RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

N. BRUNNER

A modal logic of consistency

Rendiconti del Seminario Matematico della Università di Padova, tome 93 (1995), p. 143-152
<http://www.numdam.org/item?id=RSMUP_1995__93__143_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 93 (1995)

A Modal Logic of Consistency.

N. BRUNNER (*)

ABSTRACT - The modal propositional logic which is defined by the interpretation of possibility as consistency by means of the Fraenkel-Mostowski method for proving independence results about the axiom of choice in *ZFA* set theory is Lewis' system S 5.

1. Introduction.

Mc Dermott's theory of non-monotonic reasoning motivates investigations of modal logics of consistency. As follows from Solovay [12], if in the context of *PA* the modality $\Diamond$ is interpreted as consistency, then the corresponding modal sentential calculus is **G**. As Solovay has noted, semantical restrictions lead to extensions of this system. It is possible, however, to give an interpretation of **S 5**, as has been shown by Forster [7], who in the context of *NF* has interpreted the modality $\Diamond$ as consistency by means of Bernays-Rieger permutation models. The purpose of this note is a proof of a similar result for Fraenkel-Mostowski permutation models. The major step in this proof is the observation, that iterations of this construction again give Fraenkel-Mostowski models.

The main result of this paper might appear to be in contrast with the proof of Solovay's theorem, where the use of a formalized notion of consistency enforces the validity of **W** $(\Box(\Box\phi \to \phi) \to \Box\phi)$. In the present context, however, models are proper classes for which only local notions of truth are available. The assertion «α is true in the permutation model *PM*» is expressed by the relativization of α to *PM*. It is investigated in the set theory *FM* of all *ZF*-sentences which are valid in

(*) Indirizzo dell'A.: Dept. Math., Univ. Bodenkultur, Gregor - Mendel-Str. 33, A-1180 Wien.

each permutation model (as a class of the real world V of ZFC.) The failure of **W** is not due to a peculiarity of FM, but rather to an extended interpretation of the word «model» (c.f. Boolean valued «models».)

1.1 NOTATION. Unless stated otherwise, we shall work in ZFA. ZFA is a variant of ZF set theory without the axiom of choice (AC) which permitts the existence of a set of atoms (objects without elements; c.f. Jech [9].) In the real world V of ZFA from $X \in V$ a ZFA universe $V(X)$ is constructed as follows (our construction is due to J. Truss): $V_0 = X \times \{0\}$, $V_\alpha = \{(A, \alpha) : \alpha \in \mathbf{On}$ minimal such that $A \subseteq \bigcup\{V_\beta : \beta \in \alpha\}\}$, $V(X) = \bigcup\{V_\alpha : \alpha \in \mathbf{On}\}$, and in $V(X)$ $x \in y$, iff in V $y = (z, \alpha)$, $\alpha > 0$ and $x \in z$. The elements of V_0 are atoms, the empty set is $(\emptyset, 1)$. Given a group generated (the unit element has a neighbourhoodbase consisting of open groups) T_2-group $(G, \mathbf{G})$ and an injective homomorphism d: $G \rightarrow S(X)$ into the symmetric group (in V), then a Fraenkel-Mostowski model $PM \subseteq V(X)$ of ZFA is constructed as follows: The group action d is extended recursively to $\hat{d}$ on $V(X)$ via $\hat{d}(g)x = \hat{d}(g)''x = \{\hat{d}(g)y : y \in x\}$ and for $x \in V(X)$ and $x \subseteq PM$ $x \in PM$, iff its stabilizer is open, i.e. $\mathrm{stab}(x) = \{g \in G : (\hat{d}g)x = x\} \in \mathbf{G}$. (We refer to Brunner [4] for more details.) We shall always assume, that PM contains all the atoms in V_0 of $V(X)$; i.e. the topology of pointwise convergence $\mathbf{G}_{\mathrm{fin}} \subseteq \mathbf{G}$ in the notation of [4].

