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On Total Differential Inclusions.

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ABSTRACT - We show the existence of a solution to the total differential inclusion:

$$\nabla u(x) \in \operatorname{ext} F(x, u(x)) \quad x \in \Omega, \qquad u = u_0 \quad \text{on } \partial\Omega,$$

assuming that the convexified problem

$$\nabla u(x) \in \operatorname{int} \overline{\operatorname{co}}\ F(x, u(x)) \quad x \in \Omega, \qquad u = u_0 \quad \text{on } \partial\Omega,$$

admits a smooth solution. The proof relies on a Baire category argument. Some examples are given, showing that in general our hypotheses cannot be relaxed.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be open, and let $F\colon \Omega \times \mathbb{R} \mapsto 2^{\mathbb{R}^n}$ be a continuous multifunction. Given a map $u_0\colon \partial\Omega \mapsto \mathbb{R}$, we are concerned with the boundary value problem

$$\nabla u(x) \in F(x, u(x)) \quad x \in \Omega, \tag{1.1}$$

$$u(x) = u_0(x) \quad x \in \partial\Omega. \tag{1.2}$$

By a solution of (1.1), (1.2) we mean a locally Lipschitz continuous function $u\colon\overline{\Omega} \mapsto \mathbb{R}$ which coincides with U_0 on $\partial\Omega$ and satisfies (1.1) almost everywhere.

We remark that, if each set $F(x, u)$ is contained in some hyperplane,

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say

$$F(x, u) \subset \left\{ y = (y_1, \ldots, y_n);\ \sum_{i=1}^{n} a_i(x, u)\, y_i = c(x, u) \right\},$$

then our problem is essentially overdetermined. Indeed, the first order quasilinear P.D.E.

$$\sum_{i=1}^{n} a_i(x, u)\, u_{x_i}(x) = c(x, u) \qquad x \in \Omega \tag{1.3}$$

with boundary conditions (1.2) usually has at most one solution, which can be obtained by the classical method of characteristics. When this solution is found, one only has to check whether it satisfies the additional conditions (1.1).

The previous observation indicates that differential inclusions involving the gradient of u, such as (1.1), may have independent interest only when the sets $F(x, u)$ have maximal dimension, i.e. $\mathrm{int}\, \overline{\mathrm{co}}\, F(x, u) \neq \emptyset$. By $\overline{K}$, $\mathrm{ext}\, K$ and $\mathrm{int}\, \overline{\mathrm{co}}\, K$ we denote here the closure, the extreme points and the interior of the closed convex hull of a set K, respectively. The main result of this paper establishes the existence of extremal solutions to (1.1), assuming that the convexified problem

$$\nabla u(x) \in \mathrm{int}\, \overline{\mathrm{co}}\, F(x, u) \qquad x \in \Omega \tag{1.4}$$

has a smooth solution. More precisely, one has:

THEOREM 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, F: $\Omega \times \mathbb{R} \mapsto 2^{\mathbb{R}^n}$ be a bounded continuous multifunction with compact values, u_0: $\partial\Omega \mapsto \mathbb{R}$ be continuous. If the boundary value problem* (1.4), (1.2) *has a solution which is continuously differentiable on Ω, then the inclusion*

$$\nabla u(x) \in \mathrm{ext}\, F(x, u(x)) \qquad x \in \Omega \tag{1.5}$$

also has a solution satisfying (1.2).

The proof, given in § 2, relies on a Baire category argument similar to [4]. As an easy consequence, this yields a local existence result:

COROLLARY 1. *Let F: $\Omega \times \mathbb{R} \mapsto 2^{\mathbb{R}^n}$ be a continuous multifunction with compact values. If* $\mathrm{int}\, \overline{\mathrm{co}}\, F(x_0, \omega) \neq \emptyset$, *then there exists a neighborhood V of x_0 and a solution u: $V \mapsto \mathbb{R}$ of* (1.5) *with $u(x_0) = \omega$.*

Indeed, choosing any $y \in \mathrm{int}\, \overline{\mathrm{co}}\, F(x_0, \omega)$ the linear function

$$\tilde{u}(x) = \omega + y \cdot (x - x_0)$$

is a smooth solution of (1.4), defined on some open neighborhood V of x_0. We can thus apply Theorem 1 to the set $\Omega = V \setminus \{x_0\}$.

