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On the Rank-One-Convexity Domain of the Saint Venant-Kirchhoff Stored Energy Function.

BERNARD BRIGHI - MAHMOUD BOUSSELSAL (*)

ABSTRACT - The goal of this paper is to give a characterization of the rank-one-convexity domain of the Saint Venant-Kirchhoff stored energy function.

1. Introduction.

For a homogeneous, isotropic, elastic material, whose reference configuration is a natural state, one can show that the response function associated to the second Piola-Kirchhoff stress tensor is of the form

$$(1.1) \quad \Sigma(F) = \lambda(\operatorname{tr} E)I + 2\mu E + o(E)$$

for a matrix $F \in \mathbb{M}_+^3$ neighbouring of the identity matrix $I \in \mathbb{M}^3$ and where

$$(1.2) \quad E = E(F) = \frac{1}{2} (F^T F - I).$$

For $n \in \mathbb{N}^*$, we denote by \mathbb{M}^n the set of all $n \times n$ real matrices and by \mathbb{M}_+^n the subset of \mathbb{M}^n of the matrices A verifying $\det A > 0$. For $A \in \mathbb{M}^n$, A^T is the transpose of the matrix A and $\operatorname{tr} A$ the trace of A .

The *positive* constants λ and μ are called the *Lamé constants* of the material under consideration (see [Ci.₁] or [Ci.₂] for a complete point of view about these notions).

Now, we call *Saint Venant-Kirchhoff material*, a homogeneous, isotropic, elastic material, whose response function $\bar{\Sigma}$ associated to the second Piola-Kirchhoff stress tensor is defined by (1.1) where we have

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neglected the term $o(E)$, that is to say such that

$$\widehat{\Sigma}(F) = \lambda(\operatorname{tr} E)I + 2\mu E .$$

One can prove that such a material is a hyperelastic material and that its stored energy function \widehat{W} is given by

$$(1.3) \quad \widehat{W}(F) = \frac{\lambda}{2} (\operatorname{tr} E)^2 + \mu \operatorname{tr} (E^2)$$

(see [Ci.₁] Th. 4.4-3, p. 155 or [Ci.₂] Th. 1.4-7, p. 76).

«... Since Saint Venant-Kirchhoff materials are the simplest among the nonlinear models (in the sense that they are the simplest that are compatible with (1.1)), they are quite popular in actual computations ... but the relative simplicity of their practical implementation is more than compensated by various shortcomings.

... their associated stored energy function is not polyconvex...» (P. G. Ciarlet - 1988).

See [B.], [Ci.₁], [Ci.₂] or [D.] for details about polyconvexity, and [Ra.] for the non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material.

On the other hand, if $W: M_+^3 \rightarrow \mathbb{R}$ is the stored energy function associated to some hyperelastic material, it is suitable for physical reasons, that one has

$$(1.4) \quad \lim_{\det F \rightarrow 0^+} W(F) = +\infty$$

(see [Ci.₁] or [Ci.₂]). In this case, it is usual to extend W to a continuous function defined of M^3 into $\mathbb{R} \cup \{+\infty\}$ by

$$W(F) = +\infty \quad \text{if } \det F \leq 0 .$$

The Saint Venant-Kirchhoff stored energy function does not satisfy (1.4) and thus we can not obtain an extension as above.

«... At their best, Saint Venant-Kirchhoff materials can be only expected to be useful in a narrow range of 'small' strains E , as indeed they should be from their very definition ; this is why such materials are often referred to as 'large displacement-small strain' models. In spite of these various inadequacies, Saint Venant-Kirchhoff materials can be nevertheless expected to perform better than the linearized models that are so often used...» (P. G. Ciarlet - 1988).

It turns out that we can consider the equality (1.3) as a definition of \widehat{W} which is not only valid on a neighbourhood of I , but on M_+^3 and even on the whole set M^3 .

As we have said above, the function \widehat{W} is not polyconvex on M_+^3 ; in fact, we know more since the Saint Venant-Kirchhoff stored energy function is not rank-one-convex on M_+^3 (see [A.] or [Br.]). In this paper, we are interested in *the rank-one convexity domain* of \widehat{W} , in other terms in the greatest subset of M^n on which \widehat{W} is rank-one-convex, and we are going to characterize exactly this domain.

So, from now on, we will denote by \widehat{W} the function defined on M^n (with $n \geq 2$) by the equality (1.3).

2. The «Legendre-Hadamard» condition for \widehat{W} .

First, let us recall the definition of rank-one-convexity on a subset U of M^n . A function $W: M^n \rightarrow \mathbb{R}$ is said *rank-1-convex on U* if

$$\forall F, G \in U \quad \text{such that } [F, G] \subset U \quad \text{and } \text{rank}(F - G) = 1$$

one has

$$\forall \lambda \in [0, 1], \quad W(\lambda F + (1 - \lambda)G) \leq \lambda W(F) + (1 - \lambda)W(G).$$

It is well known that, if W is twice continuously differentiable, then W is rank-1-convex on U if and only if one has

$$(2.1) \quad \inf_{a, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 W(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geq 0, \quad \forall F \in U.$$

(*Legendre-Hadamard condition on U*) (see [B.] Th. 3.3, p. 352).

Now, let us introduce some notation; for $F, G \in M^n$ we set $F: G = \text{tr}(F^T G)$ the matrix inner product in M^n and $|F| = (F: F)^{1/2}$ the corresponding norm.

