RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ANNE-MARIE SIMON

A corollary to the Evans-Griffith Syzygy theorem

Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 71-77

<http://www.numdam.org/item?id=RSMUP_1995__94__71_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

A Corollary to the Evans-Griffith Syzygy Theorem.

ANNE-MARIE SIMON (*)

ABSTRACT - Height two ideals of finite projective dimension in a Cohen-Macaulay or Gorenstein local ring are investigated, providing slight extensions of results of Serre and Evans-Griffith concerning the problem to know when they are two-generated, when the quotient ring is Cohen-Macaulay if they are three-generated.

Introduction.

This note is concerned with the height two ideals of a Cohen-Macaulay noetherian local ring: when are they two-generated, what can we say about them when they are three-generated?

In the second direction we have an important theorem of Evans-Griffith.

THEOREM 1. ([E.G.81] *Theorem* 2.1, *or* [E.G.85] *Theorem* 4.4). *Let A be a regular local ring containing a field. If I is an unmixed three generated ideal of height two, then the ring A/I is Cohen-Macaulay.*

In the first direction we have first a Serre's theorem, one formulation of it is the following.

(*) Indirizzo dell'A.: Service d'Algèbre, Université Libre de Bruxelles, Boulevard du Triomphe CP.211, B-1050 Brussel, Belgium.

THEOREM 2. ([Se, *Proposition* 5] [B-E]). *Let I be a height two ideal of a regular local ring A. If the quotient ring A/I is Gorenstein, then the ideal I is two-generated.*

However, another formulation of Serre was a little bit different.

THEOREM 2′. [Se, *corollaire à la Proposition* 2]. *Let A be a noetherian domain such that all projective A-modules of rank one or two are free, and let I be a non-zero ideal of projective dimension less or equal to one. Then the ideal I is two-generated if and only if the A-module* $\mathrm{Ext}^1_A(I, A)$ *is principal.*

We note that in Theorem 2′ only the case where I is an ideal of projective dimension one and of height two is of some interest, at least to us.

In Theorem 2, the hypothesis on I imply that the projective dimension of I is one, they imply also that the A-module $\mathrm{Ext}^1_A(I, A)$ is principal, since $\mathrm{Ext}^1_A(I, A) \simeq \mathrm{Ext}^2_A(A/I, A)$. Indeed, the last module $\mathrm{Ext}^2_A(A/I, A)$ is the canonical module of the ring A/I, hence it is principal since A/I is Gorenstein (a local ring is Gorenstein if and only if it is Cohen-Macaulay and if its canonical module is principal).

Then we have another Evans-Griffith's theorem.

THEOREM 3. ([E.G.81], *Theorem* 2.2, *or* [E.G.85], *Theorem* 4.7). *Let A be a regular local ring containing a field and let I be a prime ideal of height two such that the A-module* $\mathrm{Ext}^2_A(A/I, A)$ *is principal. Then I is two generated.*

There is an evident analogy between Theorem 3 and Theorem 2′, in the hypothesis, the conclusions and even the proofs. Both proofs use an extension $0 \to A \to M \to I \to 0$ whose class generates the principal A-module $\mathrm{Ext}^1_A(I, A)$, and one observes that $\mathrm{Ext}^1_A(M, A) = 0$. From this observation the conclusion of Theorem 2′ follows rather quickly, while for theorem 3 one has to use the syzygy theorem; and the hypothesis that I is a prime ideal of a regular ring is strongly used. Concerning that last Theorem 3 we have also to mention [Br.E.G.].

The aim of this note is to provide slight generalizations of the above-mentioned theorems. The generalizations of Theorem 1 and 2 are straightforward, indeed the proofs are essentially the same. The generalization we give of Theorem 3 requires not only the preceding extensions but also a rather different proof, using the linkage theory as developped in [P.S] or [Ul] as well as the syzygy theorem.

As a general reference for homological background we quote [E.G.85], [St], [Ul].

1. Preliminaries.

To state the syzygy theorem we must recall the Serre k-condition.

DEFINITION. *An A-module M is said to be S_k if, for all prime ideals p of A one has* depth $M_p \geqslant \min\{k, htp\}$.