1.2. The axiom of choice for pure sets is preserved in the passage from V to PM. If V satisfies AC, then the validity of choice principles (Boolean combinations of Jech-Sochor bounded statements) depends only on the generating topological group of PM, but not on its group action ([4]). Thus for instance PM satisfies AC, iff its automorphism group is discrete, and PM satisfies the axiom of multiple choice, iff its automorphism group is locally compact (Mathias and [3].) The relevant topology for these results is $\mathbf{G}_{\mathrm{nat}}$ which is generated by the subgroups $\mathrm{stab}(x)$, $x \in PM$: In view of [3] the automorphism group with this topology generates PM (and it contains an isomorphic copy of each generating group) and in view of [4] for each group generated T_2 group $(G, \mathbf{G})$ there is a permutation model such that $\mathbf{G} = \mathbf{G}_{\mathrm{nat}}$ generates that model. This correspondence is not one to one, since a group G and its subgroup H generate the same model, if H is not nowhere dense ([3]).

2. Iterated models.

We now investigate the models which are obtained, if the above constructions are performed within a given Fraenkel-Mostowski model PM. Such models QM come in three steps:

i) Construction of a model $PM \subseteq V(X)$ from a group G as above.

ii) For a set $Y \in PM$ construction of the model $V(Y)$ within PM; the resulting model is $PM(Y) \subseteq V(X)(Y)$.

iii) For a topological group $H \in PM$ and a group action t: $H \rightarrow \rightarrow S^{PM}(Y)$ in PM (S^{PM} is the symmetric group within PM) the construction of a Fraenkel-Mostowski model $QM \subseteq PM(Y)$ within PM; as in (i) we assume $\mathbf{H}_{\text{fin}}^{PM} \subseteq \mathbf{H}^{PM}$ ($\mathbf{H}^{PM}$ denotes the group topology on H in PM).

We shall imitate these construction in V by means of quotients and semidirect products to prove the following result.

THEOREM. *In V there is a Fraenkel-Mostowski model QM' which is $\in$-isomorphic with QM.*

2.1. We first investigate step (ii). A converse of the following lemma is true, too, and will be proved in section 2.2.

LEMMA. *For $Y \in PM$ the structure $PM(Y)$ is $\in$-isomorphic with a Fraenkel-Mostowski model PM' in $V(Y)$ which is generated by a quotient of an open subgroup of G.*

PROOF. By means of Mostowski's collapsing lemma (c.f. Blass, Scedrov [2], Theorem 1B1) the structure $V(X)(Y)$ is isomorphically embedded into $V(Y)$. Since an open subgroup generates the same model as the supergroup (c.f. Brunner, Rubin [3]), for the ease of the notation we shall assume, that PM is generated by $G = \text{stab}\, Y$. We show, that the collapsing image of $PM(Y)$ (also denoted by $PM(Y)$) equals the model $PM' \subseteq V(Y)$ which is generated by the quotient $G' = G | p\text{-stab}\, Y$, where $p\text{-stab}(Y) = \bigcap \{\text{stab}(y) \colon y \in Y\}$ is the pointwise stabilizer of Y, and the induced action d'; i.e. $d'([g])y = \hat{d}(g)y$ for $y \in Y$, $g \in \text{stab}\, Y = G$ and $[g] = g \cdot p\text{-stab}\, Y$. We note, that d' is a welldefined injective homomorphism $G' \rightarrow S(Y)$ and $p\text{-stab}\, Y$ is a closed and normal subgroup of G, whence G' is a T_2-group. Moreover, since for $x \in V(Y)$ by the definition of the quotient topology $\text{stab}_{G'}\, x$ is open, iff $\bigcup \text{stab}_{G'}\, x \in \mathbf{G}$ (c.f. Hewitt, Ross [8], Definition 5.15) and the latter set is $\text{stab}_G\, x$ (definition of d'), it follows, that PM' contains all the atoms in V_0 of $V(Y)$, as does $PM(Y)$, and by induction on the rank it follows, that $PM' = PM(Y)$. For if $x \in V(Y)$ and $x \subseteq PM' \cap PM(Y)$ then $x \in PM(Y)$, iff $\text{stab}_G\, x \in \mathbf{G}$, iff $\text{stab}_{G'}\, x \in \mathbf{G}'$, iff $x \in PM'$. e.o.p.

