1, \ldots)$ *and* $S = \{s^n\}_{n \geq 0}$ *with* s *non nilpotent.*

We are now ready to prove the main fact:

PROPOSITION 4.3. *Let* S *be a multiplicative part of* A. *Let, for each* $\sigma \in \boldsymbol{N}^{\infty}_{+}$, $(\boldsymbol{dR}_{\sigma,\,A/K})_S$, *respectively* $\boldsymbol{dR}_{\sigma,\,A_S/K}$, *denote the complex* (13), *resp. the complex* $\boldsymbol{dR}_{\sigma}(A_S\text{-}\mathbf{Mod})$. *Then the family of* A_S-**Mod**-*morphisms* $\{\varphi^{\sigma(n)}_S \mid n \geq 0\}$ *defined in Proposition* 4.1, *realizes an isomorphism:*

$$(\boldsymbol{dR}_{\sigma,\,A/K})_S \simeq \boldsymbol{dR}_{\sigma,\,A_S/K}$$

of complexes in $\mathbf{DIFF}_{A_S}$.

PROOF. By functoriality of $\boldsymbol{dR}$-complexes, every k-algebras morphism $h: A \to B$ defines a morphism of complexes

$$h^{\bullet}: \boldsymbol{dR}_{\sigma,\,A/K} \to \boldsymbol{dR}_{\sigma,\,B/K},$$

such that $h^{\emptyset} = h$ in the following way. Fix an $n > 0$ and consider the A-module $\mathcal{C}_{A\backslash B}(\boldsymbol{\Lambda}^{\sigma(n)}_{B/K})$; since $D_{\sigma(n),\,A/K}: A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ is strictly representable by $\boldsymbol{\Lambda}^{\sigma(n)}_{A/K}$, we have an isomorphism of A-modules

$$D_{\sigma(n),\,A/K}(\mathcal{C}_{A\backslash B}(\boldsymbol{\Lambda}^{\sigma(n)}_{B/K})) \simeq \operatorname{Hom}_A(\boldsymbol{\Lambda}^{\sigma(n)}_{A/K},\, \mathcal{C}_{A\backslash B}(\boldsymbol{\Lambda}^{\sigma(n)}_{B/K})). \tag{17}$$

In the l.h.s. of (17) there is a distinguished element $\partial_{\sigma(n),\,B/A/K}$

$$((\partial_{\sigma(n),\,B/A/K}(a_1))\dots)(a_n) \doteq$$

$$\doteq d_{\sigma(n),\,B/K}\big(h(a_n)\,d_{\sigma(n-1),\,B/K}\big(h(a_{n-1})\,d_{\sigma(n),\,B/K}\big(\dots(h(a_2)\,d_{\sigma(1),\,B/K}(h(a_1)))\dots\big)\big)\big)$$

whose image via (17) we define to be

$$h^{\sigma(n)}\colon \Lambda^{\sigma(n)}_{A/K} \to \Lambda^{\sigma(n)}_{B/K}\ ;$$

then, explicitly

$$h^{\sigma(n)}\big(a_0\,(d_{\sigma(n),\,A/K}(a_1\dots d_{\sigma(1),\,A/K}(a_n)\dots))\big) =$$

$$= h(a_0)\cdot d_{\sigma(n),\,B/K}\big(h(a_1)\dots(h(a_{n-1})\,d_{\sigma(1),\,B/K}(h(a_n)))\dots\big)\,.$$

It is then immediate to verify that $h^\bullet \doteq \{h^{\sigma(n)}\,|\,n \geqslant 0\}$ is a morphism of complexes as we claimed.