So, if an A-module is S_k for some $k > 0$, then all the associated prime ideals of M are minimal in Spec A.

A key result is the theorem of Auslander-Bridger which shows that a finitely generated A-module of finite projective dimension is S_k if and only if it is a k^{th} syzygy (see [E.G.85] Theorem 3.8) when the ring itself is S_k.

THE SYZYGY THEOREM. ([E.G.81], *Theorem* 1.1, *or* [E.G.85], *Theorem* 3.15, *see also* [Br] *or* [Og]). *Let A be a noetherian local ring containing a field and let M be a finitely generated S_k-module over A of finite projective dimension. Then if M is not free, it has rank at least k.*

We note that the rank of a finitely generated A-module of finite projective dimension is a well-defined natural number: if $0 \to A^{n_s} \to A^{n_{s-1}} \to \dots \to A^{n_0} \to M \to 0$ is a free resolution of M, then rank $M = r = \sum_{i=0}^{s} (-1)^i n_i$; and, for all minimal prime ideals p of Spec A one has $M_p \simeq A_p^r$.

In the syzygy theorem, the hypothesis that the local ring contains a field is still essential. Indeed, the proof uses a Big Cohen-Macaulay module, only available by now in equal characteristic.

Here is the straightforward generalization of theorem 1.

PROPOSITION 1. *Let A be a Cohen-Macaulay noetherian local ring containing a field, and let I be an unmixed three generated ideal of height two of finite projective dimension. Then the ideal I is perfect, i.e. the quotient ring A/I is Cohen-Macaulay. Moreover, the module* $\mathrm{Ext}_A^2(A/I, A)$ *is also a perfect module, i.e. it is a Cohen-Macaulay module of projective dimension two.*

PROOF We resolve A/I and have an exact sequence $0 \to M \to A^3 \to A \to A/I \to 0$, where M is a second syzygy of A/I.

We need to show that M is free. (If M is free, using the Auslander-Buchsbaum equality we obtain depth $A/I = \dim A - pd\, A/I = \dim A - 2 = \dim A/I$).

We observe that M is a finitely generated A-module of finite projective dimension and of rank two.

On the other hand, the A-module M is S_3. Indeed, let p be a prime ideal of A at which we localize.

If $p \not\supset I$, then $M_p \simeq A_p^2$ and depth $M_p = ht\,p \geqslant \min\{3, ht\,p\}$.

If $p \supset I$ and $ht\,p = 2$, the above exact sequence localized at p shows that depth $M_p = 2 \geqslant \min\{3, 2\}$.

If $p \supset I$ and $ht\,p > 2$, the above exact sequence localized at p shows that depth $M_p \geqslant 3$, because depth $(A/I)_p \geqslant 1$, I being unmixed.

The freeness of the module M follows now from the syzygy theorem and $M \simeq A^2$.

For the second assertion, we apply the functor $\mathrm{Hom}_A(\cdot, A) = (\cdot)^*$ to the exact sequence $0 \to A^2 \to A^3 \to A \to A/I \to 0$. We obtain a complex $0 \to A^* \to A^{3^*} \to A^{2^*} \to 0$ whose homology is concentrated in degree 2, where it is $\mathrm{Ext}_A^2(A/I, A)$.

So $pd\ \mathrm{Ext}_A^2(A/I, A) = 2$, and again the conclusion follows from the Auslander-Buchsbaum equality: $\dim A - 2 = \text{depth}\ \mathrm{Ext}_A^2(A/I, A) \leqslant$ $\leqslant \dim \mathrm{Ext}_A^2(A/I, A) \leqslant \dim A - 2$.

We give now the straightforward generalization of Theorem 2, though this has nothing to do with the syzygy theorem.

Proposition. 2. *Let I be a height two ideal of a Gorenstein local ring. If the projective dimension of I is finite and if the quotient ring A/I s Gorenstein, then the ideal I is two-generated*

Proof. The hypotheses imply that the ideal I is perfect, i.e. the projective dimension of A/I is two, the height of I.