Next, if $a, b \in \mathbb{R}^n$ let us denote by $a \otimes b$ the $n \times n$ matrix defined by

$$(a \otimes b)_{ij} = a_i b_j.$$

Therefore, if the vectors a and b are column matrices one has

$$(2.2) \quad a \otimes b = ab^T$$

and the euclidean inner product in \mathbb{R}^n (respectively the euclidean norm in \mathbb{R}^n) can be written

$$(2.3) \quad a \cdot b = a^T b \quad (\text{respectively } |a| = (a^T a)^{1/2}).$$

Using (2.2) et (2.3) it is easy to verify that, for $F \in \mathbb{M}^n$ and $a, b, c, d \in \mathbb{R}^n$, the following identities hold:

$$(2.4) \quad (a \otimes b)^T = b \otimes a,$$

$$(2.5) \quad F(a \otimes b) = (Fa) \otimes b,$$

$$(2.6) \quad (a \otimes b)F = a \otimes (F^T b),$$

$$(2.7) \quad (a \otimes b)(c \otimes d) = (b \cdot c)(a \otimes d),$$

$$(2.8) \quad \text{tr}(a \otimes b) = a \cdot b,$$

$$(2.9) \quad F : (a \otimes b) = Fb \cdot a = F^T a \cdot b,$$

$$(2.10) \quad (a \otimes b) : (c \otimes d) = (a \cdot c)(b \cdot d),$$

$$(2.11) \quad |a \otimes b|^2 = |a|^2 |b|^2.$$

Since it is clear that the Saint Venant-Kirchhoff stored energy function \widehat{W} is indefinitely differentiable we can compute $\frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}}$ and use the criterion (2.1) to study the rank-1-convexity of \widehat{W} . We have the following result:

LEMMA 2.1. *For $F \in \mathbb{M}^n$ and $a, b \in \mathbb{R}^n$ one has*

$$(2.12) \quad \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l =$$

$$= \left(\frac{\lambda}{2} (|F|^2 - n) |b|^2 + \mu (|Fb|^2 - |b|^2) \right) |a|^2 (\lambda + \mu) (F^T a \cdot b)^2 + \mu |F^T a|^2 |b|^2.$$

PROOF. Let us consider $F \in \mathbb{M}^n$ and $a, b \in \mathbb{R}^n$. First, since $E = E^T$ (where E is defined by (1.2)), we can write

$$\widehat{W}(F) = \frac{\lambda}{2} (\operatorname{tr} E)^2 + \mu |E|^2.$$

Next, for a matrix $H \in \mathbb{M}^n$ the quantity

$$\frac{1}{2} \sum_{i,j,k,l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} H_{ij} H_{kl} \stackrel{\text{def}}{=} \frac{1}{2} \varphi_F(H)$$

is the second-order term of the Taylor expansion in H of $\widehat{W}(F + H)$.

We have

$$(2.13) \quad E(F + H) = \frac{1}{2} (F^T F - I + F^T H + H^T F + H^T H)$$

thus

$$\operatorname{tr}(E(F + H)) = \frac{1}{2} (|F|^2 - n + 2F:H + |H|^2)$$

then the second-order term in H of $(\operatorname{tr}(E(F + H)))^2$ is

$$\frac{1}{4} ((2F:H)^2 + 2(|F|^2 - n) |H|^2).$$

On the other hand, by using (2.13) and the equality $|A + B|^2 = |A|^2 + |B|^2 + 2A:B$ we see that the second-order term in H of $|E(F + H)|^2$ is

$$\frac{1}{4} (|F^T H + H^T F|^2 + 2(F^T F - I):H^T H)$$

therefore we deduce that for any matrix $H \in \mathbb{M}^n$, one has

$$(2.14) \quad \varphi_F(H) = \frac{\lambda}{2} (2(F:H)^2 + (|F|^2 - n) |H|^2) + \frac{\mu}{2} (|F^T H + H^T F|^2 + 2(F^T F - I):H^T H).$$

Now, for $a, b \in \mathbb{R}^n$ and $H = a \otimes b$, (2.14) can be written

$$\begin{aligned} \varphi_F(a \otimes b) &= \frac{\lambda}{2} (2(F^T(a \otimes b))^2 + (|F|^2 - n) |a \otimes b|^2) + \\ &+ \frac{\mu}{2} (|F^T(a \otimes b) + (a \otimes b)^T F|^2 + 2(F^T F - I):(a \otimes b)^T(a \otimes b)) \end{aligned}$$

which becomes, by using the equalities (2.4)-(2.11)

$$\begin{aligned} (2.15) \quad \varphi_F(a \otimes b) &= \frac{\lambda}{2} (2(F^T a \cdot b)^2 + (|F|^2 - n) |a|^2 |b|^2) + \\ &+ \frac{\mu}{2} (|(F^T a) \otimes b + b \otimes (F^T a)|^2 + 2(F^T F - I): |a|^2(b \otimes b)). \end{aligned}$$

But, one has

$$\begin{aligned} |(F^T a) \otimes b + b \otimes (F^T a)|^2 &= |(F^T a) \otimes b|^2 + |b \otimes (F^T a)|^2 + \\ &+ 2((F^T a) \otimes b):(b \otimes (F^T a)) = 2|F^T a|^2 |b|^2 + 2(F^T a \cdot b)^2 \end{aligned}$$

and

$$\begin{aligned} (F^T F - I): |a|^2(b \otimes b) &= |a|^2(F^T F:(b \otimes b) - I:(b \otimes b)) = \\ &= |a|^2(|Fb|^2 - |b|^2) \end{aligned}$$

in such way that (2.15) gives

$$\begin{aligned} \varphi_F(a \otimes b) &= \left(\frac{\lambda}{2} (|F|^2 - n) |b|^2 + \mu(|Fb|^2 - |b|^2) \right) |a|^2 + \\ &+ (\lambda + \mu)(F^T a \cdot b)^2 + \mu |F^T a|^2 |b|^2. \end{aligned}$$

The proof is now complete. ■

Before to go on, let us recall what are the singular values of a matrix. Let $F \in \mathbb{M}^n$; if we denote by $\alpha_1, \dots, \alpha_n$ the eigenvalues of the symmetric matrix $F^T \cdot F$, one has $\forall i = 1, \dots, n$, $\alpha_i \in \mathbb{R}_+$ and the n non-negative numbers $v_i = \sqrt{\alpha_i}$ are called the *singular values* of F . Then, for some orthogonal matrices Q_1 and Q_2 , we have

$$(2.16) \quad F = Q_1 \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{pmatrix} Q_2.$$

Next, we set $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$.