Taking $h = \mathrm{loc}_S$, we get a diagram ($\forall n \geqslant 0$):

$$\begin{tikzcd}
 & \Lambda^{\sigma(n)}_{A/K} \arrow[dl, "\mathrm{loc}_S"'] \arrow[dd, "\mathrm{loc}^\bullet_S"'] \arrow[r, "d_{\sigma(n+1),\,A/K}"] & \Lambda^{\sigma(n+1)}_{A/K} \arrow[dr, "\mathrm{loc}_S"] \arrow[dd, "\mathrm{loc}^\bullet_S"] & \\
(\Lambda^{\sigma(n)}_{A/K}) \arrow[dr, "\varphi^{\sigma(n)}_S"'] & & & (\Lambda^{\sigma(n+1)}_{A/K})_S \arrow[dl, "\varphi^{\sigma(n+1)}_S"] \\
 & \Lambda^{\sigma(n)}_{A_S/K} \arrow[r, "d_{\sigma(n+1),\,A_S/K}"'] & \Lambda^{\sigma(n+1)}_{A_S/K} &
\end{tikzcd}$$

in which the square is commutative and also the two lateral triangles are proved, by a straightforward calculation using formula (16), to be commutative. Therefore, by the uniqueness part of Proposition 3 the family $\{\varphi^{g(n)}_S\,|\,n \geqslant 0\}$ defines a (iso)morphism of complexes. ∎

Therefore there is *only one* natural way to study, via the higher de Rham complexes, an algebraic singularity: we can either localize the «global» $\boldsymbol{dR}_\sigma$-complex or equivalently localize the algebra A and then consider its «global» $\boldsymbol{dR}_\sigma$-complex.

5. – Geometric modules.

We prove that Proposition 4.3 still holds if we restrict ourselves to the subcategory of (prime) geometric modules. This case is crucial for differentiable manifolds (see Example 2.4 (ii) or [15]).

Definition 5.1. *If A is a K-algebra, an A-module P is (prime) geometric if*

$$\bigcap_{\mathfrak{p}\in \operatorname{Spec} A} \mathfrak{p}P = (0). \tag{18}$$

Therefore, for a geometric A-module each element is uniquely defined by its «values» at every prime of A. The full subcategory of A-**Mod**, whose objects are the geometric modules is denoted by $A\text{-}\mathbf{Mod}_{\mathrm{geom}}$. If A is reduced then A is geometric as an A-module and $A\text{-}\mathbf{Mod}_{\mathrm{geom}}$ is differentially closed ([11]): we will suppose from now on that A is reduced.

There is an obvious *geometrization functor*:

$$(\cdot)_{\mathrm{geom},\,A} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}_{\mathrm{geom}}$$

$$\mathrm{P} \mapsto (\mathrm{P})_{\mathrm{geom},\,A} \doteq \frac{P}{\bigcap_{\mathfrak{p}\in \operatorname{Spec} A} \mathfrak{p}P}$$

and moreover we have

$$\boldsymbol{A}^{\sigma(n)}_{A\text{-}\mathbf{Mod}_{\mathrm{geom}}} \simeq (\boldsymbol{A}^{\sigma(n)}_{A\text{-}\mathbf{Mod}})_{\mathrm{geom},\,A}, \qquad \forall \sigma \in \boldsymbol{N}^{\infty}_{+}, \quad \forall n > 0,$$

$$\boldsymbol{J}^{\mathrm{k}}_{A\text{-}\mathbf{Mod}_{\mathrm{geom}}}(\mathrm{P}) \simeq ((\boldsymbol{J}^{k}_{A\text{-}\mathbf{Mod}}(P))_{\mathrm{geom},\,A}, \qquad \forall k > 0, \quad \forall P \in \mathrm{Ob}(A\text{-}\mathbf{Mod}_{\mathrm{geom}}).$$

Thus the geometrization functor can be used to build all the representative objects we need. We will then denote by $\boldsymbol{dR}_{\sigma,\,\mathrm{geom},\,A/K}$ the complex $\boldsymbol{dR}_{\sigma}(A\text{-}\mathbf{Mod}_{\mathrm{geom}})$, for any $\sigma \in \boldsymbol{N}^{\infty}_{+}$.