A minimal resolution of A/I has the form

$$0 \to A^{m-1} \xrightarrow{\alpha} A^m \to A \to A/I \to 0,$$

and $\mathrm{Ext}_A^2(A/I, A) = \mathrm{coker}\ \mathrm{Hom}_A(\alpha, A)$: the sequence $A^m \xrightarrow{\alpha^t} A^{m-1} \to$ $\to \mathrm{Ext}_A^2(A/I, A) \to 0$ is exact.

As the resolution of A/I is minimal the entries of the matrix associated to α and to its transposed α^t are in the maximal ideal of A. This shows that the minimal number of generators of $\mathrm{Ext}_A^2(A/I, A)$ is $m-1$.

On the other hand, this number is one since A/I is assumed to be Gorenstein, so $1 = m - 1$, $m = 2$ and the ideal I is two generated.

2. An extension of Theorem 3.

Proposition 3. *Let A be a Gorenstein noetherian local ring containing a field and let I be an unmixed ideal of finite projective dimen-*

sion and of height two. If the A-module $\operatorname{Ext}_A^2(A/I, A)$ *is principal, then the ideal* I *can be generated by* 2 *elements.*

PROOF. We choose in I a regular sequence x_1, x_2 and use it to make an algebraic link: if $J = (x_1, x_2): I$, then $I = (x_1, x_2): J$ since I is unmixed and since the ring A is Gorenstein [P.S.]; moreover, the ideal J is also unmixed. The isomorphisms $\operatorname{Ext}_A^2(A/I, A) \simeq \simeq \operatorname{Hom}_A(A/I, A/(x_1, x_2)) \simeq J/(x_1, x_2)$ and the hypothesis on I imply that the linked ideal J is a height two ideal three generated: $J = (x_1, x_2, y)$ for some y in A.

As $I = (x_1, x_2): J$, we have an exact sequence

$$0 \to I/(x_1, x_2) \to A/(x_1, x_2) \xrightarrow{\cdot y} J/(x_1, x_2) \to 0$$

which shows that $A/I \simeq J/(x_1, x_2)$. So we have also an exact sequence

$$0 \to A/I \to A/(x_1, x_2) \to A/J \to 0,$$

this shows that the A-module A/J is of finite projective dimension.

By proposition 1, we conclude that the ring A/J is Cohen-Macaulay; but then A/I is also Cohen-Macaulay by the linkage theory in a Gorenstein ring A ([P.S], or [Ul]). Consequently the ring A/I is Gorenstein since it is Cohen-Macaulay and since its canonical module $\operatorname{Ext}_A^2(A/I, A)$ is principal, and the conclusion follows from proposition 2.

NOTE. the above proposition is to be compared with a geometric result of Fiorentini and Lascu ([Fi.La.], Theorem 2 (iii)).

The hypothesis in Proposition 3 are slightly weaker then in Proposition 2; in Proposition 3, the ring A/I is not assumed to be Cohen-Macaulay in advance.

3. Some Examples.

EXAMPLE 1. To the twisted cubic curve (s^3, s^2t, st^2, t^3) of the projective space $\mathbb{P}_K^3$ is associated an ideal I of the regular local ring $A = = K[\![X_0, X_1, X_2, X_3]\!]$. This ideal is a height two prime ideal which is three-generated: $I = (X_0X_3 - X_1X_2, X_1^2 - X_0X_2, X_2^2 - X_1X_3)$. Hence the ring A/I is Cohen-Macaulay, not Gorenstein. In fact the ideal I is the ideal of the 2×2 minors of the matrix

$$\phi = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \\ X_0 & X_1 \end{bmatrix}.$$

A minimal projective resolution of the A-module A/I is given by

$$0 \to A^2 \xrightarrow{\phi} A^3 \to A \to A/I \to 0,$$

this shows that the canonical module of A/I, $\mathrm{Ext}_A^2(A/I, A) = \operatorname{coker} \phi^t$ (where ϕ^t is the transposed of ϕ) is minimally generated by 2 elements.

EXAMPLE 2. To the quartic curve (s^4, s^3t, st^3, t^4) of the projective space $\mathbb{P}_K^3$ is associated a height two prime ideal I of the ring $A = K[\![X_0, X_1, X_2, X_3]\!]$, this ideal I is four-generated: $I = (X_0X_3 - X_1X_2, X_1^3 - X_0^2X_2, X_2^3 - X_1X_3^2, X_0X_2^2 - X_1^2X_3)$. The quotient ring A/I is not Cohen-Macaulay, however it is a Buchsbaum local ring.