2.2 REMARK. *If* $\mathbf{G} = \mathbf{G}_{\text{nat}}$, *then under the assumption of AC in V for each closed and normal subgroup H of G there exists a set* $X \in PM$ *such that* $\operatorname{stab} X = G$ *and* $p\text{-}\operatorname{stab} X = H$.

PROOF. $\mathbf{G}_{\text{nat}}$ is the topology which is generated by the subgroups $\operatorname{stab} x$, $x \in PM$. If K is a subgroup of G, then $K \cdot H$ is a subgroup, too, since H is normal. Moreover, if K is open, so is $K \cdot H$. Since the topology $\mathbf{G}$ is group generated, $H = \bigcap \{K \cdot H : K \text{ is an open subgroup of } G\}$, for if $g \in \bigcap \{\dots\}$, say $g = g_K \cdot h_K$ such that $g_K \in K$ and $h_K \in H$, then if the open subgroups K are viewed as a net, ordered by reverse inclusion, it follows that $\lim g_K = 1$, whence $\lim h_K = \lim g_K^{-1} \cdot g = g$ and so $g \in H$, since H is closed. Since $\mathbf{G} = \mathbf{G}_{\text{nat}}$, for each open subgroup K of G there exists a set $x_K \in PM$ such that $\operatorname{stab}(x_K) = K \cdot H$ (c.f. Brunner [4]). We set $X = \bigcup \{\operatorname{orb}_G x_K : K \text{ is an open subgroup of } G\}$, where $\operatorname{orb}_G x = \{\hat{d}(g)x : g \in G\} \in PM$. Since $\operatorname{stab} X = G$, $X \in PM$. We next calculate

$$\begin{aligned} p\text{-}\operatorname{stab} X := {} & \bigcap \{\operatorname{stab}(\hat{d}(g)x_K) : g \in G, K < G \textit{ open}\}, \\ = {} & \bigcap \{g^{-1} \cdot K \cdot H \cdot g : g \in G, K < G \textit{ open}\}, \\ = {} & \bigcap \{g^{-1} \cdot (\bigcap \{K \cdot H : K < G \textit{ open}\}) \cdot g : g \in G\}, \\ = {} & \bigcap \{g^{-1} \cdot H \cdot g : g \in G\} = H, \end{aligned}$$

since H is a normal subgroup of G. e.o.p.

It follows, that each factor group generates some model $PM(X)$. For example, there exists a countable $\aleph_0$-categorical relational structure $\mathfrak{A}$, such that the finite support model PM which is generated by the automorphism group $\operatorname{Aut} \mathfrak{A}$ contains a model $PM(X)$ which is essentially the second Fraenkel model (in the terminology of Jech [9], i.e. it is generated by $\mathbb{Z}_2^\omega$); c.f. the discussion in Cameron [5], p. 108.

2.3. If G, H are groups and $\tau : G \to \operatorname{Aut} H$, $\operatorname{Aut} H$ the group of group-automorphisms of H, is a homomorphism, then the semidirect product (wreath product) is the following group $H \rtimes_\tau G = H \times G$ with the multiplication $(h, g) \cdot (h', g') = (h \cdot \tau(g)(h'), g \cdot g')$. If $(G, \mathbf{G})$ and $(H, \mathbf{H})$ are topological groups, then $H \rtimes_\tau G$ is a topological group with the product topology, if the map $H \times G \to H$, $(h, g) \to \tau(g)(h)$, is onto and continuous (c.f. Hewitt, Ross [8], Examples 2.6. and 6.20).