PROPOSITION 2.1. *Let $F \in \mathbb{M}^n$ and v_1, \dots, v_n the singular values of F . Then*

$$(2.17) \quad \inf_{a, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geq 0 \Leftrightarrow \inf_{a, b \in S^{n-1}} f_{v_1, \dots, v_n}(a, b) \geq 0$$

where $f_{v_1, \dots, v_n}(a, b)$ is given by

$$f_{v_1, \dots, v_n}(a, b) = \frac{\lambda}{2} (v_1^2 + \dots + v_n^2) + \mu (v_1^2 (a_1^2 + b_1^2) + \dots + v_n^2 (a_n^2 + b_n^2)) + (\lambda + \mu) (v_1 a_1 b_1 + \dots + v_n a_n b_n)^2 - \left(\frac{n\lambda}{2} + \mu \right).$$

PROOF. Since the functions

$$a \mapsto \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \quad \text{and} \quad b \mapsto \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l$$

are homogeneous polynomials of degree 2, we can write

$$(2.18) \quad \inf_{a, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \inf_{a, b \in S^{n-1}} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \inf_{a, b \in S^{n-1}} \left[\frac{\lambda}{2} (|F|^2 - n) + \mu (|F^T a|^2 + |Fb|^2 - 1) + (\lambda + \mu) (F^T a \cdot b)^2 \right] \geq 0$$

by using, for $a, b \in S^{n-1}$, the identity (2.12) of the previous lemma.

Now, if we use (2.16) and denote by V the matrix

$$\begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{pmatrix},$$

we get

$$\begin{aligned} (2.18) &\Leftrightarrow \inf_{a, b \in \mathcal{S}^{n-1}} \left[\frac{\lambda}{2} (|\mathcal{Q}_1 V \mathcal{Q}_2|^2 - n) + \mu (|\mathcal{Q}_2^T V \mathcal{Q}_1^T a|^2 + |\mathcal{Q}_1 V \mathcal{Q}_2 b|^2 - 1) + \right. \\ &\qquad \qquad \qquad \left. + (\lambda + \mu) (\mathcal{Q}_2^T V \mathcal{Q}_1^T a \cdot b)^2 \right] \geq 0 \\ &\Leftrightarrow \inf_{a, b \in \mathcal{S}^{n-1}} \left[\frac{\lambda}{2} (|V|^2 - n) + \mu (|V(\mathcal{Q}_1^T a)|^2 + |V(\mathcal{Q}_2 b)|^2 - 1) + \right. \\ &\qquad \qquad \qquad \left. + (\lambda + \mu) (V(\mathcal{Q}_1^T a) \cdot (\mathcal{Q}_2 b))^2 \right] \geq 0 \\ &\Leftrightarrow \inf_{a, b \in \mathcal{S}^{n-1}} \left(\frac{\lambda}{2} (|V|^2 - n) + \mu (|Va|^2 + |Vb|^2 - 1) + (\lambda + \mu) (Va \cdot b)^2 \right) \geq 0 \\ &\Leftrightarrow \inf_{a, b \in \mathcal{S}^{n-1}} f_{v_1, \dots, v_n}(a, b) \geq 0. \end{aligned}$$

So, we have (2.17). \blacksquare

Now, we are able to obtain some results about the rank-1-convexity domain of \widehat{W} .

3. The rank-one convexity domain of \widehat{W} .

Let us denote by $\widehat{\mathcal{Q}}$ the rank-1-convexity domain of \widehat{W} . The Legendre-Hadamard condition (2.1) implies that

$$\widehat{\mathcal{Q}} = \left\{ F \in \mathbb{M}^n; \inf_{a, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geq 0 \right\}$$

which can be written, by using proposition 2.1

$$(3.1) \quad \widehat{\mathcal{Q}} = \{F \in \mathbb{M}^n; \inf_{a, b \in \mathbb{S}^{n-1}} f_{v_1, \dots, v_n}(a, b) \geq 0\}$$

where v_1, \dots, v_n are the singular values of F .

For a matrix $F \in \mathbb{M}^n$ we would like to obtain necessary and sufficient conditions, in terms of its singular values, in order to have $F \in \widehat{\mathcal{Q}}$.

Our main result will rely on the following lemma:

LEMMA 3.1. *Let v_1, \dots, v_n be some real numbers such that $0 \leq v_1 \leq \dots \leq v_n$, and g be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by*

$$g(a, b) = \mu(v_1^2(a_1^2 + b_1^2) + \dots + v_n^2(a_n^2 + b_n^2)) + (\lambda + \mu)(v_1 a_1 b_1 + \dots + v_n a_n b_n)^2.$$

Next, let us denote by K the following set

$$K = \{(i, j) \in \mathbb{N}^2; 1 \leq i < j \leq n \text{ and } (\lambda + 2\mu)v_i \geq \mu v_j\}.$$

If $\mathfrak{J}_g = \inf_{x, y \in \mathbb{S}^{n-1}} g(x, y)$ then

$$\mathfrak{J}_g = \begin{cases} (\lambda + 3\mu)v_1^2 & \text{if } (1, 2) \notin K, \\ \min_{(i, j) \in K} \frac{\mu}{\lambda + \mu} (-\mu v_i^2 + 2(\lambda + 2\mu)v_i v_j - \mu v_j^2) & \text{if } (1, 2) \in K. \end{cases}$$

PROOF. First of all, let us denote by e^1, \dots, e^n the canonical basis of \mathbb{R}^n , and recall that the constantes λ, μ are positive.