In § 3 we provide counterexamples showing that the assumptions of Theorem 1 cannot be relaxed. In particular, denoting by *relint* the relative interior of a convex set, we prove that the existence of smooth solutions to

$$\nabla u(x) \in \operatorname{relint} \overline{\operatorname{co}}\, F(x, u(x)) \tag{1.6}$$

or

$$\nabla u(x) \in \overline{\operatorname{int} \overline{\operatorname{co}}\, F(x, u(x))} \tag{1.7}$$

does not guarantee the solvability of (1.1).

2. Proof of Theorem 1.

Since $\operatorname{ext} \overline{\operatorname{co}}\, F(x, u) = \operatorname{ext} F(x, u)$, without loss of generality we can assume that all values of F are convex. We begin by defining a set $\mathcal{S}$ of piecewise continuously differentiable solutions of (1.1), (1.4):

$$\begin{aligned}\mathcal{S} \doteq \{u \in \mathcal{C}(\overline{\Omega}, \mathbb{R});\ \ u = u_0 \text{ on } \partial\Omega, \text{ there exist hyperplanes } H_1, \ldots, H_\nu \\ \text{such that } u \in \mathcal{C}^1(\Omega \setminus (H_1 \cup \ldots \cup H_\nu)),\ \ \nabla u(x) \in \operatorname{int} \overline{\operatorname{co}}\, F(x, u(x)) \\ \forall x \in \Omega \setminus (H_1 \cup \ldots \cup H_\nu)\}.\end{aligned}$$

By assumption, $\mathcal{S} \neq \emptyset$. We will apply Baire's category theorem to the complete metric space $\overline{\mathcal{S}}$, where the overline denotes closure in the topology of $\mathcal{C}^0(\overline{\Omega})$.

Because of the convexity of the values of F, every $u \in \overline{\mathcal{S}}$ is a Lipschitz continuous solution of (1.1), (1.2). In analogy with [1], we now introduce the *likelihood* of a solution u of (1.1) by setting

$$L(u) \doteq \int_{\Omega} h(\nabla u(x), F(x, u(x)))\, dx,$$

with

$$h(y, K) \doteq \inf \left\{ \left(\int_0^1 |\phi(\xi) - y|^2 d\xi \right)^{1/2} ;\ \ \phi \colon [0, 1] \mapsto K,\ \int_0^1 \phi(\xi)\, d\xi = y \right\}$$

for every compact convex $K \subset \mathbb{R}^n$ and every $y \in K$. Some basic properties of the function h were proved in [1].

LEMMA 1. *The map* $(y, K) \mapsto h(y, K)$ *is upper semicontinuous in both variables; for each K, the function* $y \mapsto h(y, K)$ *is strictly concave down. Moreover, one has*

$$h(y, K) = 0 \quad \textit{if and only if } y \in \operatorname{ext} K, \tag{2.1}$$

$$h^2(y, K) \leq r^2(K) - |y - c(K)|^2, \tag{2.2}$$

where c(K) and r(K) denote the Cebyshev center and the Cebyshev radius of K, respectively.

From [3, p.74] and the above lemma, it follows that the likelihood functional L is upper semicontinuous on $\overline{S}$. Therefore, for every $\eta > 0$, the set

$$A_\eta \doteq \{u \in \overline{S};\ L(u) < \eta\}$$

is relatively open in $\overline{S}$. In the next step of the proof, we will show that each A_η is also dense in $\overline{S}$.

Let $\tilde{u} \in S$, η, $\varepsilon > 0$ be given. We will construct a function $u \in S$ such that

$$\|u - \tilde{u}\|_{\mathcal{C}^0(\Omega)} \leq \varepsilon, \qquad L(u) < \eta. \tag{2.3}$$

By definition of S, there exist finitely hyperplanes $H_1, \ldots, H_\nu$ such that $\tilde{u}$ is continuously differentiable and satisfies (1.4) at every point of the open set

$$\Omega^\dagger \doteq \Omega \setminus (H_1 \cup \ldots \cup H_\nu).$$

Let M be a constant so large that all images $F(x, u)$ are contained in the closed ball $\overline{B}(0, M)$. By (2.2) this implies

$$h(\nabla\tilde{u}(x), F(x, \tilde{u}(x))) \leq M \qquad \forall x \in \Omega^\dagger. \tag{2.4}$$