We apply this construction to a topological group $H \in PM$, where as before PM is generated by the topological group $(G, \mathbf{G})$ and the group action d. We let $\mathbf{H}^{PM}$ be the group topology of H in PM. It is a base of a group topology $\mathbf{H}^V$ on H in $V(X)$. The mapping τ is induced from the group action of G: $\tau(g)(h) = \hat{d}(g)(h)$. In order to ensure, that this definition is meaningful, we assume, that $G \subseteq \operatorname{stab}(H, \mathbf{H}^{PM})$. Then obviously in V $\tau\colon G \to \operatorname{Aut} H$ is a homomorphism such that $(h, g) \to \tau(g)(h)$ is onto H.

LEMMA. *The semidirect product $H \rtimes_\tau G$ of the topological groups $(H, \mathbf{H}^V)$ and $(G, \mathbf{G})$ is a topological group.*

PROOF. We show, that in V the map $(h, g) \to \hat{d}(g)(h)$ is continuous. Since $\mathbf{H}^{PM}$ is a base for $\mathbf{H}^V$, for $h_0, h_1 \in H$, $g_0 \in G$ and a subgroup $H_1 \in \mathbf{H}^{PM}$ such that $\hat{d}(g_0)(h_0) \in h_1 \cdot H_1$ it suffices to find subgroups $G_2 \in \mathbf{G}$ and $H_2 \in \mathbf{H}^V$, such that for all $g_2 \in G_2$, $h_2 \in H_2$ the image $h = \hat{d}(g_0 g_2)(h_0 h_2) \in h_1 H_1$. We let H_2 be the preimage $H_2 = \hat{d}(g_0)^{-1''} H_1$ which is an open subgroup in PM, since $G \subseteq \operatorname{stab} \mathbf{H}^{PM}$. Hence $G_2 = \operatorname{stab} h_0 \cap \operatorname{stab} H_2$ is open, too. Thus

$$\hat{d}(g_0 g_2)(h_0 h_2) = \hat{d}(g_0)[\hat{d}(g_2)(h_0 h_2)] =$$

$$= \hat{d}(g_0)[\hat{d}(g_2)(h_0) \cdot \hat{d}(g_2)(h_2)] = \hat{d}(g_0)[h_0 \cdot h_3],$$

where $h_3 = \hat{d}(g_2)(h_2) \in H_2$. Hence

$$h = [\hat{d}(g_0)(h_0)] \cdot [\hat{d}(g_0)(h_3)] \in [h_1 \cdot H_1] \cdot [\hat{d}(g_0)'' \hat{d}(g_0)^{-1''} H_1] =$$

$$= [h_1 \cdot H_1] \cdot [H_1] = h_1 \cdot H_1. \qquad e.\ o.\ p.$$

2.4. The natural action of the semidirect product $H \rtimes_\tau G$ on the iterated model $QM \subseteq PM(Y)$ appears to be $(h, g)x = \hat{t}(h)(\tilde{d}(g)(x))$, where $x \in QM$, $\tilde{d}(g)$ is the extension of $\hat{d}(g)$ from Y to $PM(Y)$ and $\hat{t}(h)$ is the extension of $t(h)$ from Y to QM. It is, however, not faithful, whence we shall consider a factor group $(H \rtimes_\tau G)/K$. In order to ensure, that the action makes sense, we need to assume, that $G \subseteq \operatorname{stab}(H, \mathbf{H}^{PM}, t)$. Our discussion can be simplified further by assuming $t = id$, i.e. $H < S^{PM}(Y)$. Moreover in view of 2.1 we may assume, that G is replaced by an appropriate factor group such that $Y = V_0$ is the set of the relevant atoms. Then in V we define the action $s\colon H \rtimes_\tau G \to S(V_0)$ as above; $s(h, g)(x, 0) = h(gx, 0)$ for $x \in X$.

