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An Algebraic Summation Over the Set of Partitions and Some Strange Evaluations.

CHU WENCHANG (*) (**)

ABSTRACT - By means of formal power series operation, a general algebraic summation formula over the set of partitions is established. Several combinatorial identities are demonstrated as special cases.

Let Ω be a subset of non-negative integers and x_0, x_1, \dots, x_n the indeterminates. Evaluate

$$(1) \quad E_{m \dots n}(x|\Omega) := \sum_{\kappa \in \tau(m, n|\Omega)} \{x_{k_1}(x_{k_1} + x_{k_2}) \dots (x_{k_1} + x_{k_2} + \dots + x_{k_n})\}^{-1}$$

where $\tau(m, n|\Omega)$ is the set of all n -tuples $\kappa = (k_1, k_2, \dots, k_n) \in \Omega^n$ with sum m .

For each solution $\kappa = (k_1, k_2, \dots, k_n) \in \tau(m, n|\Omega)$ of the equation $k_1 + k_2 + \dots + k_n = m$ ($k_i \in \Omega$), define its type by the partition $\rho = [0^{p_0} 1^{p_1} \dots m^{p_m}]$, if number k appears p_k times in this solution (k_1, k_2, \dots, k_n) for $0 \leq k \leq m$ (it is obvious that $p_k = 0$ if $k \notin \Omega$). Then the solutions with the same type ρ are generated by the different permutations $S(\rho)$ of multi-set $\mathfrak{p} = \{0^{p_0}, 1^{p_1}, \dots, m^{p_m}\}$. Thus, we can classify the solution-set $\tau(m, n|\Omega)$ of that equation according to the partitions $\sigma(m, n|\Omega) = \{\rho = [0^{p_0} 1^{p_1} \dots m^{p_m}]: \sum k p_k = m, \sum p_k = n \text{ and } p_k = 0 \text{ for } k \notin \Omega\}$, of number m into n -parts with each part restricted in Ω . Based on this observation, the summation defined by (1) can be decom-

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posed as

$$(2) \quad E_{m \cdot n}(x|\Omega) = \\ = \sum_{\rho \in \sigma(m, n|\Omega)} \sum_{\pi \in S(\rho)} \{x_{\pi(1)}(x_{\pi(1)} + x_{\pi(2)}) \dots (x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n)})\}^{-1}.$$

As a crucial lemma, it is an easy exercise (Chu [2], 1989) to show, by the induction principle on $n = \sum p_k$, that.

LEMMA.

$$(3) \quad \sum_{\pi \in S(\rho)} \{x_{\pi(1)}(x_{\pi(1)} + x_{\pi(2)}) \dots (x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(n)})\}^{-1} = \\ = \left\{ \prod_{k=0}^m p_k! x_k^{p_k} \right\}^{-1}.$$

It follows from substituting (3) into (2), that

$$(4) \quad E_{m \cdot n}(x|\Omega) = \sum_{\rho \in \sigma(m, n|\Omega)} \left\{ \prod_{k=0}^m p_k! x_k^{p_k} \right\}^{-1}$$

which could be expressed equivalently as

$$(5) \quad E_{m \cdot n}(x|\Omega) = (n!)^{-1} \sum_{\kappa \in \tau(m, n|\Omega)} \left\{ \prod_{i=1}^n x_{k_i} \right\}^{-1}$$

in view of the fact that the number of different permutations $S(\rho)$ of multi-set $\mathbf{p} = \{0^{p_0}, 1^{p_1}, \dots, m^{p_m}\}$ is equal to $n!/\prod p_i!$. Both (4) and (5) lead to a general algebraic summation formula if we denote by $[t^m]P(t)$ the coefficients of t^m in the power series expansion of function $P(t)$.

THEOREM.

$$(6) \quad E_{m \cdot n}(x|\Omega) := [t^m] \left\{ \sum_{k \in \Omega} t^k / x_k \right\}^n / n!.$$

Let Γ be the Gamma function. Denote the lower and upper factorials, respectively, by

$$[z]_n = z(z-1)\dots(z-n+1) = \Gamma(1+z)/\Gamma(1+z-n)$$

and

$$(z)_n = z(z+1)\dots(z+n-1) = \Gamma(z+n)/\Gamma(z).$$

Now we are ready to exhibit some special evaluations. For convenience, we denote by $\Phi(t, x|\Omega)$ the power series inside the bracket in equation (6).