We have a result similar to that of Proposition 4.3, in $A\text{-}\mathbf{Mod}_{\mathrm{geom}}$:

Proposition 5.2. *Let A be a reduced K-algebra, S a multiplicative part of A and $\sigma \in \boldsymbol{N}^{\infty}_{+}$. Then, keeping the notation of Proposition* 4.3, *we have an isomorphism in $\boldsymbol{K}(\mathbf{DIFF}_{A_S,\,A_S\text{-}\mathbf{Mod}_{\mathrm{geom}}})$*:

$$(\boldsymbol{dR}_{\sigma,\,\mathrm{geom},\,A/K})_S \simeq (\boldsymbol{dR}_{\sigma,\,A_S/K})_{\mathrm{geom}} \equiv \boldsymbol{dR}_{\sigma}(A_S\text{-}\mathbf{Mod}_{\mathrm{geom}}).$$

The proof of Proposition 5.2 is a direct consequence of Proposition 4.3 and of the following

Lemma 5.3. *The S-localization functor «commutes» with the ge-*

ometrization functor, i.e. the diagram of functor

$$\begin{tikzcd} A\text{-}\mathbf{Mod} \arrow[r, "(\cdot)_{\mathrm{geom},A}"] \arrow[d, "\mathrm{Loc}_S"'] & A\text{-}\mathbf{Mod}_{\mathrm{geom}} \arrow[d, "\mathrm{Loc}_{S|A\text{-}\mathbf{Mod}_{\mathrm{geom}}}"] \\ A_S\text{-}\mathbf{Mod} \arrow[r, "(\cdot)_{\mathrm{geom},A_S}"'] & A_S\text{-}\mathbf{Mod}_{\mathrm{geom}} \end{tikzcd}$$

is commutative.

PROOF. Since the S-localization functor Loc_S commutes with quotients, is exact and commutes with inductive limits (and then with arbitrary intersections, [8] p. 16), we have canonical isomorphisms

$$\mathrm{Loc}_S\left((P)_{\mathrm{geom},A}\right) \simeq \frac{P_S}{\left(\bigcap_{\underline{\mathfrak{p}} \in \mathrm{Spec}A} \underline{\mathfrak{p}}P\right)_S} \simeq \frac{P_S}{\bigcap_{\underline{\mathfrak{p}} \in \mathrm{Spec}A} \underline{\mathfrak{p}}_S P_S}.$$

Since $\underline{\mathfrak{p}} \mapsto \underline{\mathfrak{p}}_S$ establishes a bijection between $\{\underline{\mathfrak{p}} \in \mathrm{Spec}A \,|\, \underline{\mathfrak{p}} \cap S = \emptyset\}$ and $\mathrm{Spec}A_S$ and for each ideal $\mathfrak{a}$ in A, $\mathfrak{a}_S = A_S$ iff $\mathfrak{a} \cap S \neq \emptyset$, we get

$$\bigcap_{\underline{\mathfrak{p}} \in \mathrm{Spec}A} \underline{\mathfrak{p}}_S P_S = \bigcap_{\mathfrak{p} \in \mathrm{Spec}A_S} \mathfrak{p} P_S$$

and the thesis of the Lemma immediately follows. ∎

REMARK 5.4. *In fact, Propositions* 4.3 *and* 5.2 *are concrete instances of a more general principle which may be loosely stated in the following way: if* $\mathfrak{D} \subset A$-**Mod** *is a «good» (i.e. preserved under «appropriate»* (12) *changes of algebras) differentially closed subcategory then the S-localizations of the higher jet-Spencer* ([11], [10]), *higher de Rham's etc. complexes are isomorphic to the higher jet-Spencer, higher de Rham's etc. complexes of the localized algebra* A_S.

6. – Appendix.

PROPOSITION 6.1. *Let* T_1, T_2 *and* T_3 *be strictly representable functors* A-**Mod** $\to A$-**Mod** *with representative objects* τ_1, τ_2 *and* τ_3, *respectively. Then:*

(12) What is «appropriate» depends on the geometry we are dealing with: if we are doing algebraic geometry we may allow all changes of algebras, only flat ones, only étale ones, etc.; if we are doing differential geometry, we may limit ourselves to changes of algebras which arise as pullbacks of smooth morphisms of manifolds.