Using Carathéodory's theorem and the properties of h, for every $x \in \Omega^\dagger$ we can find points $v_0, \ldots, v_n \in \operatorname{int} F(x, \tilde{u}(x))$ and coefficients $p_0, \ldots, p_n \in [0, 1]$ such that

$$\nabla\tilde{u}(x) = \sum_{i=0}^{n} p_i v_i, \qquad \sum_{i=0}^{n} p_i = 1, \tag{2.5}$$

$$\nabla\tilde{u}(x) \in \operatorname{int} \overline{\operatorname{co}}\{v_0, \ldots, v_n\}, \tag{2.6}$$

$$h(v_i, F(x, \tilde{u}(x))) < \frac{\eta}{2\operatorname{meas}(\Omega)}. \tag{2.7}$$

Using the continuity of F and $\nabla\tilde{u}$, and the upper semicontinuity of h, we can find $\delta > 0$ such that

$$v_i + \nabla\tilde{u}(x') - \nabla\tilde{u}(x) \in F(x', \xi), \tag{2.8}$$

$$h(v_i + \nabla\tilde{u}(x') - \nabla\tilde{u}(x), F(x', \xi)) < \frac{\eta}{2\,\mathrm{meas}\,(\Omega)} \tag{2.9}$$

whenever $|x' - x| < \delta$, $|\xi - \tilde{u}(x)| < \delta$, $i = 0, \ldots, n$.

Now consider the set

$$\Gamma \doteq \overline{\mathrm{co}}\,\{v_i - \nabla\tilde{u}(x);\quad i = 0, \ldots, n\}$$

and its polar

$$\Gamma^* = \{z \in \mathbb{R}^n;\quad z \cdot v \leqslant 1 \ \ \forall v \in \Gamma\}.$$

Because of (2.6), both Γ and Γ^* are bounded and contain the origin as an interior point. By Lemma 1 in [2], there exists a continuous, piecewise linear function $w\colon \Gamma^* \mapsto \mathbb{R}$ satisfying

$$w(z) = 0 \ \ \forall z \in \partial\Gamma^*, \qquad \nabla w(z) \in \{v_i - \nabla\tilde{u}(x);\quad i = 0, \ldots, n\} \quad \text{for a.e. } z \in \Gamma^*. \tag{2.10}$$

We then choose $\hat{\rho} > 0$ such that

$$\hat{\rho} \cdot \max\{|z| + |w(z)|;\ z \in \Gamma^*\} < \min\{\delta, \varepsilon\}, \qquad x + \hat{\rho}\Gamma^* \subset \Omega^\dagger. \tag{2.11}$$

The previous construction can be repeated for every $x \in \Omega^\dagger$ obtaining a family of polar sets Γ_x^*, piecewise linear functions w_x and constants $\hat{\rho}_z > 0$ as in (2.5)-(2.11). Observe that the family of sets

$$\{x + \rho\Gamma_x^*;\quad x \in \Omega^\dagger,\quad 0 < \rho \leqslant \hat{\rho}_x\}$$

is a Vitali covering of the bounded open set $\Omega^\dagger$. Applying Vitali's theorem [5, p. 109], we thus obtain a finite family of disjoint sets, say

$$\{x_j + \rho_j\Gamma_j^*;\quad j = 1, \ldots, N\},$$

such that

$$\mathrm{meas}\left(\Omega^\dagger \setminus \bigcup_{j=1}^{N}(x_j + \rho_j\Gamma_j^*)\right) < \frac{\eta}{2M}. \tag{2.12}$$

Let $w_j\colon \Gamma_j^* \mapsto \mathbb{R}$ be the corresponding piecewise linear functions. We now define $u\colon \Omega \mapsto \mathbb{R}$ by setting

$$u(x) = \begin{cases} \tilde{u}(x)^* \rho_j w_j\left(\dfrac{x - x_j}{\rho_j}\right) & \text{if } x \in x_j + \rho_j \Gamma_j^* \text{ for some } j\,, \\ \tilde{u}(x) & \text{otherwise}\,. \end{cases}$$

From the above definition it is clear that u is $\mathcal{C}^1$ on the complement of finitely many hyperplanes. Since $\rho_j \leqslant \hat{\rho}_{x_j}$, (2.11) implies

$$\left| \rho_j w_j\left(\frac{x - x_j}{\rho_j}\right)\right| < \varepsilon \qquad \forall x \in x_j + \rho_j \Gamma_j^*\,,$$

hence $\|u - \tilde{u}\|_{\mathcal{C}^0} < \varepsilon$. Moreover, (2.10) and (2.9) together imply $\nabla u(x) \in F(x, u(x))$ and

$$\text{(2.13)} \qquad h(\nabla u(x), F(x, u(x))) \leqslant \frac{\eta}{2\,\mathrm{meas}\,(\Omega)} \qquad \forall x \in x_j + \rho_j \Gamma_j^*\,.$$