(A) Let $\Omega = \mathbb{N}_0$, the set of non-negative integers, and $x_k = y^{-k} / \binom{x}{k}$. Then the corresponding Φ -function is $\Phi(t, x|\Omega) = (1 + yt)^x$ which results in

$$(7) \quad E_{m \dots n} \left(x_k = y^{-k} / \binom{x}{k} \mid \mathbb{N}_0 \right) = y^m \binom{nx}{m} / n!.$$

Taking $x = -1$ and $y = -1/a$ in (6), we get

$$(8) \quad E_{m \dots n} (x_k = a^k \mid \mathbb{N}_0) = a^{-m} \binom{m + n - 1}{m} / n!$$

whose special case corresponding to $a = 2$ is due to Knuth and Pittel[4].

$$\sum_{k_1 + k_2 + \dots + k_n = m} \{2^{k_1} (2^{k_1} + 2^{k_2}) \dots (2^{k_1} + 2^{k_2} + \dots + 2^{k_n})\}^{-1} = 2^{-m} (n)_m / m! n!.$$

Alternatively, the limiting case of (6) for $1/x$ and y tending to zero under condition $xy = 1/c$ yields another formula

$$(9) \quad E_{m \dots n} (x_k = k! c^k \mid \mathbb{N}_0) = n^m c^{-m} / m! n!$$

which gives, for $c = 1$, another evaluation of Knuth and Pittel[4].

$$\sum_{k_1 + k_2 + \dots + k_n = m} \{k_1! (k_1! + k_2!) \dots (k_1! + k_2! + \dots + k_n!)\}^{-1} = n^m / m! n!.$$

(B) If we take $\Omega = \mathbb{N}$, the positive integers. Then for $x_k = y / \binom{x}{k-1}$, it holds that $\Phi(t, x|\Omega) = yt(1 + yt)^x$. From this, the shifted version of (7) follows

$$(10) \quad E_{m \dots n} \left(x_k = y^{-k} / \binom{x}{k-1} \mid \mathbb{N} \right) = y^m \binom{nx}{m-n} / n!, \quad (m \geq n)$$

which reduces when $x = -1$, $y = -1/a$ and $x = -2$, $y = -1/c$, re-

spectively, to the formulae:

$$(11) \quad E_{m..n}(x_k = a^k | \mathbb{N}) = a^{-m} \binom{m-1}{n-1} / n!, \quad (m \geq n),$$

$$(12) \quad E_{m..n}(x_k = c^k/k | \mathbb{N}) = c^{-m} \binom{m+n-1}{m-n} / n!, \quad (m \geq n).$$

Similarly, one can compute the following summations:

$$(13) \quad E_{m..n}(x_k = (-1)^{k-1} kx^k | \mathbb{N}) = x^{-m} S_1(m, n)/m!, \quad (m \geq n),$$

$$(14) \quad E_{m..n}(x_k = kx^k | \mathbb{N}) = x^{-m} S_1^*(m, n)/m!, \quad (m \geq n),$$

$$(15) \quad E_{m..n}(x_k = k!x^k | \mathbb{N}) = x^{-m} S_2(m, n)/m!, \quad (m \geq n),$$

where $S_1(m, n)$, $S_1^*(m, n)$ and $S_2(m, n)$ are the Stirling numbers defined by

$$[z]_m = \sum_{n \leq m} S_1(m, n) z^n,$$

$$(z)_m = \sum_{n \leq m} S_1^*(m, n) z^n,$$

and

$$z^m = \sum_{n \leq m} S_2(m, n) [z]_n.$$

(C) Recall the generating function of Hagen-Rothe coefficients (Gould [3], 1956)

$$\sum_{k \geq 0} \frac{a}{a+bk} \binom{a+bk}{k} u^k = v^a, \quad u = (v-1)v^{-b}.$$

We can extend the results displayed in (A) as an identity on binomial coefficients

$$(16) \quad E_{m..n} \left(x_k = c^{-k} \left\{ \frac{a}{a+bk} \binom{a+bk}{k} \right\}^{-1} | \mathbb{N}_0 \right) = \\ = \frac{an}{an+bm} \binom{an+bm}{m} \frac{c^m}{n!}$$

as well as its Abel-analogue

$$(17) \quad E_{m..n}(x_k = ac^{-k}(a+bk)^{-1}/k! | \mathbb{N}_0) = anc^m (an+bm)^{m-1}/n!$$

where the latter is the limiting version of the former under replacements $a \rightarrow aM$, $b \rightarrow bM$ and $c \rightarrow cM$ when $M \rightarrow \infty$.

More generally, for any Sheffer-sequences generated by

$$\sum_{k \in \Omega} t^k / X_k(\lambda) = t^\delta \exp[\lambda\psi(t)], \quad (\delta \in \Omega)$$

all the formulas exhibited above could be formally unified as

$$E_{m..n}(x_k = y^{-k} X_k(\lambda) | \Omega) = y^m X_{m-n\delta}(n\lambda) / n!.$$

For particular settings of $\psi(t)$, this identity could be used to create numerous other evaluations. But the resulting relations are too messy to be stating unless when necessary.

REMARK. The evaluations (7)-(17) demonstrated in this note may also be reformulated in the summations of (4)-(5). Some of them can be found in Chu [2] (1989).

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