(i) $0 \to T_1 \xrightarrow{F} T_2$ *is exact iff* $\tau_2 \xrightarrow{F^\vee} \tau_1 \to 0$ *is exact and right-splits (i.e. there exists* $r\colon \tau_1 \to \tau_2$ *such that* $F^\vee \circ r = \mathrm{id}_{\tau_1}$*);*

(ii) $T_1 \xrightarrow{G} T_2 \to 0$ *is exact iff* $0 \to \tau_2 \xrightarrow{G^\vee} \tau_1$ *is exact and left-splits (i.e. there exists* $l\colon \tau_1 \to \tau_2$ *such that* $l \circ G^\vee = \mathrm{id}_{\tau_2}$*);*

(iii) $0 \to T_1 \xrightarrow{F} T_2 \xrightarrow{G} T_3 \to 0$ *is exact iff* $0 \to \tau_3 \xrightarrow{G^\vee} \tau_2 \xrightarrow{F^\vee} \tau_1 \to 0$ *is exact and splits (left or right, since left* $\Leftrightarrow$ *right).*

REMARK 6.2. *Observe, however, that none of these splittings is canonical.*

PROOF. (i) is dual to (ii) and (iii) follows from known facts about $\mathrm{Hom}_A(\cdot, \cdot)$ and (i). Let us prove (ii).

Suppose G is epic, then $G(\tau_2)\colon \mathrm{Hom}_A(\tau_1, \tau_2) \to \mathrm{Hom}_A(\tau_2, \tau_2)$ is epic, so it exists an $l \in \mathrm{Hom}_A(\tau_1, \tau_2)$ such that $G(\tau_2)(l) \equiv l \circ G^\vee = \mathrm{id}_{\tau_2}$.

Conversely, suppose that $G^\vee$ has a left inverse l. Let P be an A-module:

$$G(P)\colon \mathrm{Hom}_A(\tau_1, P) \to \mathrm{Hom}_A(\tau_2, P);$$

pick a $\psi \in \mathrm{Hom}_A(\tau_2, P)$, then $\psi \circ l$ is sent by $G(P)$ to ψ. ∎

PROPOSITION 6.3. *Let* $K \to A \xrightarrow{h} B$ *be morphisms of commutative rings with unit,* $n > 0$ *and* $\mathcal{C}_{A\backslash B}\colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ *be the «change-of-rings» functor. Then:*

(i) $D_{(n), A/K} \circ \mathcal{C}_{A\backslash B}\colon B\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ *is strictly representable by* $B \otimes_A \boldsymbol{\Lambda}^{(n)}_{A/K}$*;*

(ii) $\mathit{Diff}_{n, A/K} \circ \mathcal{C}_{A\backslash B}\colon B\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ *is strictly representable by* $B \otimes_A \boldsymbol{J}^n_{A/K}$.

PROOF. The two proofs are analogous: we prove (i). The canonical A-**Mod**-morphism

$$D_{(n), A/K}(\mathcal{C}_{A\backslash B}(P)) \simeq \mathrm{Hom}_A(\boldsymbol{\Lambda}^{(n)}_{A/K}, \mathcal{C}_{A\backslash B}(P))$$

is also a B-**Mod**-morphism and we conclude using the canonical B-**Mod**-isomorphism:

$$\mathrm{Hom}_A(Q, \mathcal{C}_{A\backslash B}(P)) \simeq \mathrm{Hom}_B(B \otimes_A Q, P),$$

$$\psi \mapsto \widehat{\psi}\colon b \otimes q \mapsto b\psi(q),$$

with inverse $\check{\chi} \leftarrow \chi$

$$\check{\chi}(q) \doteq \chi(1 \otimes q)$$

where Q is any A-module, $b \in B$ and $q \in Q$. ∎

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