If $v_1 = 0$, then $g(e^1, e^1) = 0$ and since $g \geq 0$, we get $\mathfrak{J}_g = 0$, in such way that the lemma is proved in this case.

So, from now on, we will assume $v_1 > 0$, and we will denote by I_1, \dots, I_p the subsets of $\{1, \dots, n\}$ defined by $I_1 \cup \dots \cup I_p = \{1, \dots, n\}$ and

$$\begin{cases} i, j \in I_k \Rightarrow v_i = v_j, \\ i \in I_k, j \in I_l \text{ and } k < l \Rightarrow v_i < v_j. \end{cases}$$

Since $S^{n-1} \times S^{n-1}$ is a compact set,

$$(3.2) \quad \mathfrak{J}_g = g(a, b)$$

for some $(a, b) \in S^{n-1} \times S^{n-1}$.

Now, let us divide the rest of the proof in two steps.

Step 1. Here we are going to give necessary conditions in order to have (3.2). So, let $(a, b) \in S^{n-1} \times S^{n-1}$ satisfying (3.2); then, there exist Lagrange multipliers α and β such that $\forall i = 1, \dots, n$ one has

$$\begin{aligned} \frac{\partial}{\partial a_i} (g(a, b) + \alpha(|a|^2 - 1) + \beta(|b|^2 - 1)) &= \\ &= \frac{\partial}{\partial b_i} (g(a, b) + \alpha(|a|^2 - 1) + \beta(|b|^2 - 1)) = 0. \end{aligned}$$

Therefore, $\forall i = 1, \dots, n$ one has

$$(3.3_i) \quad \mu v_i^2 a_i + (\lambda + \mu) c v_i b_i + \alpha a_i = 0$$

and

$$(3.4_i) \quad \mu v_i^2 b_i + (\lambda + \mu) c v_i a_i + \beta b_i = 0$$

where we have set $c = \sum_{j=1}^n v_j a_j b_j$.

Now, multiplying (3.3_i) by b_i/v_i and adding for $i = 1, \dots, n$ we get

$$(3.5) \quad (\lambda + 2\mu) c + \alpha \sum_{i=1}^n \frac{a_i b_i}{v_i} = 0.$$

From (3.4_i) we obtain, by the same way

$$(3.6) \quad (\lambda + 2\mu) c + \beta \sum_{i=1}^n \frac{a_i b_i}{v_i} = 0.$$

So, (3.5) and (3.6) imply that

$$(\alpha - \beta) \sum_{i=1}^n \frac{a_i b_i}{v_i} = 0.$$

— In a first case, let us assume that one has $\sum_{i=1}^n \frac{a_i b_i}{v_i} = 0$; then by (3.5), $c = 0$. Therefore, (3.3_i) and (3.4_i) can be written

$$(\mu v_i^2 + \alpha) a_i = 0 \quad \text{and} \quad (\mu v_i^2 + \beta) b_i = 0$$

and thus, since these equalities are valid for all $i = 1, \dots, n$, and that α (respectively β) does not depend on i , we get necessarily, for some k, l

$$a = \sum_{i \in I_k} a_i e^i \quad \text{and} \quad b = \sum_{i \in I_l} b_i e^i.$$

Consequently, since $c = 0$ and $a, b \in S^{n-1}$, we easily deduce

$$(3.7) \quad g(a, b) = \mu(v_i^2 + v_j^2) \quad \text{for some } i, j.$$

— Now, let us assume that $\alpha = \beta$; then we can suppose $c \neq 0$ (indeed, if $c = 0$, then thanks to (3.3_i), we obtain $\alpha \neq 0$ and by (3.5), $\sum_{i=1}^n \frac{a_i b_i}{v_i} = 0$ which brought us back to the previous case). Next, if we multiply (3.3_i) by b_i and (3.4_i) by a_i , we get by subtraction

$$(3.8) \quad (\lambda + \mu) c v_i (b_i^2 - a_i^2) = 0 \quad \text{implying that } b_i^2 = a_i^2 \quad \forall i.$$

Let $i \in \{1, \dots, n\}$; using (3.8), one has, either

$$(3.9) \quad \left\{ \begin{array}{l} a_i = b_i = 0 \\ \text{or} \\ a_i = b_i \neq 0 \text{ and thus } c = -\frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} \text{ by (3.3}_i) \\ \text{or} \\ a_i = -b_i \neq 0 \text{ and thus } c = \frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} \text{ by (3.3}_i). \end{array} \right.$$

Let us remark that the equality

$$\frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} = \frac{\alpha + \mu v_j^2}{(\lambda + \mu) v_j}$$

implies $(v_j - v_i)(\alpha - \mu v_i v_j) = 0$ and $v_i = v_j$ (since, if we multiply (3.3_i) by a_i , and add for $i = 1, \dots, n$, we get $\mu \sum_{i=1}^n v_i^2 a_i^2 + (\lambda + \mu) c^2 + \alpha = 0$ and $\alpha \leq 0$).

Consequently, if we denote by A and B the following subsets of $\{1, \dots, n\}$

$$A = \{i ; a_i = b_i \neq 0\} \quad \text{and} \quad B = \{i ; a_i = -b_i \neq 0\}$$

we get for some $k, l \in \{1, \dots, p\}$

$$A \subset I_k \quad \text{and} \quad B \subset I_l.$$

$$(3.10) \quad \rightarrow \text{ If } A = \emptyset \text{ then } g(a, b) = (\lambda + 3\mu) v_j^2 \text{ for } j \in I_l.$$

$$(3.11) \quad \rightarrow \text{ If } B = \emptyset \text{ then } g(a, b) = (\lambda + 3\mu) v_j^2 \text{ for } j \in I_k.$$

\rightarrow If $A \neq \emptyset$ and $B \neq \emptyset$ then $k \neq l$ (indeed, if $k = l$, then (3.9) implies $c = 0$, but we have supposed the opposite). So, a and b are of the form

$$(3.12) \quad a = \sum_{i \in I_k} a_i e^i + \sum_{i \in I_l} a_i e^i \quad \text{and} \quad b = \sum_{i \in I_k} a_i e^i - \sum_{i \in I_l} a_i e^i$$

and thus, if $i \in I_k$ and $j \in I_l$, we have $i \neq j$ and

$$(3.13) \quad g(a, b) = 2\mu(v_i^2 x^2 + v_j^2 y^2) + (\lambda + \mu) c^2$$

where $x^2 = \sum_{s \in I_k} a_s^2$ and $y^2 = \sum_{s \in I_l} a_s^2$.

