RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

PATRIZIA LONGOBARDI
MERCEDE MAJ
HOWARD SMITH

A note on locally graded groups

Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 275-277
<http://www.numdam.org/item?id=RSMUP_1995__94__275_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

A Note on Locally Graded Groups.

PATRIZIA LONGOBARDI - MERCEDE MAJ - HOWARD SMITH (*)

A group G is locally graded if every nontrivial finitely generated subgroup of G has a nontrivial finite image. The class of locally graded groups is not closed under forming homomorphic images, since free groups (but not all groups) are locally graded. It was shown in [HS] that, if G is locally graded, then G/H is locally graded if, for instance, H is a G-invariant subgroup of the hypercentre of G. A few other results of this kind were established there, but the question as to whether G/H is locally graded if H is abelian was left open. It is not difficult to show that the answer to this question is in the affirmative. Our results are as follows.

THEOREM. *Let G be a locally graded group and H a G-invariant subgroup of the Hirsch-Plotkin radical of G. Then G/H is locally graded.*

COROLLARY. *Suppose G is locally graded and H is a normal subgroup of G. If H is an ascending union of G-invariant subgroups $H_\lambda (\lambda < \mu)$ such that $H_{\lambda+1}/H_\lambda$ is locally nilpotent for each λ, then G/H is locally graded. In particular, if H is soluble then G/H is locally graded.*

(*) Indirizzo degli A.A.: P. LONGOBARDI and M. MAJ: Dipartimento di Matematica, Università di Napoli, Monte S. Angelo, Via Cintia, 80126 Napoli; H. SMITH: Dept. of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.

The proof of our theorem requires the following result.

LEMMA. *Let G be a finitely generated group and suppose that there exists a finitely generated subgroup T of the Hirsch-Plotkin radical H of G such that $G = G'T$. Then $\gamma_j(G)$ is finitely generated, for all $j \geq 1$, and, if c is the nilpotency class of T, then $\gamma_{c+1}(G) = \gamma_{c+i}(G)$ for all $i \geq 1$.*

PROOF. Let $L = \gamma_{c+1}(G)$. Then G/L is nilpotent and $G = G'T$ and so $G = LT$. As G is finitely generated, we may write $G = \langle X, Y \rangle$, for some finite subsets X of L and Y of T, and we may as well assume that Y is closed under forming inverses. Since H is locally nilpotent and normal in G, there is a positive integer k such that $[x, y_1, \ldots, y_k] = 1$ for all x in X, $y_1, \ldots, y_k$ in Y.

Let $V = \langle X, [x, y_1], [x, y_1, y_2], \ldots, [x, y_1, \ldots, y_k] : x \in X, y_i \in Y$ for all $i \rangle$. Then $\langle X \rangle \leq V$ and $V^y \leq V$ for all $y \in Y$. Since $Y = Y^{-1}$, it follows that V is normal in G. Then $G = VT$ and G/V is nilpotent of class at most c, giving $L \leq V \leq L$. Hence $L = V$, which of course is finitely generated. Since G/L satisfies the maximal condition, all terms of the lower central series of G are thus finitely generated. Finally, since $G = \gamma_{c+2}(G)T$, we see that $G/\gamma_{c+2}(G)$ has class at most c. Thus $L = \gamma_{c+i}(G)$ for all $i \geq 1$.

PROOF OF THE THEOREM. Suppose that the pair (G, H) satisfies the hypotheses of the theorem and suppose, for a contradiction, that G/H is not locally graded. Then, as in the proof of Lemma 1 of [HS], we may assume that G is finitely generated and that G/H is nontrivial but has no nontrivial finite images. This implies that $G = G'H$ and hence that $G = G'T$ for some finitely generated subgroup T of H.

By the lemma, $L = \gamma_{c+1}(G)$ is finitely generated, where c is the nilpotency class of T. Certainly $L \neq 1$, and so it contains a proper normal subgroup N of finite index, and N may be chosen to be normal in G. But now we have $G = HN$ and hence $G/N \cong H/H \cap N$, which is locally nilpotent. Thus G/N is nilpotent. The lemma now gives the contradiction that $L \leq N$.

PROOF OF THE COROLLARY. Assuming the result false, we may choose the ordinal μ to be minimal such that G/H is not locally graded. By the theorem, μ is a limit ordinal. As before, we may now assume that G is finitely generated and that G/H has no nontrivial finite images. As in the proof of Corollary 6 of [HS], we may further suppose that H is a direct product of G-invariant, finite simple groups. But this of course implies that H is abelian, and the theorem gives us a contradiction.

REFERENCES

[HS] H. Smith, *On homomorphic images of locally graded groups*, Rend. Sem. Mat. Univ. Padova, **91** (1994), pp. 53-60.

Manoscritto pervenuto in redazione l'8 aprile 1994
e, in forma revisionata, il 20 ottobre 1994.