LEMMA. *The action s is a homomorphism whose kernel $K = \ker(s)$ is closed.*

PROOF. Since in $S(V_0)$ the action $\hat{d}(g)(h) = \hat{g} \cdot h \cdot \hat{g}^{-1}$ for $g \in G$ and $h \in H$, where $\hat{g}(x, 0) = (gx, 0)$ is the restriction of $\hat{d}(g)$ to V_0, it easily follows, that s is a homomorphism; e.g.:

$$s[(h_1, g_1)\cdot(h_2, g_2)](a) = s(h_1\cdot\hat{g}_1\cdot h_2\cdot\hat{g}_1^{-1}, \hat{g}_1\cdot\hat{g}_2)(a) =$$

$$= (h_1\cdot\hat{g}_1)\cdot(h_2\cdot\hat{g}_2)(a) = s(h_1, g_1)(s(h_2, g_2)(a))$$

for $h_i \in H$, $g_i \in G$ and $a \in V_0$. That K is closed follows from the assumptions in section 1, that $\mathbf{G}_{\text{fin}} \subseteq \mathbf{G}$ and $\mathbf{H}_{\text{fin}}^{PM} \subseteq \mathbf{H}^{PM}$. We let $(h_\alpha, g_\alpha) \in K$ be a net (no AC is needed in this argument) such that $\lim_\alpha (h_\alpha, g_\alpha) = (h, g)$. Since in view of Lemma 2.3 the mapping $(h, g) \to \hat{d}(g)(h) = \hat{g}\cdot h\cdot\hat{g}^{-1}$ is continuous, it follows from $\mathbf{H}_{\text{fin}} \subseteq \mathbf{H}^V$, that in the discrete topology on V_0 $\lim \hat{g}_\alpha\cdot h_\alpha\cdot\hat{g}_\alpha^{-1}(a) = \hat{g}\cdot h\cdot\hat{g}^{-1}(a)$ for $a \in V_0$; since $(h_\alpha, g_\alpha) \in K$, $\hat{g}_\alpha\cdot h_\alpha\cdot\hat{g}_\alpha^{-1} = \hat{g}_\alpha\cdot(h_\alpha\cdot\hat{g}_\alpha)\cdot\hat{g}_\alpha^{-2} = \hat{g}_\alpha^{-1}$. From the definition of the product topology it follows, that $\lim g_\alpha = g$ in G, whence $\mathbf{G}_{\text{fin}} \subseteq \mathbf{G}$ implies $\lim \hat{g}_\alpha^{-1}(a) = \hat{g}^{-1}(a)$ for $a \in V_0$. Hence $\hat{g}\cdot h\cdot\hat{g}^{-1}(a) = \lim \hat{g}_\alpha^{-1}(a) = \hat{g}^{-1}(a)$ and $\hat{g}\cdot h(a) = a$, whence also $h\cdot\hat{g}(a) = a$ (since $\hat{g}\cdot h\cdot\hat{g} = \hat{g}$) and $(h, g) \in K$. e.o.p.

2.5. From Lemma 2.4 it follows, that $(H \rtimes_\tau G)/K$ is a T_2 topological group. Its topology is group generated, since the sets $H_1 \times G_1$, where $H_1 \in \mathbf{H}^{PM}$ and $G_1 \in \mathbf{G}$ such that $G_1 \subseteq \operatorname{stab} H_1$ are subgroups, form a neighbourhoodbase of the identity in $H \rtimes_\tau G$ consisting of open subgroups (c.f. the proof of Lemma 2.3). We let $\sigma: (H \rtimes_\tau G)/K \to S(V_0)$, $\sigma((h, g)\cdot K) = s(h, g) = h\cdot\hat{g}$ be the induced injective homomorphism and define the Fraenkel-Mostowski model $QM' \subseteq V(V_0)$ from the topological group $(H \rtimes_\tau G)/K$ with the action σ. Concerning the model QM we keep the assumptions from 2.4. Thus the following lemma proves the theorem.