But, using (3.9), we can write

$$c = -\frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} = \frac{\alpha + \mu v_j^2}{(\lambda + \mu) v_j}$$

which gives easily

$$(3.14) \quad \alpha = -\mu v_i v_j \quad \text{and} \quad c = \frac{\mu}{\lambda + \mu} (v_j - v_i).$$

On the other hand, by definition of c and since $y^2 = 1 - x^2$, we have

$$c = v_i x^2 - v_j y^2 = (v_i + v_j) x^2 - v_j$$

implying that

$$(3.15) \quad x^2 = \frac{-\mu v_i + (\lambda + 2\mu)v_j}{(\lambda + \mu)(v_i + v_j)} \quad \text{and} \quad y^2 = \frac{(\lambda + 2\mu)v_i - \mu v_j}{(\lambda + \mu)(v_i + v_j)}.$$

By using (3.13), (3.14) and (3.15) we get

$$(3.16) \quad g(a, b) = \frac{\mu}{\lambda + \mu} (-\mu v_i^2 + 2(\lambda + 2\mu)v_i v_j - \mu v_j^2).$$

So, we have proved that, if $\mathfrak{J}_g = g(a, b)$ then necessarily, $g(a, b)$ is given by (3.7), (3.10), (3.11) or (3.16).

Step 2. Now, we can compute \mathfrak{J}_g according to the values of v_1/v_2 .

First, let us denote by h the function

$$h(x, y) = \frac{\mu}{\lambda + \mu} (-\mu x^2 + 2(\lambda + 2\mu)xy - \mu y^2).$$

Now, let us remark that, if $K = \emptyset$, then (3.15) is not possible for $i \neq j$ and thus (3.12) does not hold. Consequently, thanks to (3.7), (3.10), (3.11) and (3.16) we get

$$(3.17) \quad \mathfrak{J}_g = \begin{cases} \min [(\lambda + 3\mu)v_1^2; \mu(v_1^2 + v_2^2)] & \text{if } K = \emptyset, \\ \min [(\lambda + 3\mu)v_1^2; \mu(v_1^2 + v_2^2); \min_{(i,j) \in K} h(v_i, v_j)] & \text{if } K \neq \emptyset. \end{cases}$$

— If $(1, 2) \in K$ then

$$\min_{(i,j) \in K} h(v_i, v_j) \leq h(v_1, v_2) \leq \min [(\lambda + 3\mu)v_1^2; \mu(v_1^2 + v_2^2)]$$

the last inequality arising from the following identities

$$\begin{cases} h(v_1, v_2) - \mu(v_1^2 + v_2^2) = -\frac{\mu(\lambda + 2\mu)}{\lambda + \mu} (v_1 - v_2)^2, \\ h(v_1, v_2) - (\lambda + 3\mu)v_1^2 = -\frac{1}{\lambda + \mu} (\mu v_1 - (\lambda + 2\mu)v_2)^2. \end{cases}$$

So, by using (3.17) we obtain

$$(1, 2) \in K \Rightarrow \mathfrak{J}_g = \min_{(i,j) \in K} h(v_i, v_j).$$

— If $(1, 2) \notin K$ then

$$(\lambda + 3\mu)v_1^2 - \mu(v_1^2 + v_2^2) = (\lambda + 2\mu)v_1^2 - \mu v_2^2 < 0$$

(indeed, $\frac{v_1}{v_2} < \frac{\mu}{\lambda + 2\mu} < 1$ and thus $\frac{v_1^2}{v_2^2} < \frac{\mu}{\lambda + 2\mu}$ which implies that $(\lambda + 2\mu)v_1^2 < \mu v_2^2$).

Therefore, taking into account of (3.17), we can write when $(1, 2) \notin K$

$$\mathfrak{J}_g = \begin{cases} (\lambda + 3\mu)v_1^2 & \text{if } K = \emptyset, \\ \min [(\lambda + 3\mu)v_1^2; \min_{(i,j) \in K} h(v_i, v_j)] & \text{if } K \neq \emptyset. \end{cases}$$

Finally, to conclude, it remains to show that

$$(3.18) \quad \forall (i, j) \in K, \quad h(v_i, v_j) - (\lambda + 3\mu)v_1^2 \geq 0$$

holds when $K \neq \emptyset$ and $(1, 2) \notin K$.