Since means $(\Omega \setminus \Omega^\dagger) = 0$, using (2.4), (2.12), (2.13), the likelihood of u can be estimated by

$$L(u) = \sum_{j=1}^{N} \int_{x_j + \rho_j \Gamma_j^*} h(\nabla u(x), F(x, u(x)))\,dx +$$

$$+ \int_{\Omega \setminus \bigcup_j (x_j + \rho_j \Gamma_j^*)} h(\nabla \tilde{u}(x), F(x, \tilde{u}(x)))\,dx <$$

$$< \mathrm{meas}\,(\Omega) \cdot \frac{\eta}{2\,\mathrm{meas}\,(\Omega)} + \frac{\eta}{2M} \cdot M = \eta\,.$$

This establishes the density of each A_η in $\mathcal{S}$, and hence in $\bar{\mathcal{S}}$, as claimed. Consider now the countable intersection

$$A \doteq \bigcap_{k \geqslant 1} A_{1/k}\,.$$

Since every $A_{1/k}$ is open and dense in the complete metric space $\bar{\mathcal{S}}$, by Baire's theorem A is nonempty. If $u \in A$, then u is a solution of (1.1), (1.2). Moreover, $L(u) = 0$. The definition of L and (2.1) now imply $\nabla u(x) \in \mathrm{ext}\,F(x, u(x))$ almost everywhere, proving the theorem.

3. Examples.

EXAMPLE 1. On $\mathbb{R}^2$, consider the multifunction

$$F(x_1, x_2, u) = \{(0, x_1), (0, -x_1)\}.$$

Then the constant function $\tilde{u}(x_1, x_2) \equiv 0$ satisfies (1.6), but the inclusion (1.1) does not have solutions on any open set. Indeed, if $\nabla u(x) \in F(x)$ on some open ball Ω, then $u_{x_1} \equiv 0$ and $u(x_1, x_2) = \psi(x_2)$ for some Lipschitz continuous function ψ. If $\nabla u(a, b) = (0, \psi'(b))$ exists at some point $(a, b) \in \Omega$, then $\nabla u(x_1, b) = (0, \psi'(b))$ for all $(x_1, b) \in \Omega$. Since $\psi'(b) \notin \{-x_1, x_1\}$ for almost every x_1, the map u cannot satisfy (1.1).

EXAMPLE 2. On $\mathbb{R}^2$, consider the circular crown given (in polar coordinates) by

$$\Omega = \{(\rho, \theta);\ 1 < \rho < 2,\ 0 \leqslant \theta \leqslant 2\pi\}.$$

Using again polar coordinates, define the multifunction $F: \Omega \mapsto 2^{\mathbb{R}^2}$ as

$$F(\rho, \theta) = \{(r, \alpha);\ r = 2 + \sin\theta,\ \theta \leqslant \alpha \leqslant \theta + \pi\}. \tag{3.1}$$

Observe that the values of $\overline{\mathrm{co}}\, F$ are half-circles, with nonempty interior, and that the function $\tilde{u} \equiv 0$ is a solution of (1.7). However, the equation (1.1) has no solution on Ω. Indeed, if $u = u(\rho, \theta)$ were such a solution, the definition (3.1) of F implies $\partial u/\partial\theta \geqslant 0$. Since $u(\rho, 0) = u(\rho, 2\pi)$, we must have

$$\frac{\partial u}{\partial \theta} \equiv 0, \qquad u = \psi(\rho), \tag{3.2}$$

for some Lipschitz continuous function $\psi:]1, 2[\mapsto \mathbb{R}$. This and (3.1) in turn imply

$$\frac{\partial u}{\partial \rho}(\rho, \theta) \in \{2 + \sin\theta,\ -2 - \sin\theta\}. \tag{3.3}$$

We now reach a contradiction because on one hand, by (3.2), $\partial u/\partial\rho = \psi'(\rho)$ is a function of ρ alone. On the other hand, by (3.3), $\partial u/\partial\rho$ must vary with θ.

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