LEMMA. *QM and QM' are $\in$-isomorphic.*

PROOF. As in Lemma 2.1 the isomorphism is defined via the Mostowski collapsing lemma: QM is isomorphic with $QM_1 \subseteq PM$ and QM' with $QM_2 \subseteq V(X)$, whereby $V_0 \subseteq QM_i$ by the definition of the collapsing mapping for rank zero objects as $F((x, 0), 0) = (x, 0)$, $x \in X$. We shall prove by induction on the rank, that $QM_1 = QM_2$ and assume, that $x \subseteq QM_1 \cap QM_2$ for some $x \in V(X)$. For a topological group $T = G, H, F = H \rtimes_\tau G$ or $E = (H \rtimes_\tau G)/K$ the stabilizer corresponding to the action d, t, s or σ will be denoted by stab_T. Then by the definition of QM, $x \in QM_1$, iff $x \in PM$ and $\operatorname{stab}_H(x) \in \mathbf{H}^{PM}$. Since H and t are in PM, $\operatorname{stab}_H(x) \in PM$ for $x \in PM$, whence $x \in QM_1$, iff $x \in PM$ and $\operatorname{stab}_H(x) \in \mathbf{H}^V$, iff $(\operatorname{stab}_H(x)) \times (\operatorname{stab}_G(x))$ is open in the product topology on $H \times G$. Then $\operatorname{stab}_F(x) \in \operatorname{stab}_H(x) \times \operatorname{stab}_G(x)$ is open. If conversely $\mathrm{stab}_F(x)$ is open, then so are $\mathrm{stab}_H(x)$ and $\mathrm{stab}_G(x)$, since F carries the product topology and $\mathrm{stab}_F(x) \cap (H \times \{1_G\}) = (\mathrm{stab}_H(x)) \times \{1_G\}$ and similarily for $\mathrm{stab}_G(x)$, where 1_G is the unit element. Now it follows as in Lemma 2.1 from the definition of the quotient topology, that $x \in QM_1$, iff $\mathrm{stab}_F(x) = \bigcup \mathrm{stab}_E(x)$ is open, iff $\mathrm{stab}_E(x)$ is open, iff $x \in QM_2$. e.o.p.

If $QM \subseteq PM(Y)$ is generated from H and the action t in PM, and if PM is generated from G with the action d, then our lemmas combine to yield the following reconstruction of QM in V. We start with $G_1 = \mathrm{stab}(H, \mathbf{H}^{PM}, Y, t)$ which generates PM, too, and set $G_2 = G_1/p\text{-}\mathrm{stab}_{G_1}(Y)$. Then d induces an action δ of G_2 on $PM(Y)$ (restriction of $\hat{d}$). This action induces a homomorphism τ: $G_2 \to \mathrm{Aut}\, H$ from which we define $H \rtimes_\tau G_2$. This group acts on QM via the product action $s(h, g)(x) = \hat{t}(h)(\delta(g)(x))$ whose kernel K we factor out to obtain a generating topological group $(H \rtimes_\tau G_2)/K$ for QM.

3. Modal interpretations.

Modal formulas are built up from propositional variables v_i, $i \in \omega$, the constant F for false, the connective $\to$ for implication and the modal operator $\Diamond$ of possibility. An interpretation $(\cdot)^*$ of the modal language is a function, that assigns to each modal formula ϕ a ZF sentence ϕ^*. The interpretation of the variable v_i is arbitrary but fixed. It is prolongated inductively through the clauses $F^* =$ «$\emptyset \neq \emptyset$», $(\phi \to \psi)^* =$ «$\phi^* \to \psi^*$» and the key clause $(\Diamond\phi)^* =$ «$\exists G, \cdot, \mathbf{G}, X, d$: $\alpha \wedge Rel(\mu, \phi^*)$», where α and μ are ZF-formulas such that $\alpha(G, \cdot, \mathbf{G}, X, d)$ expresses «G is a group generated T_2 topological group and d: $G \to S(X)$ is an injective homomorphism», $\mu(G, \cdot, \mathbf{G}, X, d, x)$ is the ZF sentence from section 1, that $x \in PM \subseteq V(X)$, and $Rel(\mu, \phi^*)$ is the relativization of ϕ^* to PM (i.e. $\exists x : \beta$ in ϕ^* is replaced by $\exists x : \mu(x) \wedge \beta$). Thus $(\Diamond\phi)^*$ says, that ϕ^* is Fraenkel-Mostowski consistent. (We note, that in view of [13], pp. 287-289, the notion of satisfaction is absolute: $PM \models Rel(QM, \phi)$ for some permutation model QM which is constructed within PM, iff $QM' \models \phi$, QM' the model of Theorem 2.) A modal formula ϕ is FM_S-valid, if for the system S of set theory (extending ZFA) ϕ^* is provable for all interpretations $(\cdot)^*$. The set of all FM_S-valid formulas is a propositional modal logic, provided that S is consistent. For it includes the tautologies of classical propositional logic, it is closed under the rules of detachment under material implication and uniform substitution of modal formulas for propositional variables and in view of $S \supseteq ZFA$ and the definition of validity it contains the axioms $\boldsymbol{K}$ and *