Let $(i, j) \in K$; then

$$(3.19) \quad (\lambda + 2\mu)v_i - \mu v_j \geq 0.$$

Next, $(1, 2) \notin K$ and thus

$$(3.20) \quad (\lambda + 2\mu)v_1 \leq \mu v_2 \leq \mu v_j.$$

By using (3.19) and (3.20) we get

$$((\lambda + 2\mu)v_1 - \mu v_j)((\lambda + 2\mu)v_i - \mu v_j) \leq 0$$

that we can write

$$\mu^2 v_j^2 - \mu(\lambda + 2\mu)v_i v_j \leq -(\lambda + 2\mu)^2 v_1 v_i + \mu(\lambda + 2\mu)v_1 v_j.$$

Consequently,

$$\begin{aligned} h(v_i, v_j) - (\lambda + 3\mu)v_1^2 &= \\ &= -\frac{1}{\lambda + \mu}(\mu^2 v_i^2 - 2\mu(\lambda + 2\mu)v_i v_j + \mu^2 v_j^2 + (\lambda + 3\mu)(\lambda + \mu)v_1^2) \geq \end{aligned}$$

$$\begin{aligned}
&\geq -\frac{1}{\lambda + \mu} (\mu^2 v_i^2 - \mu(\lambda + 2\mu) v_i v_j + (\lambda + 3\mu)(\lambda + \mu) v_1^2 - \\
&\qquad\qquad\qquad - (\lambda + 2\mu)^2 v_1 v_i + \mu(\lambda + 2\mu) v_1 v_j) = \\
&= -\frac{1}{\lambda + \mu} (\mu^2 v_i^2 - \mu(\lambda + 2\mu) v_i v_j + (\lambda + 2\mu)^2 v_1^2 - \mu^2 v_1^2 - \\
&\qquad\qquad\qquad - (\lambda + 2\mu)^2 v_1 v_i + \mu(\lambda + 2\mu) v_1 v_j) = \\
&= -\frac{1}{\lambda + \mu} (\mu^2 (v_i - v_1)(v_i + v_1) - \mu(\lambda + 2\mu) v_j (v_i - v_1) - \\
&\qquad\qquad\qquad - (\lambda + 2\mu)^2 v_1 (v_i - v_1)) = \\
&= \frac{v_i - v_1}{\lambda + \mu} (\mu(\lambda + 2\mu) v_j + (\lambda + 2\mu)^2 v_1 - \mu^2 (v_i + v_1)) = \\
&= \frac{v_i - v_1}{\lambda + \mu} ((\lambda + 3\mu)(\lambda + \mu) v_1 + \mu(\lambda + 2\mu) v_j - \mu^2 v_i) \geq 0
\end{aligned}$$

since $v_j \geq v_i$ and $v_i \geq v_1$.

So, (3.18) holds, and the proof is complete. ■

The following theorem gives the requested characterization of $\widehat{\mathfrak{Q}}$.

THEOREM 3.1. *Let $F \in \mathbb{M}^n$ and $v_1 \leq \dots \leq v_n$ be the singular values of F . Let us denote by $K(F)$ the following set*

$$K(F) = \{(i, j) \in \mathbb{N}^2; 1 \leq i < j \leq n \text{ and } (\lambda + 2\mu) v_i \geq \mu v_j\}.$$

(i) *If $(1, 2) \notin K(F)$ then, $F \in \widehat{\mathfrak{Q}}$ if and only if*

$$(\lambda + 3\mu) v_1^2 \geq \left(\frac{n\lambda}{2} + \mu \right) - \frac{\lambda}{2} (v_1^2 + \dots + v_n^2).$$

(ii) *If $(1, 2) \in K(F)$ then, $F \in \widehat{\mathfrak{Q}}$ if and only if*

$$\begin{aligned}
\min_{(i, j) \in K(F)} \frac{\mu}{\lambda + \mu} (-\mu v_i^2 + 2(\lambda + 2\mu) v_i v_j - \mu v_j^2) &\geq \\
&\geq \left(\frac{n\lambda}{2} + \mu \right) - \frac{\lambda}{2} (v_1^2 + \dots + v_n^2).
\end{aligned}$$

PROOF. This follows immediatly from (3.1), Proposition 2.1 and Lemma 3.1. ■

REMARK 3.1. Clearly, one can see that $\widehat{\mathcal{D}}$ is a neighbourhood of the identity matrix of \mathbb{M}^n .

4. Characterization of $\widehat{\mathcal{D}}$ in the two-dimensional case.

In this last section, we will assume that $n = 2$. We would like to precise the results of the previous part, and to give a representation in $\mathbb{R}_+ \times \mathbb{R}_+$ of the following set

$$\widehat{\mathcal{V}} = \left\{ (v_1, v_2) \in \mathbb{R}_+ \times \mathbb{R}_+ ; \forall Q_1, Q_2 \in O(2), Q_1 \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} Q_2 \in \widehat{\mathcal{D}} \right\}$$

(where $O(2)$ is the set of all the 2×2 orthogonal matrices).

The first theorem is the two-dimensional version of Theorem 3.1.

THEOREM 4.1. *Let $F \in \mathbb{M}^2$ and v_1, v_2 be the singular values of F .*

(i) *If $\frac{v_1}{v_2} \in \left[0, \frac{\mu}{\lambda + 2\mu} \right]$ then,*

$$F \in \widehat{\mathcal{D}} \Leftrightarrow 3 \left(\frac{\lambda}{2} + \mu \right) v_1^2 + \frac{\lambda}{2} v_2^2 \geq \lambda + \mu.$$

(ii) *If $\frac{v_1}{v_2} \in \left[\frac{\mu}{\lambda + 2\mu}, \frac{\lambda + 2\mu}{\mu} \right]$ then,*