EXAMPLE 3. Bertini constructed an example of a non Cohen-Macaulay factorial ring B which is an image of a regular local ring A: $B = A/I$, the height g of I is greater than 3. Since B is factorial, the module $\mathrm{Ext}_A^g(B, A)$ is principal. This illustrates the fact that the hypothesis in Proposition 3 are weaker than those in Proposition 2. On the other hand, Theorem 1 is concerned with unmixed ideals I of height g generated by $g + 1$ element. When $g = 2$, when the ring A is regular, the quotient A/I is Cohen-Macaulay. When $g > 2$, this conclusion is not valid anymore. Indeed, in Bertin's example we can choose in the ideal I a regular sequence $x_1, \ldots, x_g$ such that $IA_I = (x_1, \ldots, x_g)A_I$ (I is a prime ideal of the regular ring A). This gives us a link (even a geometric link): $J = (x_1, \ldots, x_g): I$. The ideal J is an unmixed ideal of height g of the ring A, the quotient ring A/J is not Cohen-Macaulay (since A/I is not), but J can be generated by $g + 1$ elements: the module $J/(x_1, \ldots, x_g) = \mathrm{Hom}_A(A/I, A/x_1, \ldots, x_g) = \mathrm{Ext}_A^g(A/I, A)$ is principal.

Schenzel gave other examples of prime ideal of height g in a regular local ring, minimally generated by $g + 1$ elements, and such that the quotient ring is not Cohen-Macaulay.

This work was done while the author was visiting the University of Ferrara.

REFERENCES

[Be] M. J. BERTIN, *Anneaux d'invariants d'anneaux de polynômes, en caractéristique p*, C. R. Acad. Sci. Paris 264 (1967), 653-656.

[Br] W. BRUNS, *The Evans-Griffith syzygy theorem and Bass numbers*, Proc. of the Amer. Math. Soc. 115 n°4 (1992), pp. 939-946.

[Br.E.G] W. BRUNS - E. EVANS - P. GRIFFITH, *Syzygies, ideals of height two and vector bundles*, Jour. Alg. 67 (1980), pp. 143-162.

[B.E.] A. BUCHSBAUM - D. EISENBUD, *Gorenstein ideals of height 3*, in Seminar Eisenbud Singh Vogel, Teubner texte zur Mathematik, band 48 (1982), 30-47.

[Fi.La.] M. FIORENTINI - A. LASCU, *Projective embeddings and linkage*, in Rendiconti del Seminario Matematico e Fisico di Milano, vol. LXII (1987).

[E.G.81] E. G. EVANS - P. GRIFFITH, *The syzygy problem*, Annals of Math., **114** (1981), pp. 323-333.

[E.G.85] E. G. EVANS - P. GRIFFITH, *Syzygies*, London Math. Soc. Lect. Notes Ser. 106, Cambridge Univ. Press, Cambridge (1985).

[Og] T. OGOMA, *A note on the syzygy problem*, Comm. Algebra, **17** (1989), pp. 2061-2066.

[P.S.] C. PESKINE - L. SZPIRO, *Liaison des variétés algébriques I*, Invent. Math., **26** (1974), pp. 271-302.

[Sch] P. SCHENZEL, *A note on almost complete intersections*, in Seminar D. Eisenbud, B. Singh, W. Vogel, vol. 2, Teubner-Texte zur Mathematik, Band 48, Leipzig (1982), pp. 49-54.

[Se] J. P. SERRE, *Sur les modules projectifs*, dans Séminaire Dubreil-Pisot (Algèbre et théorie des nombres) 1960/1961, n° 2.

[St] J. STROOKER, *Homological questions in local Algebra*, London Math. Soc. Lect. Notes Ser., **145**, Cambridge Univ. Press, Cambridge, 1990.

[Ul] B. ULRICH, *Lectures on Linkage and deformation*, preprint, workshop on commutative algebra, Trieste 1992.

Manoscritto pervenuto in redazione il 15 dicembre 1993
e, in forma revisionata, il 31 gennaio 1994.