($\boldsymbol{K}$: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, $*$: $\neg \Diamond F$). In order to construct a system which is compatible with the rule of necessitation we set $\mathcal{S} = FM$, the set of all sentences which are true in all Fraenkel-Mostowski models over a fixed ZFC universe V.

THEOREM. *If ZF is consistent, then a modal formula ϕ is FM_{FM}-valid, iff ϕ is in* S5.

The major step in the proof is the soundness of S5. As in the case of the Bernays-Rieger models the completeness then follows from the pretabularity of S5.

3.1 LEMMA. *If ZF is consistent, then the FM_{FM}-valid formulas form a normal extension of* S5.

PROOF. Theorem 2 implies the rule of necessitation. For if ϕ^* is true in all Fraenkel-Mostowski models but PM does not satisfy $(\Box\phi)^*$, then for some model QM which is constructed within PM ϕ^* is false, but QM is isomorphic to a Fraenkel-Mostowski model, whence this is impossible. A similar argument proves the soundness of axiom **4**: $\Box\phi \to \Box\Box\phi$, since the proof of Theorem 2 does not depend on AC. Since $PM(V_0)$ is a Fraenkel-Mostowski model in PM which is isomorphic to PM, the axiom $\boldsymbol{T}$ is sound: $\Box\phi \to \phi$. As any model QM may be interpreted in $V(\emptyset) \subseteq PM$, the axiom $\boldsymbol{E}$ is sound, too: $\Diamond\phi \to \to \Box\Diamond\phi$. e.o.p.

We note, that FM_{FM} does not contain the axiom $\boldsymbol{Tr}$: $\phi \leftrightarrow \Box\phi$. For otherwise the $ZFA + AC$ model $V(\emptyset)$ would satisfy $(\Box\phi)^*$ for the formula $\phi =$ «v_0» and the interpretation $(v_0)^* =$ «AC». This contradicts the fact, that if ZF is consistent, then AC fails in all Fraenkel-Mostowski submodels of $V(\emptyset)$ whose generating groups are not discrete.

3.2 LEMMA. *If ZF is consistent, then all FM_{FM}-valid modal formulas are in* S5.

PROOF. According to Scroggs [11] a proper normal extension of S5 satisfies one of Dugundji's axioms $\boldsymbol{D}_n$: $\bigvee_{i<j\leq n} v_i \leftrightarrow v_j$ for some $n \geq 1$. If ZF is consistent, then $\boldsymbol{D}_n$ cannot even be FM_{ZFC}-valid, as follows from Easton [6] when applied to the interpretation $v_i^* =$ «$2^{\aleph_i} = \aleph_{i+1}$». For FM_{FM}-validity Mostowski's results on finite choice axioms suffice (Jech [9], Theorem 7.15). e.o.p.

The above interpretation $v_i^* =$ «AC for families of all n-element sets, $0 < n \leqslant p_i$» where p_i is the i^{th} prime does not suffice to improve Theorem 3 by defining validity with respect to this fixed interpretation only. Any such system FM^*_{FM} would either contain the inconsistency $\boldsymbol{D}_1$ or the substitution rule would fail ($(\lozenge(\neg v_i \rightarrow v_i)$ is valid for some $0 \leqslant$ $\leqslant i \leqslant 1$ but $\lozenge(\neg F \rightarrow F)$ is invalid, if ZF is consistent). This is in contrast with the uniform version of the completeness theorem due to Artemov and Montagna (et alii) for the Gödel-Löb system **G**.