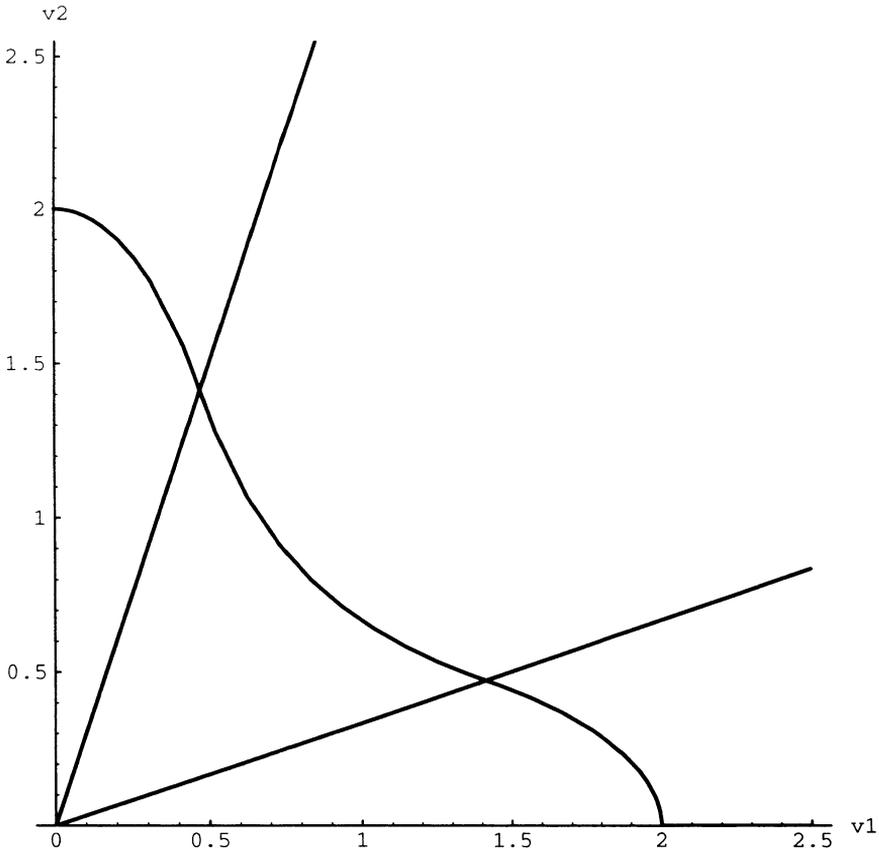
$$F \in \widehat{\mathcal{D}} \Leftrightarrow \frac{\lambda - \mu}{2} (v_1^2 + v_2^2) + 2\mu v_1 v_2 \geq \frac{(\lambda + \mu)^2}{\lambda + 2\mu}.$$

(iii) *If $\frac{v_1}{v_2} \in \left[\frac{\lambda + 2\mu}{\mu}, +\infty \right]$ then,*

$$F \in \widehat{\mathcal{D}} \Leftrightarrow \frac{\lambda}{2} v_1^2 + 3 \left(\frac{\lambda}{2} + \mu \right) v_2^2 \geq \lambda + \mu.$$

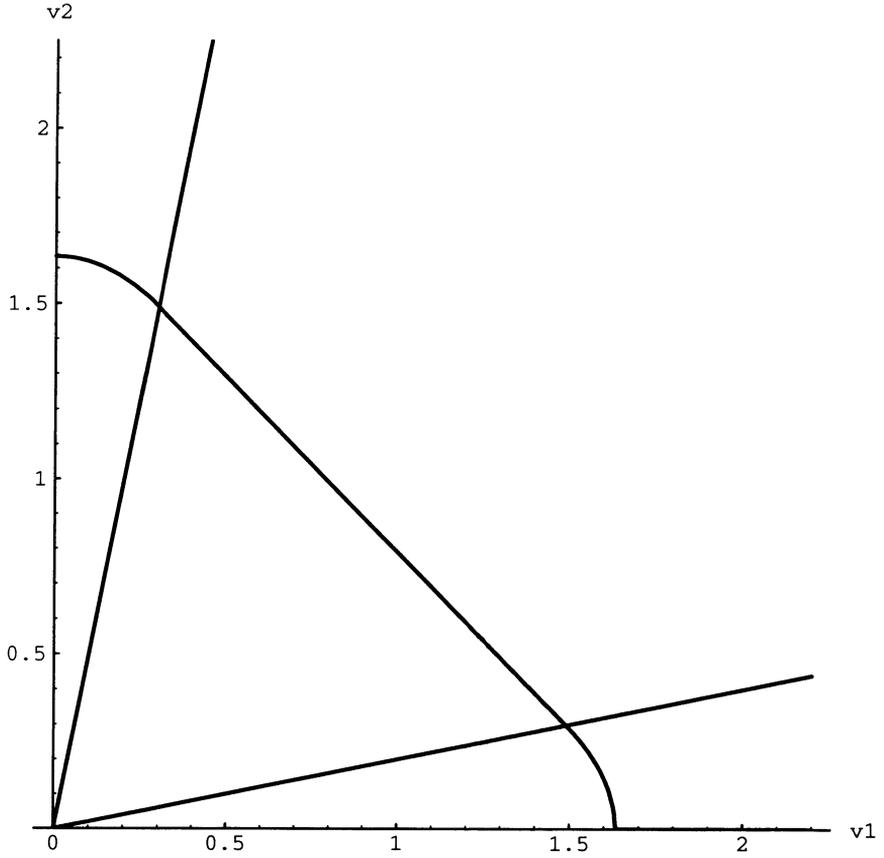
PROOF. This follows immediatly from Theorem 3.1. Let us note that we have not assumed $v_1 \leq v_2$, which explains the three cases (i), (ii) and (iii). ■

On each of the following figures, the region located above the curve represent the set $\widehat{\mathcal{V}}$ for some values of the quotient λ/μ .



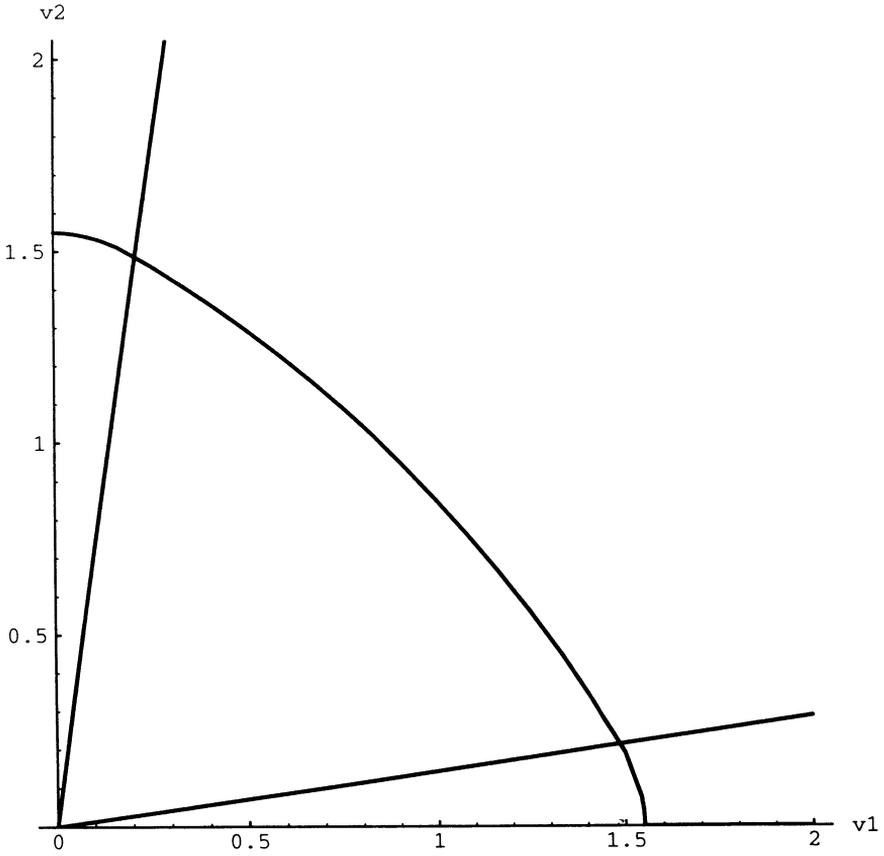
$$(v_1, v_2) \in \widehat{\mathcal{V}} \Leftrightarrow \begin{cases} 9v_1^2 + v_2^2 \geq 4 & \text{when } 0 \leq \frac{v_1}{v_2} \leq \frac{1}{3}, \\ v_1 v_2 \geq \frac{2}{3} & \text{when } \frac{1}{3} \leq \frac{v_1}{v_2} \leq 3, \\ v_1^2 + 9v_2^2 \geq 4 & \text{when } 3 \leq \frac{v_1}{v_2} \leq +\infty. \end{cases}$$

Figure 1. - $\frac{\lambda}{\mu} = 1$.



$$(v_1, v_2) \in \widehat{\mathcal{V}} \Leftrightarrow \begin{cases} 15v_1^2 + 3v_2^2 \geq 8 & \text{when } 0 \leq \frac{v_1}{v_2} \leq \frac{1}{5}, \\ v_1 + v_2 \geq \frac{4}{\sqrt{5}} & \text{when } \frac{1}{5} \leq \frac{v_1}{v_2} \leq 5, \\ 3v_1^2 + 15v_2^2 \geq 8 & \text{when } 5 \leq \frac{v_1}{v_2} \leq +\infty. \end{cases}$$

Figure 2. - $\frac{\lambda}{\mu} = 3$.



$$(v_1, v_2) \in \widehat{\mathfrak{V}} \Leftrightarrow \begin{cases} 21v_1^2 + 5v_2^2 \geq 12 & \text{when } 0 \leq \frac{v_1}{v_2} \leq \frac{1}{7}, \\ v_1^2 + v_2^2 + v_1 v_2 \geq \frac{18}{17} & \text{when } \frac{1}{7} \leq \frac{v_1}{v_2} \leq 7, \\ 5v_1^2 + 21v_2^2 \geq 12 & \text{when } 7 \leq \frac{v_1}{v_2} \leq +\infty. \end{cases}$$

Figure 3. - $\frac{\lambda}{\mu} = 5$.

Now, for $F \in \mathbb{M}^2$ and $v_1 \leq v_2$ its singular values, one has

$$(4.1) \quad v_1^2 + v_2^2 = |F|^2 \quad \text{and} \quad v_1 v_2 = \det F.$$

Moreover, if we denote by $\|\cdot\|$ the matrix norm defined on \mathbb{M}^2 by

$$\|F\|^2 = \sup_{|x|=1} |Fx|^2$$

where $|\cdot|$ denotes the euclidean norm (see (2.3)). It is well known that $\|F\| = v_2$ and thus

$$(\lambda + 2\mu)v_1 \geq \mu v_2 \Leftrightarrow (\lambda + 2\mu) \det F \geq \mu \|F\|^2$$

in such way that Theorem 4.1 leads to:

THEOREM 4.2. *Let $F \in \mathbb{M}^2 \setminus \{0\}$.*

(i) *If $(\lambda + 2\mu) \det F \leq \mu \|F\|^2$, then $F \in \widehat{\mathcal{Q}}$ if and only if*

$$\frac{\lambda}{2} \|F\|^4 - (\lambda + \mu) \|F\|^2 + 3\left(\frac{\lambda}{2} + \mu\right)(\det F)^2 \geq 0.$$

(ii) *If $(\lambda + 2\mu) \det F \geq \mu \|F\|^2$, then $F \in \widehat{\mathcal{Q}}$ if and only if*

$$\frac{\lambda - \mu}{2} |F|^2 + 2\mu \det F \geq \frac{(\lambda + \mu)^2}{\lambda + 2\mu}.$$

PROOF. It is immediate since

$$v_1 = \frac{\det F}{\|F\|} \quad (\text{if } F \neq 0) \quad \text{and} \quad v_2 = \|F\|. \quad \blacksquare$$

REMARK 4.1. This can be express also without using the norm $\|\cdot\|$; indeed (4.1) implies that v_1^2 and v_2^2 are the solutions of the following equations

$$X^2 - |F|^2 X + (\det F)^2 = 0$$

and thus

$$v_i^2 = \frac{1}{2} \left(|F|^2 \pm \sqrt{|F|^4 - 4(\det F)^2} \right).$$

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