3.3 CONCLUSION. Motivated by Luce [10], Baaz, Brunner, Svozil [1] propose the thesis, that the notion of empirically meaningful concepts may be represented by means of Fraenkel-Mostowski models. The modal interpretation of this section therefore describes the logic of concepts. The fact, that it is **S5**, enhances the proposed philosophical thesis, since it means, that the restrictions on the perception and the language of observers as given by the models do not affect the capacity to reason about the perception of others. This is what is expected for observers with the same mathematical background $\mathcal{S} = FM$ of knowledge which is independent of the empirical base.

Reasoning about the perception of Laplacean demons is described by the accessibility relation $PM \mathcal{R} QM$, iff $QM = PM(X)$ for some $X \in PM$. It corresponds to the frame $\mathcal{G}$ of all group-generated T_2-topological groups and the relation $G\mathcal{R}H$, iff H is a factor group of G (c.f. 2.1 and 2.2). This motivates the following modal interpretation: Variables are interpreted by Boolean combinations of Jech-Sochor bounded sentences $v^* = \alpha$. Connectives are interpreted as before and $G \models$ $\models (\lozenge \phi)^*$, iff some quotient $H \models \phi^*$. Here $(G, \mathbf{G}) \models \alpha$ means, that the permutation models PM which are generated by $(G, \mathbf{G}) = (G, \mathbf{G}_{\text{nat}})$ satisfy α (c.f. 1.2). The investigation of the system $\mathcal{M}$ of all modal formulas ϕ such that $G \models \phi^*$ for all groups G and all interpretations seems to establish a link between topological group theory and modal logic.

REFERENCES

[1] M. BAAZ - N. BRUNNER - K. SVOZIL, *Effective Quantum Observables*, e-print 9501018, QUANT-PH@XXX.LANL.GOV.

[2] A. BLASS - A. SCEDROV, *Freyd's Models for the Independence of the Axiom of Choice*, Memoirs AMS, **79**, Providence (1989).

[3] N. BRUNNER - J. RUBIN, *Permutation models and topological groups*, Rend. Sem. Mat. Univ. Padova, **76** (1986), pp. 149-161.

[4] N. BRUNNER, *The Fraenkel-Mostowski method, revisited*, Notre Dame J. Formal Logic, **31** (1990), pp. 64-75.

[5] P. CAMERON, *Oligomorphic Permutation Groups*, LMS Lecture Notes, **152**, Cambridge (1990).
[6] W. EASTON, *Powers of regular cardinals*, Ann. Math. Logic, **1** (1970), pp. 139-178.
[7] T. FORSTER, *Permutation models in the sense of Rieger-Bernays*, Zeitschrift Math. Logik Grundl. Math., **33** (1987), pp. 201-210.
[8] E. HEWITT - K. ROSS, *Abstract Harmonic Analysis, I,* Springer Grundlehren, **115**, Berlin (1963).
[9] T. JECH, *The Axiom of Choice,* North-Holland Studies in Logic, **75**, Amsterdam (1973).
[10] D. LUCE, *Dimensionally invariant laws correspond to meaningful qualitative relations*, Philosophy of Science, **45** (1978), pp. 81-95.
[11] S. SCROGGS, *Extensions of the Lewis system* S5, J. Symbolic Logic, **16** (1951), pp. 112-120.
[12] R. SOLOVAY, *Provability interpretations of modal logic*, Israel J. Math., **25** (1976), pp. 287-304.
[13] Y. SUZUKI - G. WILMERS, *Non-standard models for set theory*, in J. BELL et. al. (ed.): *Proceedings of the Bertrand Russell Memorial Logic Conference, Leeds* (1973), pp. 278-314.

Manoscritto pervenuto in redazione il 30 aprile 1993
e, in forma revisionata il 4 ottobre 1993.